The Slow-Coloring Game
(On-line Sum-Paintability)

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slides available on DBW preprint page

Joint work with
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Def. sum-color cost $\mathring{s}(G)$ - score under optimal play.

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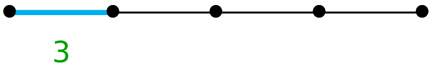

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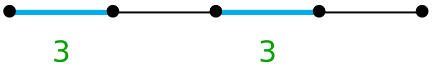

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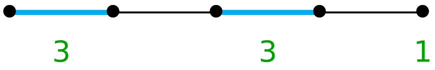

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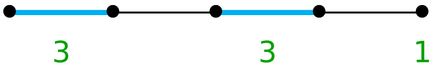

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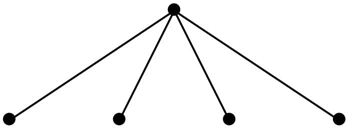

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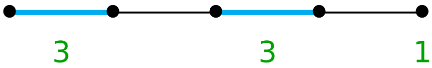

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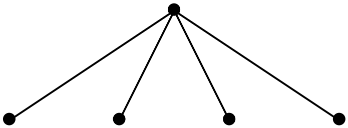

Lister marking all or just an edge gets only 6.

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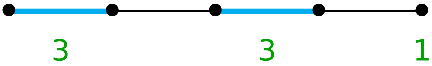

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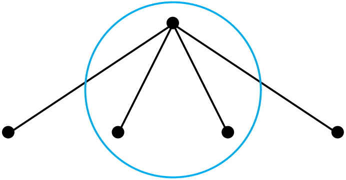

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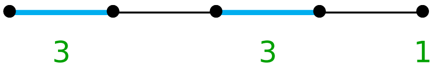

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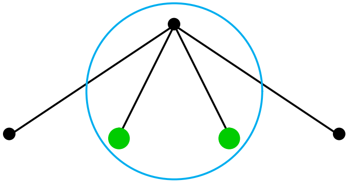

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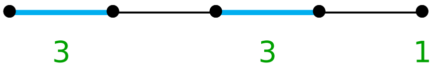

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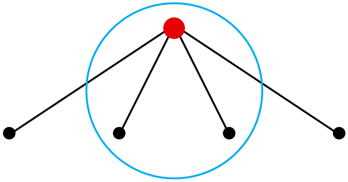

Score is $3 + \mathring{s}(P_3)$ or $3 + 4$.

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Pf. Using the sets in an optimal coloring, in size order, the average index of color used is at most $\frac{\chi(G)+1}{2}$. ■

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Ques. How to improve the upper bound $\chi(G)n$ for special classes of graphs?

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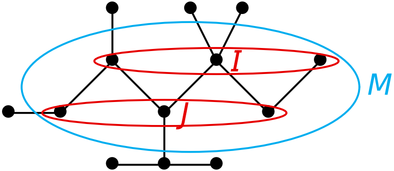
Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter can delete I or J.



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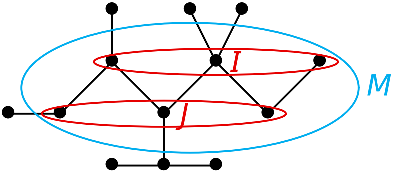

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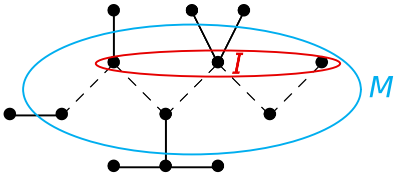

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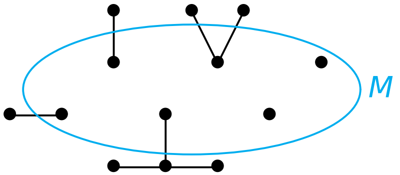

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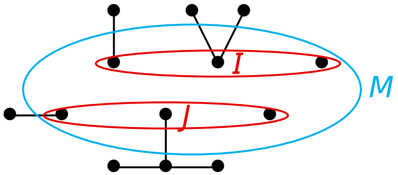
Whether the component of T' containing $v \in M$ is even or odd, it saves $\frac{1}{2}$ when Painter colors one of I and J.



Trees - Upper Bound

Thm. $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter can delete I or J.

Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\mathring{s}(T-I) + \mathring{s}(T-J)\right]$.

Inductively, $\mathring{s}(T-I) \le \frac{3}{2}(n-|I|) - \frac{1}{2}o(T-I)$.

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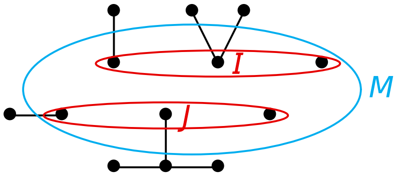
Thus $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\frac{3}{2}(n-|I|) + \frac{3}{2}(n-|J|) - \frac{|M|}{2}\right] = \frac{1}{2}3n$. ■



Computation for Forests

Def. Let $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

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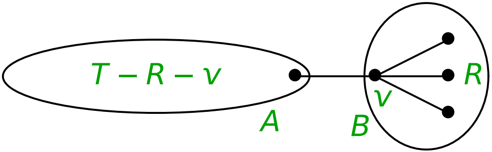
$$\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v) + r + 1 + u_r, & \text{if } r+1 \text{ is not triangular,} \\ \mathring{s}(T-R) + r + u_r, & \text{if } r+1 \text{ is triangular.} \end{cases}$$



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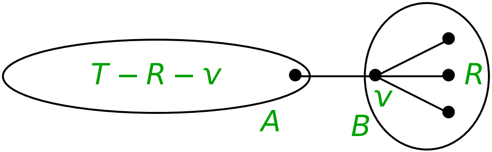

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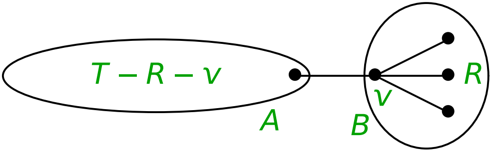

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Brief sketch for planar graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

Recall lemma $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$.

Letting $n = |V(G)|$, a numerical consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.

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For such G, we cover $V(G)$ using three independent sets (some vertices twice) such that at least one works as X.