

The Slow-coloring Game on Sparse Graphs: k -Degenerate, Planar, and Outerplanar

Grzegorz Gutowski*, Tomasz Krawczyk*, Krzysztof Maziarz*,
Douglas B. West^{†‡}, Michał Zając*, Xuding Zhu[†]

July 19, 2018

Abstract

The *slow-coloring game* is played by Lister and Painter on a graph G . Initially, all vertices of G are uncolored. In each round, Lister marks a nonempty set M of uncolored vertices, and Painter colors a subset of M that is independent in G . The game ends when all vertices are colored. The score of the game is the sum of the sizes of all sets marked by Lister. The goal of Painter is to minimize the score, while Lister tries to maximize it. We provide strategies for Painter on various classes of graphs whose vertices can be partitioned into a bounded number of sets inducing forests, including k -degenerate, acyclically k -colorable, planar, and outerplanar graphs. For example, we show that on an n -vertex graph G , Painter can keep the score to at most $\frac{3k+4}{4}n$ when G is k -degenerate, $3.9857n$ when G is acyclically 5-colorable, $3n$ when G is planar with a Hamiltonian dual, $\frac{8n+3m}{5}$ when G is 4-colorable with m edges (hence $3.4n$ when G is planar), and $\frac{7}{3}n$ when G is outerplanar.

1 Introduction

The *slow-coloring game*, introduced by Mahoney, Puleo, and West [4], models proper coloring of graphs in a scenario with restrictions on the coloring process. The game is played by *Lister* and *Painter* on a graph G . Initially, all vertices of G are uncolored. In each round, Lister marks a nonempty set M of uncolored vertices of G and scores $|M|$. Painter responds by selecting an independent set $X \subseteq M$ to receive the next color. The game ends when all

*Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland. *Email:* {gutowski,krawczyk}@tcs.uj.edu.pl, {krzysztof.s.maziarz,emzajac}@gmail.com. Research of G. Gutowski was partially supported by National Science Center of Poland, grant UMO-2016/21/B/ST6/02165. Research of T. Krawczyk, K. Maziarz, and M. Zając was partially supported by National Science Center of Poland, grant UMO-2015/17/B/ST6/01873.

[†]Department of Mathematics, Zhejiang Normal University, Jinhua, China. *Email:* xdzhu@zjnu.edu.cn. Research of X. Zhu supported in part by CNSF 00571319.

[‡]Department of Mathematics, University of Illinois, Urbana, IL, USA. *Email:* dwest@math.uiuc.edu. Research of D.B. West supported by Recruitment Program of Foreign Experts, 1000 Talent Plan, State Administration of Foreign Experts Affairs, China.

vertices in G are colored. The score is the sum of the sizes of the sets marked by Lister. Lister's goal is to maximize the score; Painter's goal is to minimize it. The result when both players play optimally (to ensure the best possible score they can guarantee) is denoted by $\mathring{s}(G)$ and called the *sum-color cost* of G .

A *proper coloring* of a graph G assigns distinct colors to adjacent vertices. The *chromatic number* $\chi(G)$ is the minimum number of colors in a proper coloring (G is k -colorable if $\chi(G) \leq k$). The sets selected by Painter during the slow-coloring game together form a proper coloring of G , using many colors. However, when G is k -colorable, every set $M \subseteq V(G)$ contains an independent set of size at least $|M|/k$, which means that if Painter selects a largest independent subset then the score on each round is at most $\chi(G)$ times the number of vertices colored on that round. Summing over all rounds yields $\mathring{s}(G) \leq \chi(G) |V(G)|$. Since in the last round all remaining vertices are colored, the inequality is strict unless $\chi(G) = 1$.

Nevertheless, Wu [8] (presented in [4]) proved that the upper bound $\mathring{s}(G) \leq \chi(G) |V(G)|$ is asymptotically sharp; his general lower bound for complete multipartite graphs shows that $\mathring{s}(G) \sim kn$ for the complete k -partite n -vertex graph with part-sizes r_1, \dots, r_k differing by at most 1. Wu's general lower bound is $n + \sum_{i < j} u_{r_i} u_{r_j}$, where $u_r = \max\{t: \binom{t+1}{2} \leq r\} \sim \sqrt{2r}$. Our first general technique yields an immediate upper bound for complete multipartite graphs that is very close to this; see Corollary 1.2 below.

Let $n = |V(G)|$. Using the greedy Painter strategy suggested above, Mahoney, Puleo, and West [4] improved the general upper bound of $\chi(G)n$ to $\max_{H \subseteq G} \frac{|V(H)|}{\alpha(H)} n$, where $\alpha(H)$ denotes the maximum size of an independent set in H (also $\mathring{s}(G) \geq \frac{1}{2}(1 + \frac{n}{\alpha(G)})n$). They proved $n + u_{n-1} \leq \mathring{s}(G) \leq \lfloor \frac{3}{2}n \rfloor$ when G is an n -vertex tree. The lower bound is exact for the star $K_{1,n-1}$, and the upper bound is exact for the path P_n (and other trees). Puleo and West [5] extended this, characterizing the trees achieving the upper and lower bounds and giving a linear-time algorithm (and inductive formula) to compute $\mathring{s}(G)$ when G is a tree.

Our general theme is to improve the bound $k|V(G)|$ for interesting special classes of k -chromatic graphs (the coefficient k cannot be improved for general k -chromatic graphs). Given a graph G and $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G induced by A . We present a general Painter strategy to prove the following result.

Theorem 1.1. *Let G be an n -vertex graph. If $\mathring{s}(G[V_i]) \leq c_i |V_i|$ for $1 \leq i \leq t$, where $V(G)$ is the disjoint union of V_1, \dots, V_t , then $\mathring{s}(G) \leq \left(\sum_i \sqrt{c_i |V_i|} \right)^2$ and $\mathring{s}(G) \leq (\sum_i c_i) n$.*

The strategy ignores edges joining V_i and V_j . They may all be present, which explains why a very good upper bound for complete multipartite graphs is an immediate corollary.

Corollary 1.2. *If G is an n -vertex complete k -partite graph with part-sizes r_1, \dots, r_k , then $\mathring{s}(G) \leq n + 2 \sum_{i < j} \sqrt{r_i r_j}$.*

Note that the bound simplifies to kn when all part-sizes equal r .

Forests are 2-colorable. Because the trivial upper bound $\mathring{s}(G) \leq 2n$ has been improved to the optimal bound $\mathring{s}(G) \leq \lfloor \frac{3}{2}n \rfloor$ when G is an n -vertex forest, Theorem 1.1 provides useful bounds for graphs whose vertices can be partitioned into t sets inducing forests. Such a

graph G is $2t$ -colorable, but Theorem 1.1 yields $\mathring{s}(G) \leq \frac{3t}{2}n$. We consider several classes of graphs admitting vertex partitions into a small number of sets inducing forests.

A graph is k -degenerate when every nonempty subgraph has a vertex of degree at most k . A graph is *planar* if it can be drawn on the plane so that edges intersect only at their endpoints, and it is *outerplanar* if it has a such a drawing with all vertices lying on the unbounded face. A graph is *acyclically k -colorable* if it has a proper k -coloring with no 2-colored cycle, meaning that the union of any two color classes induces a forest.

Inductively, every k -degenerate graph is $(k + 1)$ -colorable. Outerplanar graphs are 2-degenerate and hence 3-colorable, and planar graphs are 4-colorable [1, 6]. Acyclically k -colorable graphs by definition are k -colorable. In these classes, applying the general decomposition results improves the trivial upper bound $\mathring{s}(G) \leq \chi(G)|V(G)|$ as follows.

Corollary 1.3. *Let G be an n -vertex graph.*

- (a) *If G is k -degenerate, then $\mathring{s}(G) \leq \frac{3k+4}{4}n$ for even k ; $\mathring{s}(G) \leq \frac{3k+3}{4}n$ for odd k .*
- (b) *If G is acyclically k -colorable, then $\mathring{s}(G) \leq \frac{3k}{4}n$ for even k ; $\mathring{s}(G) \leq \frac{3k+1}{4}n$ for odd k .*
- (c) *If G is acyclically k -colorable and k is odd, then $\mathring{s}(G) \leq \frac{1}{k}[\sqrt{.75}(k-1) + 1]^2n$. In particular, $\mathring{s}(G) \leq 3.9857$ for acyclically 5-colorable graphs, which includes planar graphs.*
- (d) *If G is a plane graph and the planar dual of G has a spanning cycle, then $\mathring{s}(G) \leq 3n$.*

Except for Corollary 1.2, the bounds are most likely not sharp, since the arguments allow extra edges that in the special classes cannot all appear. Our closest constructions arise from disjoint unions of complete graphs. Note that $\mathring{s}(H) \leq \mathring{s}(G)$ when $H \subseteq G$ and that \mathring{s} is additive under disjoint union. Also, the complete graph K_n satisfies $\mathring{s}(K_n) = \binom{n+1}{2} = \frac{n+1}{2}n$, and K_n is $(n - 1)$ -degenerate. Thus, among k -degenerate graphs, a graph G consisting of disjoint copies of K_{k+1} satisfies $\mathring{s}(G) = (\frac{1}{2}k + 1)n$. Using $k = 3$ for planar graphs and $k = 2$ for outerplanar graphs yields lower bounds of $\frac{5}{2}n$ and $2n$, respectively.

The results mentioned above appear in Section 2. In Section 3 and beyond, we introduce a different technique with a more complicated algorithm for Painter that yields better upper bounds. Using appropriate “potential functions” on graphs, it provides the following bounds.

Theorem 1.4. *Let G be an n -vertex graph with m edges.*

- (a) *If G is 4-colorable, then $\mathring{s}(G) \leq \frac{8n+3m}{5}$ (in particular, $\mathring{s}(G) \leq 3.4n$ when G is planar).*
- (b) *If G is outerplanar, then $\mathring{s}(G) \leq \frac{7}{3}n$.*

2 The Decomposition Method

We introduce a way to combine strategies for Painter on disjoint induced subgraphs.

Definition 2.1. Let \mathcal{P} be a partition of the vertices of a graph G , with $\mathcal{P} = \{V_1, \dots, V_t\}$. For $1 \leq i \leq t$, let $w_i = \sqrt{c_i|V_i|} / \sum_j \sqrt{c_j|V_j|}$, where $\mathring{s}(G[V_i]) \leq c_i|V_i|$. The \mathcal{P} -composite strategy for Painter on G is as follows. When Lister marks a set M , Painter chooses any index i such that $|M \cap V_i| / |M| \geq w_i$ and responds to the move $M \cap V_i$ according to an optimal strategy on $G[V_i]$, ignoring the rest of M . Such an index exists because $\sum_i w_i = 1$. To be deterministic, Painter may choose the least such index.

Theorem 2.2. *Let \mathcal{P} be a partition of the vertex set of a graph G into sets V_1, \dots, V_t . If $\mathring{s}(G[V_i]) \leq c_i |V_i|$, then*

$$\mathring{s}(G) \leq \left(\sum_i \sqrt{c_i |V_i|} \right)^2, \quad (1)$$

and

$$\mathring{s}(G) \leq \left(\sum_i c_i \right) |V(G)|. \quad (2)$$

Proof. Our main task is to prove (1), from which (2) follows by numerical arguments.

We use the \mathcal{P} -composite strategy for Painter, with weights w_1, \dots, w_t as in Definition 2.1. When Lister marks M and Painter choose the index i to play on $G[V_i]$, we have $|M \cap V_i| \geq w_i |M|$, and hence $|M| \leq |M \cap V_i| / w_i$.

Rounds played in $G[V_i]$ form a game on $G[V_i]$ played optimally by Painter. Therefore, over those rounds $|M \cap V_i|$ sums to at most $c_i |V_i|$ and $|M|$ sums to at most $c_i |V_i| / w_i$. Over all rounds, $\mathring{s}(G) \leq \sum_i (c_i |V_i| / w_i) = \left(\sum_i \sqrt{c_i |V_i|} \right)^2$, completing the proof of (1).

Next, the Arithmetic-Geometric Mean Inequality yields $2\sqrt{c_i |V_j| c_j |V_i|} \leq c_i |V_j| + c_j |V_i|$. Using this in the expansion of (1) yields

$$\begin{aligned} \mathring{s}(G) &\leq \left(\sum_i \sqrt{c_i |V_i|} \right)^2 = \sum_i c_i |V_i| + \sum_{i < j} 2\sqrt{c_i |V_i| c_j |V_j|} \\ &= \sum_i c_i |V_i| + \sum_{i < j} 2\sqrt{c_i |V_j| c_j |V_i|} \leq \sum_i c_i |V_i| + \sum_{i < j} (c_i |V_j| + c_j |V_i|) \\ &= \sum_i c_i \sum_j |V_j| = \left(\sum_i c_i \right) |V(G)|. \quad \square \end{aligned}$$

As an immediate corollary, consider the complete k -partite graph with part-sizes r_1, \dots, r_k . The lower bound by Wu [8] is $\sum_i r_i + \sum_{i < j} u_{r_i} u_{r_j}$, where $u_r = \max\{t: \binom{t+1}{2} \leq r\}$. Note that $u_r = \lfloor (-1 + \sqrt{1 + 8r}) / 2 \rfloor \approx \sqrt{2r}$. Actually, u_r is a bit smaller than $\sqrt{2r}$, but within a small constant. Hence our corollary is very close to the lower bound. Note that (2) gives only the trivial bound $k|V(G)|$.

Corollary 2.3. *If G is an n -vertex complete k -partite graph with part-sizes r_1, \dots, r_k , then $\mathring{s}(G) \leq n + 2 \sum_{i < j} \sqrt{r_i r_j}$.*

Proof. Use the maximal independent sets as the parts V_1, \dots, V_k in a partition of $V(G)$. Since $\mathring{s}(G[V_i]) = |V_i|$, we have $c_i = 1$ for all i . Hence expanding (1) yields the claim. \square

The case $k = 2$ of this upper bound was proved in [4]. Wu [8] actually proved the better upper bound $n + \sum_{i < j} \sqrt{2r_i - 1} \sqrt{2r_j - 1}$ by a more difficult argument. That bound has the virtue of being exact for some stars.

Theorem 2.2 and the bound of [4] for forests can be combined to obtain upper bounds when the vertices of a graph can be partitioned into a small number of sets inducing forests.

Theorem 2.4. *If the vertex set of a graph G can be partitioned into t sets inducing forests, then $\mathring{s}(G) \leq \frac{3t}{2} |V(G)|$. If it can be partitioned into one independent set and $t - 1$ sets inducing forests, then $\mathring{s}(G) \leq \frac{3t-1}{2} |V(G)|$.*

Proof. Let V_1, \dots, V_t be the given vertex partition. To apply Theorem 2.2, we set $c_i = 3/2$ for each i , except $c_t = 1$ in the second case. The claim then follows immediately from (2). \square

By induction on the number of vertices, every k -degenerate graph has a vertex ordering in which every vertex has at most k earlier neighbors; call this a k -ordering. The following lemma is well-known.

Lemma 2.5. *Let G be a k -degenerate graph. For k_1, \dots, k_t such that $\sum_{i=1}^t (k_i + 1) \geq k + 1$, there is a partition of $V(G)$ into sets V_1, \dots, V_t such that $G[V_i]$ is k_i -degenerate, for each i .*

Proof. Such a partition is produced iteratively by considering the vertices in the order of a k -ordering. When we reach the j th vertex, it can be placed safely in one of the sets, because having more than k_i earlier neighbors in each evolving set V_i requires the vertex to have more than k neighbors in G that are earlier in the k -ordering. \square

This idea of Lemma 2.5 was used by Chartrand and Kronk [3] to show that a k -degenerate graph decomposes into $\lceil (k+1)/2 \rceil$ forests (forests are the 1-degenerate graphs). We use $(k+1)/2$ forests when k is odd, $k/2$ forests plus one independent set when k is even. The corollary then follows from Theorem 2.4. Similarly, when a graph is acyclically k -colorable, again the conditions of Theorem 2.4 apply, since grouping color classes in pairs provides a decomposition of the vertex set into sets inducing forests.

Corollary 2.6. *Let G be an n -vertex graph. If G is k -degenerate, then $\mathring{s}(G) \leq \frac{3k+4}{4}n$ for even k and $\mathring{s}(G) \leq \frac{3k+3}{4}n$ for odd k . If G is acyclically k -colorable, then $\mathring{s}(G) \leq \frac{3k}{4}n$ for even k and $\mathring{s}(G) \leq \frac{3k+1}{4}n$ for odd k .*

Although this upper bound for k -degenerate graphs is halfway between the trivial upper bound and the trivial lower bound, it does not seem strong, because the argument allows all edges joining vertices in distinct forests in the partition, but having all such edges would contradict the degree requirements in the full k -degenerate graph.

In general, when the coefficients c_1, \dots, c_t are the same and we do not know the sizes of V_1, \dots, V_t , the bound from (1) does not improve on (2). The reason is that the square-root function is concave, and hence the bound in (1) is largest when the parts have equal size. In that case the bound $(\sum_i \sqrt{c_i |V_i|})^2$ becomes $(t\sqrt{cn/t})^2$, which equals ctn .

When $V(G)$ is partitioned into one independent set and $t-1$ sets inducing forests (such as when G is $(2t-2)$ -degenerate or acyclically $(2t-1)$ -colorable), we can improve on the bound in Theorem 2.4 by using a result intermediate between (1) and (2) that takes advantage of the difference between the coefficients $3/2$ for the forests and 1 for the independent set.

Theorem 2.7. *Let A and B partition the vertex set of a graph G . If $\mathring{s}(G[A]) \leq c_A|A|$ and $\mathring{s}(G[B]) \leq c_B|B|$, and γ is a constant satisfying $\frac{c_A}{c_A+c_B} \leq \gamma \leq \frac{|A|}{|A|+|B|}$, then*

$$\mathring{s}(G) \leq \left(\sqrt{c_A \gamma} + \sqrt{c_B(1-\gamma)} \right)^2 |V(G)|.$$

Proof. The condition for the existence of such γ is equivalent to $c_A |B| \leq c_B |A|$; we may label A and B so that this holds and there is a claim to prove. Let $\beta = \frac{|A|}{|V(G)|}$, and let $g(x) = \sqrt{c_A x} + \sqrt{c_B(1-x)}$ for $0 \leq x \leq 1$. By (1), we have

$$\mathring{s}(G) \leq \left(\sqrt{c_A |A|} + \sqrt{c_B |B|} \right)^2 = g(\beta)^2 |V(G)|.$$

Since $g'(x) = \frac{1}{2} \left(\sqrt{c_A/x} - \sqrt{c_B/(1-x)} \right)$, we have $g'(x) \leq 0$ if and only if $x \geq \frac{c_A}{c_A+c_B}$. Also $g(x) \geq 0$ on $[0, 1]$, so we conclude that $g(x)$ and $(g(x))^2$ are nonincreasing on the interval $[\frac{c_A}{c_A+c_B}, 1]$. Since this interval contains the interval $[\frac{c_A}{c_A+c_B}, \beta]$, which contains γ , we have

$$\mathring{s}(G) \leq g(\beta)^2 |V(G)| \leq g(\gamma)^2 |V(G)| = \left(\sqrt{c_A \gamma} + \sqrt{c_B(1-\gamma)} \right)^2 |V(G)|. \quad \square$$

Since $g(x)$ is nonincreasing, the bound from Theorem 2.7 is strongest when $\gamma = |A|/|V(G)|$, where it is the same as (1), and weakest when $\gamma = c_A/(c_A+c_B)$, where it is the same as (2). When we know the coefficients c_A and c_B but do not know $|A|$ and $|B|$, we may still be able to improve on (2) if we can bound $|A|/|V(G)|$ from below.

For a k -degenerate graph with k even, we can partition the vertices into $k/2$ sets inducing forests and one independent set, but in doing this we cannot control the size of the independent set. For an acyclically k -colorable graph with k odd, we obtain the coloring first and combine the classes arbitrarily in pairs to form sets inducing forests, so in this case we can require the independent set to be the smallest of the classes and play the role of B . Since our previous bound is $2.5n$ for acyclically 3-colorable graphs, $4n$ for acyclically 5-colorable graphs, and $\frac{1}{4}(3k+1)n$ in general, we obtain an improvement.

Corollary 2.8. *If G is an acyclically k -colorable graph with n vertices, where k is odd, then $\mathring{s}(G) \leq \frac{1}{k}[\sqrt{.75}(k-1) + 1]^2 n$. In particular, the coefficient on n is less than 2.4881 when $k = 3$ and less than 3.9857 when $k = 5$.*

Proof. Let B be the smallest color class in an acyclic k -coloring, and let A be the union of the $k-1$ largest color classes. Since A can be partitioned into $(k-1)/2$ sets inducing forests in G , we have $\mathring{s}(G[A]) \leq \frac{3}{4}(k-1)|A|$. Since $\mathring{s}(G[B]) = |B|$, we have $c_B = 1$ and $c_A = \frac{3}{4}(k-1)$ in the notation of Theorem 2.7.

The choice of B guarantees $|B| \leq n/k$ and $|A| \geq (k-1)n/k$. Hence $c_A |B| \leq \frac{3}{4}(k-1)n/k$ and $c_B |A| \geq (k-1)n/k$, so Theorem 2.7 applies. Since $\frac{c_A}{c_A+c_B} \leq \frac{k-1}{k} \leq \frac{|A|}{|A|+|B|}$, we can set $\gamma = (k-1)/k$. By Theorem 2.7,

$$\mathring{s}(G) \leq \left(\sqrt{\frac{3}{4}(k-1)\frac{k-1}{k}} + \sqrt{1 \cdot \frac{1}{k}} \right)^2 n = \frac{1}{k}[\sqrt{.75}(k-1) + 1]^2 n. \quad \square$$

Grouping of color classes can be applied more generally, although doing so does not yet improve on Corollary 2.8. Let \mathcal{H} be a hereditary family of graphs. A graph is \mathcal{H} r -colorable if its vertices can be partitioned into r sets inducing subgraphs in \mathcal{H} .

Proposition 2.9. *For $r \in \mathbb{N}$, let c_r be a constant such that $\mathring{s}(G) \leq c_r|V(G)|$ whenever G is \mathcal{H} r -colorable. If G is \mathcal{H} $(p+q)$ -colorable, then*

$$\mathring{s}(G) \leq \frac{(\sqrt{pc_p} + \sqrt{qc_q})^2}{p+q} |V(G)|.$$

Proof. Consider an \mathcal{H} $(p+q)$ -coloring of G with colors $1, \dots, p+q$, indexed in nonincreasing order of the sizes of the color classes. Note that $\frac{c_p}{c_p+c_q} + \frac{c_q}{c_q+c_p} = 1 = \frac{p}{p+q} + \frac{q}{q+p}$. Therefore, exactly one of $\frac{c_p}{c_p+c_q} \leq \frac{p}{p+q}$ and $\frac{c_q}{c_q+c_p} < \frac{q}{q+p}$ holds. By symmetry, we may assume $\frac{c_p}{c_p+c_q} \leq \frac{p}{p+q}$.

Let P denote the set of vertices having colors $1, \dots, p$, and let $Q = V(G) - P$. Since $\chi_a(G[P]) \leq p$ and $\chi_a(G[Q]) \leq q$, we have $\mathring{s}(G[P]) \leq c_p|P|$ and $\mathring{s}(G[Q]) \leq c_q|Q|$. By the indexing of the color classes, $\frac{|P|}{|Q|} \geq \frac{p}{q}$, and hence $\frac{|P|}{|P|+|Q|} \geq \frac{p}{p+q}$.

With $\gamma = \frac{p}{p+q}$, we thus have $\frac{c_p}{c_p+c_q} \leq \gamma \leq \frac{|P|}{|P|+|Q|}$. Now we apply Theorem 2.7 using sets P and Q and parameter γ to obtain

$$\mathring{s}(G) \leq \left(\sqrt{c_p \frac{p}{p+q}} + \sqrt{c_q \frac{q}{p+q}} \right)^2 |V(G)| = \frac{(\sqrt{pc_p} + \sqrt{qc_q})^2}{p+q} |V(G)|. \quad \square$$

In fact, Proposition 2.9 provides a recursive upper bound for the sequence $\{c_r\}_{r \geq 1}$, by minimizing that bound over $\{p, q\}$ such that $p+q=r$.

Borodin [2] proved that planar graphs are acyclically 5-colorable, so the bound we have given for acyclically 5-colorable graphs holds also for all planar graphs. It is slightly better than the trivial lower bound of $4|V(G)|$ implied by the Four Color Theorem [1]. In Section 4 we will improve the general upper bound for planar graphs to $3.4|V(G)|$. Our results above allow us to improve the bound further for a special class of planar graphs. A graph is *Hamiltonian* if it has a spanning cycle.

Proposition 2.10. *If G is a plane graph whose vertices can be partitioned into sets A and B such that $\mathring{s}(G) \leq c|A|$ and $\mathring{s}(G[B]) \leq c|B|$, then $\mathring{s}(G) \leq 2c|V(G)|$. In particular, $\mathring{s}(G) \leq 3|V(G)|$ when the dual graph of G is Hamiltonian.*

Proof. The first claim is a special case of (2). For the second, it is well known that the vertices of a plane graph G can be partitioned into sets A and B inducing forests if and only if the dual graph of G is Hamiltonian (Stein [7] proved this for plane graphs whose faces are all triangles). Hence in this case the hypothesis holds with $c = \frac{3}{2}$. \square

We know of no n -vertex planar graph G with $\mathring{s}(G) > 5n/2$; equality holds for graphs whose components are copies of K_4 . Note also that since planar graphs are 4-colorable, the vertex set of any planar graph can be partitioned into two sets inducing bipartite graphs. Hence bounding $\mathring{s}(G)$ for planar bipartite graphs may also be of interest. Here there is a nontrivial construction. Puleo (private communication) showed that $\mathring{s}(G) = 1.75n - 1$ when G is the cartesian product of a 4-cycle and a path; this graph G is planar and bipartite.

3 The Potential Method

We introduce another technique to prove upper bounds for planar and outerplanar graphs. Painter uses a “potential function” Φ on the vertices and edges of a graph that can be thought of as summing “potential” contributions to the score in the remaining game. The total potential is the sum of these contributions. The contributions are different in our two applications, but we explain the technique first as a common generalization that can be applied to other families of sparse graphs.

Definition 3.1. Let ϕ_G be a function assigning a positive real number (called “potential”) to each vertex and edge of a graph G . For G in a given class \mathcal{G} of graphs, each edge will have potential γ , and the vertex potentials will be defined later in such a way that $\phi_H(x) \leq \phi_G(x)$ when $x \in V(H)$ and $H \subseteq G$ (that is, the potential function is *monotone*). For a graph G , define the total potential $\Phi(G)$ by $\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$.

The goal of this method is to prove $\mathfrak{s}(G) \leq \Phi(G)$ for graphs G in a given hereditary class \mathcal{G} (closed under taking induced subgraphs). Since the potential has a contribution that is linear in the number of edges, applying this to a family of k -colorable graphs can give an improvement over the trivial bound only when the number of edges is at most linear in the number of vertices, which holds for planar and outerplanar graphs.

When Lister marks a set M in a graph G in \mathcal{G} , Painter will seek an independent set $X \subseteq M$ such that $|M| \leq \Phi(G) - \Phi(G - X)$. That is, the total score in the current round should be at most the loss in potential by coloring X . Since the potential is reduced to 0 when the game is over, always being able to find such a set X yields $\mathfrak{s}(G) \leq \Phi(G)$. To consider X , we define the “utility” of X relative to M , which we also split into contributions from the various vertices of G . Let $d_G(v)$ denote the degree of vertex v in a graph G .

Definition 3.2. Let \mathcal{G} be a hereditary class \mathcal{G} on which a monotone potential function is defined, with γ being the potential of each edge. For each $M \subseteq V(G)$ and each independent set X contained in M , define the *utility* of X by $u(X) = \Phi(G) - \Phi(G - X) - |M|$. (We seek X such that $u(X) \geq 0$.) Apportion $u(X)$ among the vertices of G by letting

$$u_X(v) = \begin{cases} \phi_G(v) + \gamma d_G(v) - 1 & \text{if } v \in X, \\ \phi_G(v) - \phi_{G-X}(v) - 1 & \text{if } v \in M - X, \\ \phi_G(v) - \phi_{G-X}(v) & \text{if } v \in V(G) - M. \end{cases}$$

Lemma 3.3. For $X \subseteq M \subseteq V(G)$ with X independent, $\sum_{v \in V(G)} u_X(v) = u(X)$. Also, $u_X(v) \geq -1$ when the potential function is monotone.

Proof. The terms -1 for $v \in X$ and $v \in M - X$ count $|M|$ negatively. The contribution of edges to $\Phi(G) - \Phi(G - X)$ is γ for every edge incident to X ; since X is independent, this equals $\sum_{v \in X} \gamma d_G(v)$. When $v \in X$, vertex v contributes $\phi_G(v)$ to $\Phi(G)$ and nothing to $\Phi(G - X)$; otherwise, v contributes $\phi_G(v) - \phi_{G-X}(v)$ to $\Phi(G) - \Phi(G - X)$.

Since always $\phi_G(v) \geq 0$ and $\phi_G(v) - \phi_{G-X}(v) \geq 0$, always $u_X(v) \geq -1$. □

When G belongs to a family \mathcal{G} of k -colorable graphs and M is a marked set in G , we will cover M using k independent sets V'_1, \dots, V'_k such that $\sum_i u(V'_i) \geq 0$ (some vertices may appear in more than one set). Hence at least one set has nonnegative utility and can be chosen as the desired play for Painter. We may assume that $G[M]$ is connected, that is, M is a *connected set*. In order to produce V'_1, \dots, V'_k , we begin with a special k -coloring of $G[M]$.

Definition 3.4. Given a connected set M in a k -colorable graph G , let T be the set of vertices in M that lie in no cycle in $G[M]$, and let $S = M - T$. Call the components of $G[T]$ *tree-components* and the components of $G[S]$ *cycle-components*. A *good k -coloring* of $G[M]$ is a proper k -coloring such that every tree component is 2-colored and the color on every vertex having a neighbor in a tree component is one of the two colors assigned to that component.

In Figure 1, T_1 and T_2 are tree-components, C_1 and C_2 are cycle-components, and the marked vertices have degree at most 2 in $G[M]$. We will focus on these later.

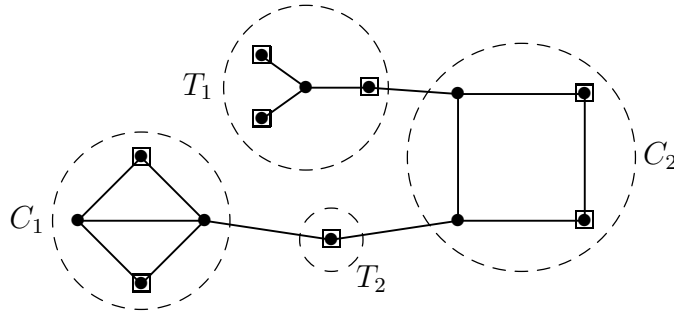


Figure 1: Decomposition of an induced subgraph of a 4-colorable graph.

Lemma 3.5. *Given a connected set M in a k -colorable graph G , the induced subgraph $G[M]$ has a good k -coloring.*

Proof. Note that no edge of $G[M]$ can join two cycle-components, and no edge can join two tree-components. Hence the graph H obtained from $G[M]$ by contracting each tree-component and each cycle-component to a single vertex is bipartite, with one part corresponding to the tree-components and the other part to the cycle-components. Furthermore, H is acyclic, since vertices in tree-components lie in no cycle in $G[M]$.

We produce a proper coloring of $G[M]$ with colors 1 through k . First choose a vertex of H and give the corresponding subgraph of $G[M]$ an optimal proper coloring. Next, for any unprocessed vertex of H whose corresponding subgraph Q has a neighbor v that is already colored, give Q an optimal coloring, using the color on v as one of the colors if Q has at least one edge. Do this until all of $G[M]$ has been colored. Since H is a tree, the process succeeds and produces a proper k -coloring. Furthermore, each set consisting of the vertices of a tree-component and their neighbors in cycle-components uses only two colors. \square

To obtain the desired independent sets V'_1, \dots, V'_k such that $\sum_i u(V'_i) \geq 0$, we will start with sets V_1, \dots, V_k forming a good k -coloring of $G[M]$ and augment the sets by allowing some vertices with low degree in $G[M]$ to receive more than one color.

4 Sparse 4-Colorable Graphs

In this section we prove $\mathring{s}(G) \leq \frac{8n+3m}{5}$ for every 4-colorable graph G with n vertices and m edges, improving $\mathring{s}(G) < 4n$ when $m < 4n$. Since n -vertex planar graphs are 4-colorable and have at most $3n - 6$ edges, we obtain $\mathring{s}(G) \leq 3.4n - 3.6$ when G is an n -vertex planar graph.

In order to apply the potential method, where $\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$, we specify ϕ_G when G is a 4-colorable graph. In this section, let

$$\phi_G(x) = \begin{cases} \frac{3}{5} & \text{for } x \in E(G) \\ 1 + \frac{1}{5} \min\{d_G(x), 3\} & \text{for } x \in V(G) \end{cases}$$

Note that ϕ is monotone. Since every vertex has potential at most $\frac{8}{5}$, proving $\mathring{s}(G) \leq \Phi(G)$ implies the desired bound $\mathring{s}(G) \leq \frac{8n+3m}{5}$ in Theorem 1.4(a). Note that $\mathring{s}(K_j) = \binom{j+1}{2} = \Phi(K_j)$ for $j \leq 4$. In order to prove $\mathring{s}(G) \leq \Phi(G)$ by induction on $|V(G)|$, we have noted in Section 3 that it suffices to find, for each connected set M , independent sets V'_1, \dots, V'_4 covering M such that $\sum u(V'_i) \geq 0$. We obtain these independent sets from the sets V_1, \dots, V_4 in a good 4-coloring of $G[M]$ by giving additional colors to some vertices of low degree.

Definition 4.1. In a connected set M in a 4-colorable graph G , let $P = \{v \in M : d_{G[M]}(v) \leq 2\}$. Let Q be the vertex set of a component of $G[P]$. Note that $G[Q]$ is a path or a cycle, and Q is contained in T or is disjoint from T .

Let R be a largest independent set in $G[Q]$, so $|R| = \lceil |Q|/2 \rceil$ when $G[Q]$ is a path and $|R| = \lfloor |Q|/2 \rfloor$ when $G[Q]$ is a cycle. For $v \in R$, add v to all color classes that do not contain v or any neighbor of v . Since R is independent in G , these additions can be made in any order. Let V'_1, \dots, V'_4 be the resulting *augmented sets* containing V_1, \dots, V_4 , respectively.

To each color i , we have added only vertices with no neighbor in color i . Hence the resulting sets V'_1, \dots, V'_4 are independent sets covering M . It remains only to prove $\sum_{i=1}^4 u(V'_i) \geq 0$. The sum is the total utility over each vertex in each set, grouped by the sets. We can also group the utility by vertices: let $u(v) = \sum_{i=1}^4 u_{V'_i}(v)$. We will prove $u(v) \geq 0$ when $v \in M - P$ and consider the vertices in P grouped by their components in $G[P]$.

For $x \in V(G)$, let $N_G(x)$ denote the set of neighbors of x in G , and let $c(x)$ denote the number of colors assigned to x . Let $s(x) = \sum_{y \in N_{G[M]}(x)} c(y)$. A lemma greatly simplifies the subsequent case analysis.

Lemma 4.2. *If V'_1, \dots, V'_4 are the augmented sets covering a connected set M in a 4-colorable graph G , and $x \in M$, then*

$$u(x) \geq \begin{cases} \left(\frac{4}{5}d_G(x) + 1\right)c(x) + \frac{1}{5}s(x) - 4 & \text{if } d_G(x) \leq 3 \\ 4c(x) - 4 & \text{if } d_G(x) \geq 4 \end{cases}$$

Proof. First suppose $d_G(x) \leq 3$. If $x \in V'_i$, then $u_{V'_i}(x) = \phi_G(x) + \frac{3}{5}d_G(x) - 1 = \frac{4}{5}d_G(x)$. When $x \notin V'_i$, the difference between $\phi_G(x)$ and $\phi_{G-V'_i}(x)$ is $\frac{1}{5}|N(x) \cap V'_i|$, since when

$d_G(x) \leq 3$ the potential decreases by $\frac{1}{5}$ for each lost neighbor. Summing over the $4 - c(x)$ values of i such that $x \notin V'_i$, we obtain

$$\sum_{i: x \notin V'_i} u_{V'_i}(x) = \sum_{i: x \notin V'_i} \left(\frac{1}{5} |N(x) \cap V'_i| - 1 \right) = \frac{1}{5} s(x) - (4 - c(x)).$$

Adding $\frac{4}{5} d_G(x)$ for each of the $c(x)$ values of i with $x \in V'_i$ yields the desired value.

When $d_G(x) \geq 4$, we have $\phi_G(x) = \frac{8}{5}$ and $\phi_G(x) - \phi_{G-V'_i}(x) \geq 0$. Hence $u_{V'_i}(x) = \frac{3}{5}(d_G(x) + 1)$ if $x \in V'_i$ and $u_{V'_i}(x) \geq -1$ if $x \notin V'_i$. Again there are $c(x)$ indices of the former type and $4 - c(x)$ of the latter type, so

$$u(x) \geq \frac{3}{5}(d_G(x) + 1)c(x) - [4 - c(x)] = \left(\frac{3}{5}d_G(x) + \frac{8}{5}\right)c(x) - 4 \geq 4c(x) - 4. \quad \square$$

As mentioned earlier, to complete the proof of Theorem 1.4(a) it suffices to prove the following lemma.

Lemma 4.3. *If V'_1, \dots, V'_4 are the augmented sets covering a connected set M in a 4-colorable graph G , then $\sum_{i=1}^4 u(V'_i) \geq 0$.*

Proof. We break the sum into its vertex contributions, with $u(v) = \sum_{i=1}^4 u_{V'_i}(v)$ for $v \in V(G)$. We first show $u(v) \geq 0$ for $v \in V(G) - P$ and then group the vertices of P by components of $G[P]$.

If $v \in V(G) - M$, then $u_{V'_i}(v) = \phi_G(v) - \phi_{G-V'_i}(v) \geq 0$ for each i , which suffices. If $v \in M - P$ with $d_G(v) \geq 4$, then Lemma 4.2 yields $u(v) \geq 4c(v) - 4 \geq 0$, since $c(v) \geq 1$. If $v \in M - P$ with $d_G(v) < 4$, then $v \notin P$ requires $d_G(v) = d_{G[M]}(v) = 3$. Hence $c(v) \geq 1$ and $s(v) \geq 3$, which by Lemma 4.2 yields $u(v) \geq 0$.

It remains to prove $\sum_{v \in Q} u(v) \geq 0$ when Q is the vertex set of a component of $G[P]$. Recall that $G[Q]$ is a path or a cycle, and vertices in a largest independent subset $R \subseteq Q$ have been assigned additional colors. Also $Q \subseteq T$ or $Q \cap T = \emptyset$.

Case 1. $Q \subseteq T$.

If $d_{G[M]}(v) = 0$, then v is in all augmented sets and $u(v) \geq 0$. Hence we may assume $d_G(v) \geq d_{G[M]}(v) \geq 1$.

If $v \in R$, then only one color is used on $N_{G[T]}(v)$, so $c(v) = 3$ and $s(v) \geq 1$. By Lemma 4.2, $u(v) \geq \frac{8}{5}$. On the other hand, if $v \in Q - R$, then $c(v) = 1$ and $s(v) \geq 3$. Now Lemma 4.2 yields $u(v) \geq -\frac{8}{5}$. Since $|R| \geq |Q - R|$, we thus obtain $\sum_{v \in Q} u(v) \geq 0$.

Case 2. $Q \cap T = \emptyset$.

For $v \in Q$ we have $d_G(v) \geq d_{G[M]}(v) = 2$, since v lies on a cycle in M .

If $v \in R$, then $c(v), s(v) \geq 2$, and Lemma 4.2 yields $u(v) \geq \frac{8}{5}$. If $v \notin R$, then $c(v) = 1$, and at least one neighbor of v lies in R ; choose $w \in N(v) \cap R$. Since $w \in P$, we have $d_{G[M]}(w) = 2$, so w receives at least one extra color. Since $d_{G[M]}(v) = 2$, there is another neighbor of v with at least one color, so $s(v) \geq 3$. Since also $d_G(v) \geq 2$, Lemma 4.2 now yields $u(v) \geq -\frac{4}{5}$.

Since always $|R| \geq \frac{1}{3}|Q|$ (with equality when $G[Q] = K_3$), we have $|R| \geq \frac{1}{2}|Q - R|$, and hence $\sum_{v \in Q} u(v) \geq 0$. \square

5 Outerplanar Graphs

For the family of outerplanar graphs, we use a different potential function. A *triangle* is a 3-vertex complete graph.

Definition 5.1. For an outerplanar graph G , let $\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$, where

$$\phi_G(x) = \begin{cases} \frac{1}{3} & \text{if } x \text{ is an edge of } G, \\ \frac{5}{3} & \text{if } x \text{ is a vertex in a triangle or having degree at least 3 in } G, \\ \frac{4}{3} & \text{if } x \text{ is a vertex in no triangle and } x \text{ has degree 1 or 2 in } G, \\ 1 & \text{if } x \text{ is an isolated vertex in } G. \end{cases}$$

Note that always $\phi_H(x) \leq \phi_G(x)$ when $H \subseteq G$; thus ϕ is monotone on this family.

Figure 2 illustrates the contributions to potential for vertices in an outerplanar graph; every edge contributes $\frac{1}{3}$. As motivation for the definition, note that if H is K_n for $n \in \{1, 2, 3\}$, then $\Phi(H) = \binom{n+1}{2} = \mathring{s}(H)$.

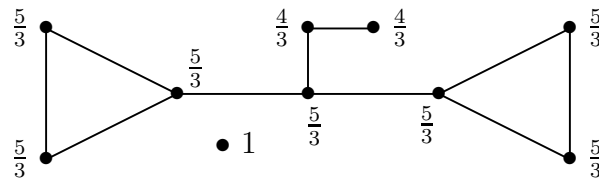


Figure 2: Potentials of vertices.

A *maximal outerplanar graph* is an outerplanar graph that is not a spanning subgraph of any other outerplanar graph. For $n \geq 3$, a maximal outerplanar graph with n vertices can be embedded in the plane so that the boundary of the unbounded face is a spanning cycle and all bounded faces are triangles.

Lemma 5.2. *If $\mathring{s}(G) \leq \Phi(G)$ whenever G is a maximal outerplanar graph, then $\mathring{s}(G) \leq \frac{7}{3}|V(G)|$ whenever G is an outerplanar graph.*

Proof. The desired bound holds by inspection when $|V(G)| \leq 2$. By the monotonicity of \mathring{s} , it suffices to prove $\mathring{s}(G) \leq \frac{7}{3}n$ when $n \geq 3$ and G is a maximal outerplanar graph with n vertices. Such a graph G has exactly $2n - 3$ edges and has every vertex in a triangle. Hence each vertex has potential $\frac{5}{3}$, and

$$\mathring{s}(G) \leq \Phi(G) = \frac{5}{3}n + \frac{1}{3}(2n - 3) < \frac{7}{3}n. \quad \square$$

To prove Theorem 1.4, we will show that $\mathring{s}(G) \leq \Phi(G)$ whenever G is an induced subgraph of a maximal outerplanar graph. The relevant consequence of maximality here is that every vertex lying on a cycle in G in fact lies on a triangle.

The approach is as in Section 3. With $\gamma = \frac{1}{3}$ for this potential function, and the connected set M in G , we have the definition of utility as in Definition 3.2. Lemma 3.3 holds, and it suffices to prove $\sum_i u(V'_i) \geq 0$ for independent sets V'_1, V'_2, V'_3 in G covering M . With *tree-components*, *cycle-components*, and *good 3-coloring* defined as in Definition 3.4, again the proof of Lemma 3.5 guaranteeing a good 3-coloring V_1, V_2, V_3 is valid. However, this time our method for obtaining the augmented sets V'_1, V'_2, V'_3 is a bit different.

Definition 5.3. Given a connected set M in an induced subgraph G of a maximal outerplanar graph G , with M split into T and S as in Definition 3.4, let $P = \{v \in T : d_{G[M]}(v) \leq 2\}$. Call each component of $G[P]$ a *path-component*. Let $N(T)$ be the set consisting of T and all neighbors in $G[M]$ of vertices in T .

The definition of P here differs from Definition 4.1 by restricting P to T . For a connected set M with $|M| \geq 2$, each component of $G[N(T)]$ is a tree with at least two vertices, on which the good 3-coloring guaranteed by Lemma 3.5 uses two colors. Since $G[M]$ is an induced subgraph of a maximal outerplanar graph, each vertex of S lies in a triangle in $G[M]$.

Definition 5.4. Given a connected set M in an induced subgraph G of a maximal outerplanar graph, let A, B, C be a good 3-coloring of $G[M]$ as provided by Lemma 3.5. Let Q be the vertex set of a path-component in $G[P]$, having vertices v_1, \dots, v_k in order. Let Z be the color among $\{A, B, C\}$ not initially used on the component of $N(T)$ containing Q . If $d_G(v_1) \geq 2$, then add to the set of vertices with color Z the odd-indexed vertices v_1, v_3, \dots . If $d_G(v_1) = 1$, then instead add the analogous set starting from the other end: $v_\ell, v_{\ell-2}, \dots$. Do this independently for each path-component to produce the *augmented sets* A', B', C' .

Note that the augmented sets are independent, since in the good 3-coloring all neighbors in $G[M]$ of vertices in Q receive colors other than Z . The final lemma completes the proof of Theorem 1.4(b).

Lemma 5.5. *If G is an induced subgraph of a maximal outerplanar graph, then $\mathring{s}(G) \leq \Phi(G)$.*

Proof. We use induction on $|V(G)|$. When $|V(G)| = 1$, both $\mathring{s}(G)$ and $\Phi(G)$ equal 1. For $|V(G)| > 1$, let M be the initial set marked by Lister, which we may assume is connected. It suffices to prove $u(A') + u(B') + u(C') \geq 0$ for the augmented sets A', B', C' in Definition 5.4.

If $|M| = 1$ with $M = \{v\}$, then setting $X = M$ satisfies $u(X) = \Phi(G) - \Phi(G - X) - 1 \geq 0$, since $\phi_G(v) \geq 1$ and vertex potentials cannot increase when taking subgraphs. Therefore, we may assume $|M| \geq 2$. Since $G[M]$ is connected, this implies $d_G(v) \geq d_{G[M]}(v) \geq 1$ for $v \in M$, so all vertices of M have potential at least $\frac{4}{3}$ in G .

We prove the desired inequality by breaking the sum into its contributions from individual vertices, as in Lemma 4.3. For $v \in V(G)$, let $u(v) = u_{A'}(v) + u_{B'}(v) + u_{C'}(v)$. Since $\sum_{v \in M} u(v) = u(A') + u(B') + u(C')$, the argument is completed by proving $u(v) \geq 0$ for $v \in V(G) - P$ and $\sum_{v \in Q} u(v) \geq 0$ for every path component $G[Q]$.

Since always $\phi_G(v) \geq \phi_{G-X}(v)$ when $v \notin X$, for every independent set $X \subseteq M$ we have $u_X(v) \geq 0$ when $v \notin M$ and $u_X(v) \geq -1$ when $v \in M$. Hence we need only consider $v \in M$.

CLAIM 1: $u(v) \geq 0$ for $v \in M - P$. Vertex v has exactly one color originally and after the augmentation; by symmetry, we may assume $v \in A \subseteq A'$. We consider cases depending on $d_G(v)$. If $d_G(v) \leq 1$, then v cannot lie in a triangle, so $v \in T$, but then $v \in P$. Hence for $v \in M - P$ we may assume $d_G(v) \geq 2$.

Case 1. $d_G(v) \geq 4$.

Since $d_G(v) \geq 3$, we have $\phi_G(v) = \frac{5}{3}$. With $v \in A$ and four incident edges, $u_{A'}(v) \geq \frac{5}{3} + 4 \cdot \frac{1}{3} - 1 = 2$. We have noted $u_X(v) \geq -1$ for $X \in \{B', C'\}$, so $u(v) \geq 0$.

Case 2. $d_G(v) = 3$.

First consider $v \in S$. Choose $x, y \in M$ so that $\{v, x, y\}$ is a triangle – see Figure 3.

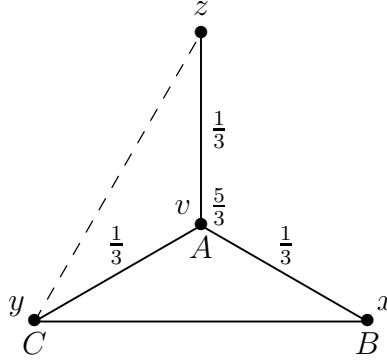


Figure 3: Neighborhood of v in Case 2 when $v \in S$

The vertices v, x, y have different colors in the good 3-coloring of $G[M]$, so by symmetry we may assume $x \in B$ and $y \in C$. Let z be the neighbor of v outside $\{x, y\}$. Since G is outerplanar and hence cannot contain K_4 , vertex z is not adjacent to both x and y . By symmetry, we may assume $xz \notin E(G)$. We have $u_{A'}(v) = \frac{5}{3} + 3 \cdot \frac{1}{3} - 1 = \frac{5}{3}$ and $u_{B'}(v) \geq -1$. For $X = C$, note that after coloring y the vertex v will be in no triangle and have degree 2, so $\phi_{G-X}(v) = \frac{4}{3}$. Thus $u_{C'}(v) = \frac{5}{3} - \frac{4}{3} - 1 = -\frac{2}{3}$, yielding $u(v) \geq 0$.

Now suppose $v \in T$. All three neighbors of v are in M , since otherwise $v \in P$. Since $v \in T$, the neighbors all have the same color; call it C . Again $u_{A'}(v) = \frac{5}{3} + 3 \cdot \frac{1}{3} - 1 = \frac{5}{3}$ and $u_{B'}(v) \geq -1$. In $G - C$, vertex v is isolated, so $u_{C'}(v) \geq \frac{5}{3} - 1 - 1 = -\frac{1}{3}$; again $u(v) \geq 0$.

Case 3. $d_G(v) = 2$.

If $v \in T$, then $v \in P$, so we may assume $v \in S$. Since $d_G(v) = 2$, exactly one triangle contains v , and its vertices have three different colors. Recall $v \in A$. When B' is deleted, v is no longer in a triangle, so $u_{B'}(v) \geq \frac{5}{3} - \frac{4}{3} - 1 = -\frac{2}{3}$. Similarly, $u_{C'}(v) \geq -\frac{2}{3}$. Since $u_{A'}(v) \geq \frac{5}{3} + 2 \cdot \frac{1}{3} - 1 = \frac{4}{3}$, we have $u(v) \geq 0$.

CLAIM 2: $\sum_{i=1}^{\ell} u(v_i) \geq 0$ for a path-component with vertices v_1, \dots, v_{ℓ} in order. Let $Q = \{v_1, \dots, v_{\ell}\}$. By symmetry, let A and B be the colors in the good 3-coloring used on the tree-component containing Q . We consider two cases.

Case 1. $\ell = 1$.

Let $Y \in \{A, B\}$ be the initial color on v_1 , with Z being the other color in $\{A, B\}$. Since v_1

is the first vertex of Q from either end, v_1 is added to C' regardless of $d_G(v_1)$. Since $G[M]$ is connected, all vertices have potential at least $\frac{4}{3}$, so $u_{Y'}(v_1) \geq \frac{4}{3} + \frac{1}{3} - 1 = \frac{2}{3}$. Similarly $u_{C'}(v_1) \geq \frac{2}{3}$. Since $u_{Z'}(v_1) \geq -1$, we have $u(v_1) \geq \frac{1}{3} > 0$.

Case 2. $\ell \geq 2$.

Consider $v \in Q$. Again let $Y \in \{A, B\}$ be the initial color on v and $Z \in \{A, B\}$ be the other initial color on Q .

First suppose $d_G(v) \geq 2$. If v is added to C' , then $u_{Y'}(v) = u_{C'}(v) \geq \frac{4}{3} + 2 \cdot \frac{1}{3} - 1 = 1$ and $u_{Z'}(v) \geq -1$, so $u(v) \geq 1$. If v is not added to C' , then $u_{Y'}(v) \geq 1$, $u_{Z'}(v) \geq -1$, and $u_{C'}(v) \geq -1$, so $u(v) \geq -1$. Thus internal vertices of $G[Q]$ alternate bounds $u(v) \geq 1$ and $u(v) \geq -1$.

Degree 1 in G is possible when v is an endpoint of $G[Q]$. In that case $u_{Y'}(v) \geq \frac{4}{3} + \frac{1}{3} - 1 = \frac{2}{3}$. Since deleting Z' isolates v , losing potential $\frac{1}{3}$, we have $u_{Z'}(v) \geq -\frac{2}{3}$. If v is added to C' , then $u_{C'}(v) \geq \frac{2}{3}$, and $u(v) \geq \frac{2}{3}$. If v is not added to C' , then its neighbor is added to C' , so $u_{Y'}(v) \geq \frac{2}{3}$, $u_{Z'}(v) \geq -\frac{2}{3}$, and $u_{C'}(v) \geq -\frac{2}{3}$, yielding $u(v) \geq -\frac{2}{3}$.

If ℓ is odd, then v_1 and v_ℓ are both added to C' , and $u(v_1) + u(v_\ell) \geq \frac{4}{3}$. For the internal vertices, $\sum_{i=1}^{\ell-1} u(v_i) \geq -1$, so $\sum_{i=1}^{\ell} u(v_i) \geq 0$.

If ℓ is even, then only one of v_1 and v_ℓ is added to C' . This endpoint contributes at least $\frac{2}{3}$, and the other endpoint contributes at least $-\frac{2}{3}$. Since the number of internal vertices is even, they also contribute at least 0, so $\sum_{i=1}^{\ell} u(v_i) \geq 0$. \square

References

- [1] K. Appel and W. Haken, Every planar map is four-colorable, *Illinois J. Math.* 21 (1977), 429–567.
- [2] O.V. Borodin, On acyclic colorings of planar graphs, *Discrete Math.* 25 (1979), 211–236.
- [3] G. Chartrand and H.V. Kronk, The point-arboricity of planar graphs, *J. London Math. Soc.* 44 (1969), 612–616.
- [4] T. Mahoney, G.J. Puleo, and D.B. West, Online sum-paintability: The slow-coloring game on graphs, *Discrete Math.* 341 (2018), 1084–1093.
- [5] G.J. Puleo and D.B. West, Online sum-paintability: Slow-coloring of trees, [arXiv:1612.04702v2](https://arxiv.org/abs/1612.04702v2)
- [6] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, A new proof of the four colour theorem, *Electron. Res. Announc. Amer. Math. Soc.* 2 (1996), 17–25.
- [7] S.K. Stein, B -sets and coloring problems. *Bull. Amer. Mat. Soc.* 76 (1970), 805–806.
- [8] H. Wu, personal communication and lecture at International Workshop on Graph Theory, Ewha Woman’s University, Seoul, Korea, January 5, 2018.