On-line Sum-Paintability: The Slow-Coloring Game

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slides and preprints available on DBW preprint page

Joint work with
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Def. sum-color cost $\mathring{s}(G)$ - score under optimal play.

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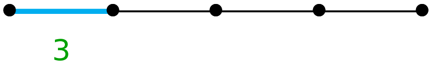

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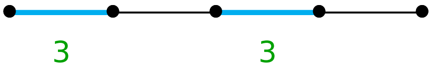

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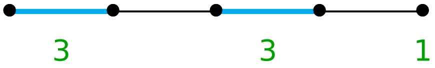

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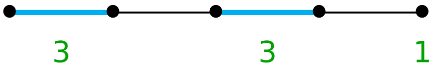

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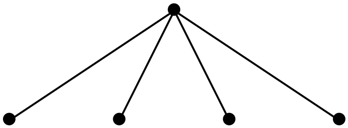

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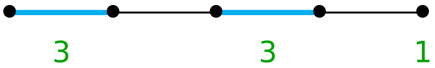

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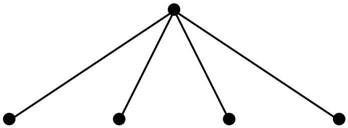

Lister marking all or just an edge gets only 6.

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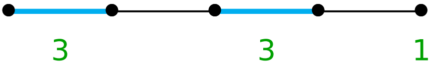

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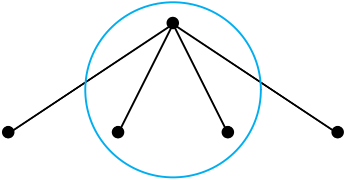

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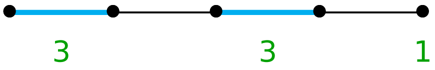

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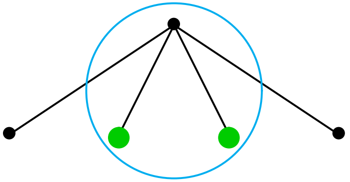

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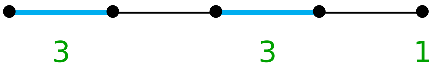

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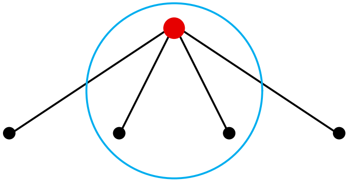

Score is $3 + \mathring{s}(P_3)$ or $3 + 4$.

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Prop. $\Sigma(G) \leq \frac{\chi(G)+1}{2} n$, with equality for Turán graphs.

Pf. Using the sets in an optimal coloring, greedily, the average color used is at most $(\chi(G)+1)/2$. ■

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Sharp only when $\chi(G) = 1$?

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Thm. $\mathring{s}(G) \leq \chi_{sp}(G)$, with equality if and only if every component of G is complete.

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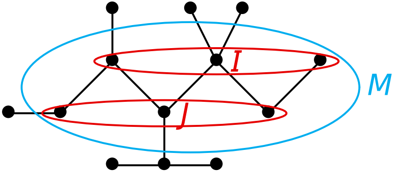
Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter can delete I or J.



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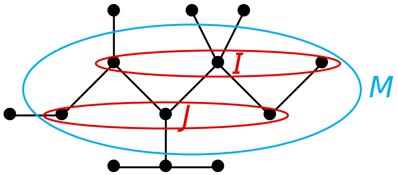

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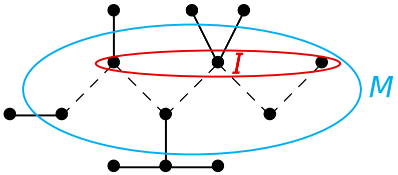

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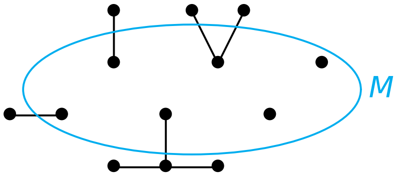

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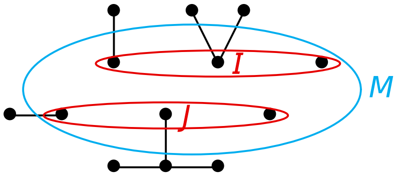

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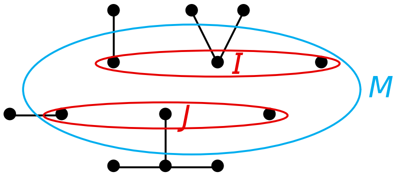
Thus $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\frac{3}{2}(n - |I|) + \frac{3}{2}(n - |J|) - \frac{|M|}{2}\right] = \frac{1}{2}3n$. ■



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Def. A stem vertex has one non-leaf neighbor.

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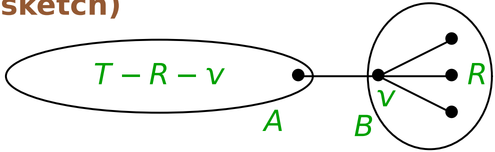

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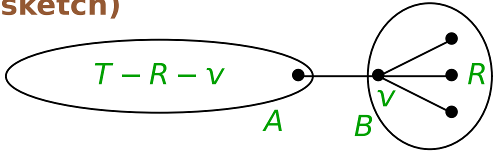

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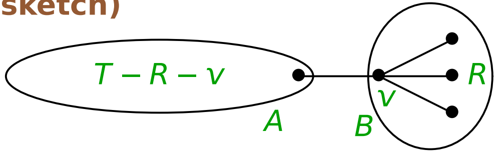

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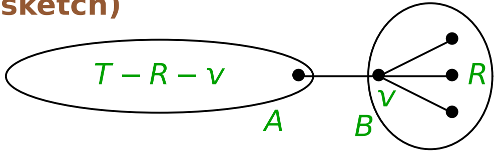

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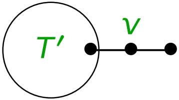

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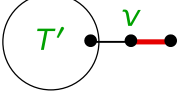

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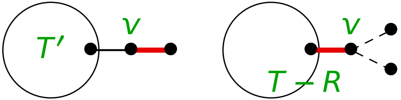

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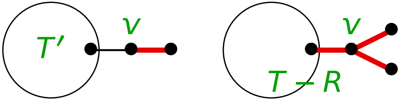

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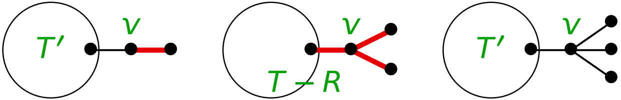

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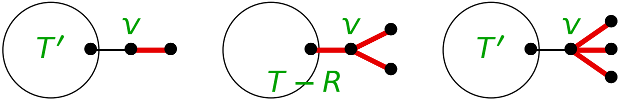

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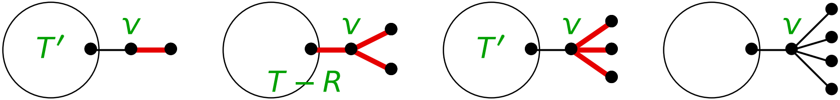

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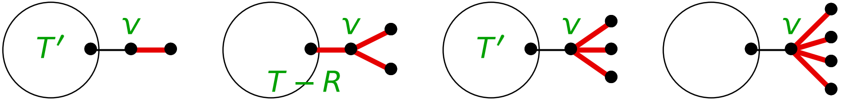
Pf. Consider cases by the degree $(r+1)$ of a stem vertex v. Let $T' = T - R - \{v\}$.

$r = 1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.

$r = 2$ ($r+1$ triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T-R) + 3$, leaf v.

$r = 3$ ($r+1$ not triangular), $u_r = 2$. $\mathring{s}(T) = \mathring{s}(T') + 6$.

$r \ge 4$. In either case, $\mathring{s}(T) < 3n/2$, and no $1,3$-forest.



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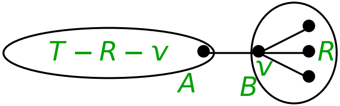

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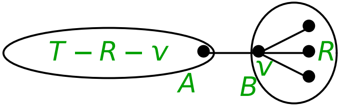

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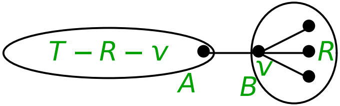

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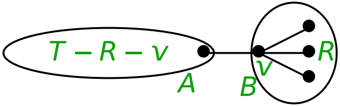

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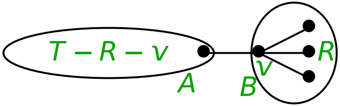

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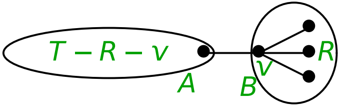

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When $n-1$ or $n-2$ is triangular, three more trees.

Complete Bipartite Graphs

Thm. $r + \frac{5s-3}{2} + u_{r-s} \leq \mathring{s}(K_{r,s}) \leq r + s + 2\sqrt{rs}$ for $r \geq s$.

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This strategy establishes the lower bound inductively.

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The upper and lower bound proofs each take one page.

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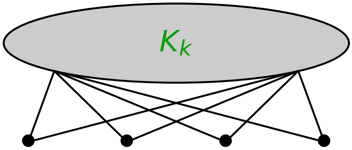

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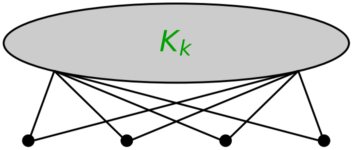

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Ques. Complexity?

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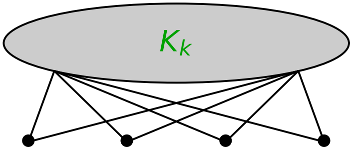

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Ques. Complexity? Algorithm on larger families?

Further Work Beyond Trees

Gutowski–Krawczyk–West–Zajac–Zhu [2017+]

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Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$.

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Thank you for your attention!