The Slow-Coloring Game

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slides available on DBW preprint page

Joint work with
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Def. sum-color cost $\mathring{s}(G)$ - score under optimal play.

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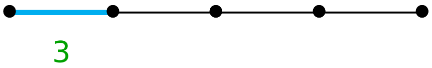

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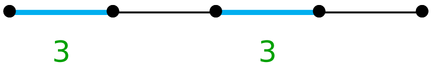

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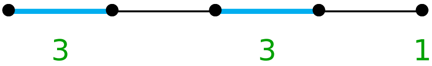

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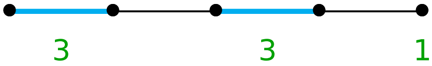

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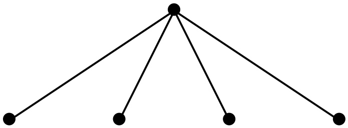

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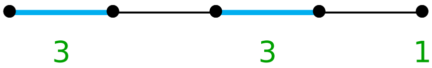

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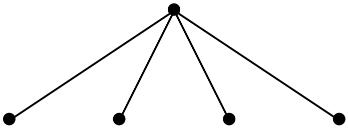

Lister marking all or just an edge gets only 6.

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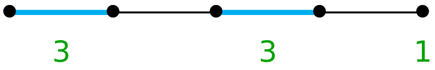

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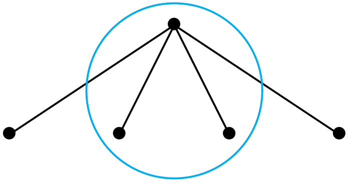

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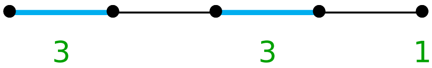

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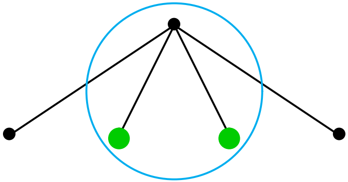

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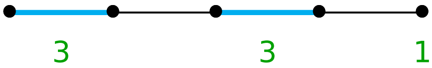

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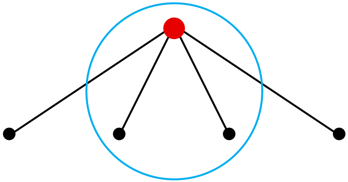

Score is $3 + \mathring{s}(P_3)$ or $3 + 4$.

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Pf. Using the sets in an optimal coloring, in size order, the average index of color used is at most $\frac{\chi(G)+1}{2}$. ■

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$\therefore$ The maximum $f_{n,k}$ of $\mathring{s}(G)$ over k-chromatic graphs with n vertices occurs on a complete k-partite graph, so

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Ques. How to improve the upper bound $\chi(G)n$ for special classes of graphs?

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d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.
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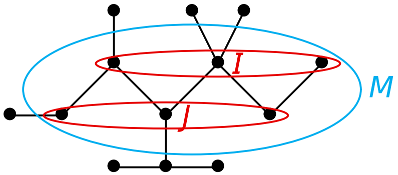
Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter can delete I or J.



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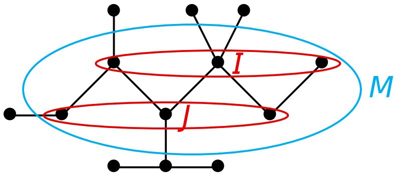

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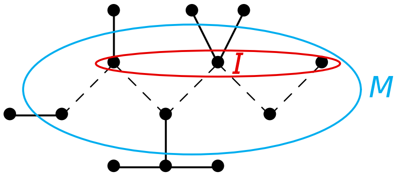

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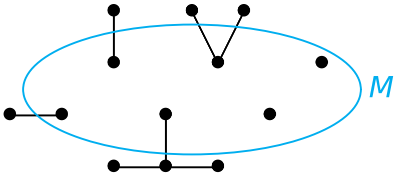
Let $T' = T - E(T[M])$; one vert. of M in each component.



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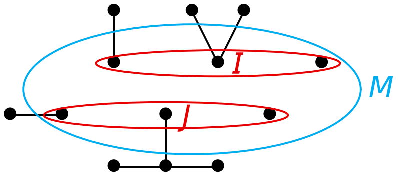
Given the bipartition (I, J) of M; Painter can delete I or J.
Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\mathring{s}(T - I) + \mathring{s}(T - J)\right]$.
Inductively, $\mathring{s}(T - I) \le \frac{3}{2}(n - |I|) - \frac{1}{2}o(T - I)$.

Let $T' = T - E(T[M])$; one vert. of M in each component.

Whether the component of T' containing $v \in M$ is even or odd, it saves $\frac{1}{2}$ when Painter colors one of I and J.



Trees - Upper Bound

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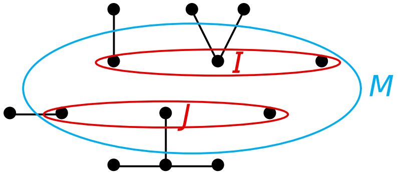
Whether the component of T' containing $v \in M$ is even or odd, it saves $\frac{1}{2}$ when Painter colors one of I and J.

Thus $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\frac{3}{2}(n-|I|) + \frac{3}{2}(n-|J|) - \frac{|M|}{2}\right] = \frac{1}{2}3n$. ■



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Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
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Def. A stem vertex has one non-leaf neighbor.

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Def. Let $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

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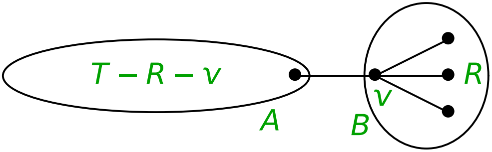

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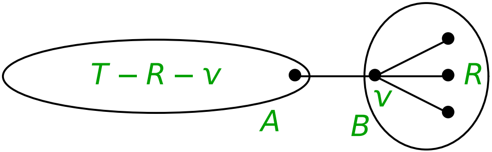

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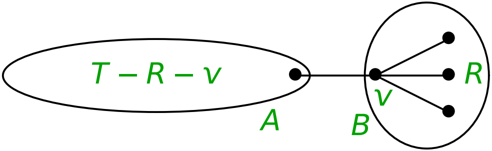

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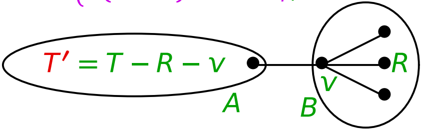

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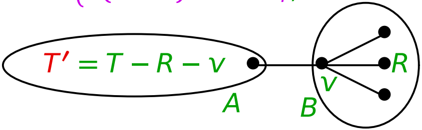

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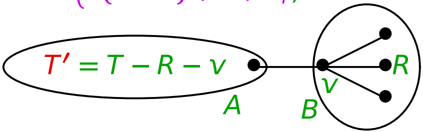

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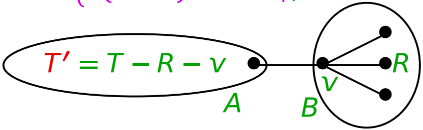

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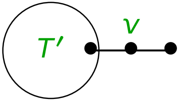

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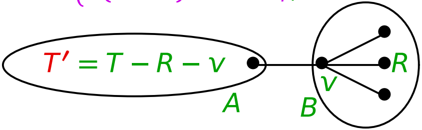

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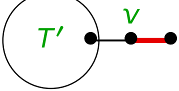

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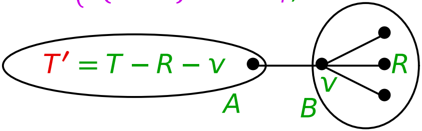

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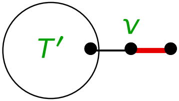

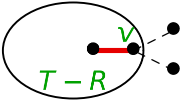

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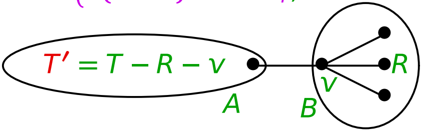

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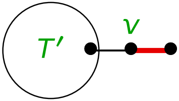

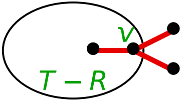

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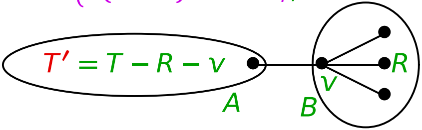

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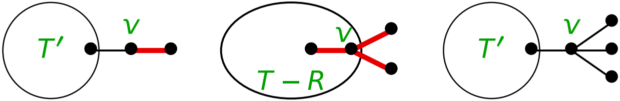

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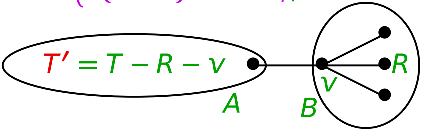

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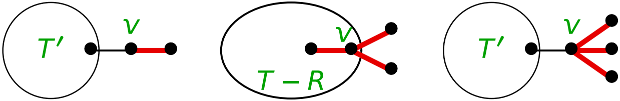

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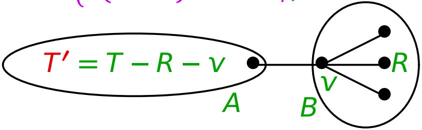

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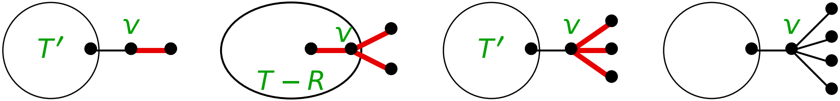

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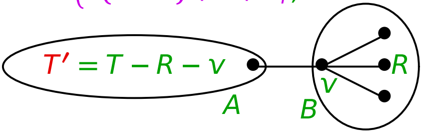

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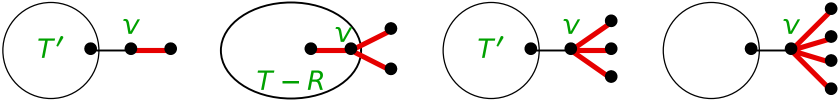

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Cor. $\mathring{s}(K_{r_1,\ldots,r_k}) \le (\sum_i \sqrt{r_i})^2 = n + \sum_{i<j} \sqrt{2r_i}\sqrt{2r_j}$.

Application to k-degenerate graphs

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Idea: For any induced subgraph G of a maximal outerplanar graph, Painter seeks an independent set X such that $|M| \leq \Phi(G) - \Phi(G-X)$; inductively this suffices.

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Start with coloring A, B, C, some vertices get extra color.

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G-X)$.

Let utility $u(X) = \Phi(G) - \Phi(G-X) - |M|$. Want $u(X) \geq 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v,G) + \frac{1}{3} \cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v,G) - \phi(v,G-X) - 1 & \text{if } v \in M - X, \\ \phi(v,G) - \phi(v,G-X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x,G) = 1/3$ if $x \in E(G)$, and X is independent.)

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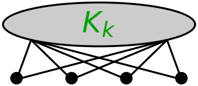

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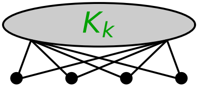

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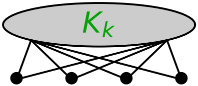

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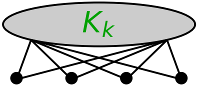

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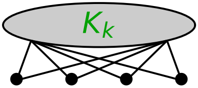

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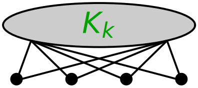

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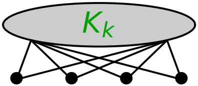

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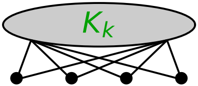

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