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SKEW CHAIN ORDERS AND SETS OF RECTANGLES

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A "skew chain order" is one which can be partitioned into saturated chains all starting at rank 0. A two-part Sperner-type theorem is derived for the direct product of two skew chain orders. This theorem is used to solve an extremal problem about certain sets of integer rectangles.

1. Introduction

We consider certain ranked partial orders. A symmetric chain order is a ranked partial order which admits a partition into chains which are saturated and symmetric about the middle rank. We introduce *skew chain orders* as those which admit a partition into consecutive chains which each contain an element of rank 0. (The dual of a skew chain order, an "anti-skew order", is one partitionable into saturated chains containing an element of maximum rank n). An antichain is a subset of a partial order in which no two elements lie on a chain. Sperner [8] proved that in a Boolean algebra the maximum-sized antichain consists of the elements of middle rank. The number of elements of rank k is called the kth Whitney number,[1] W_k. A partial order is said to have the *Sperner property* if the number of elements in a maximum-sized antichain is the largest Whitney number. Dilworth's theorem [1] states that a partial order has the Sperner property if it admits a partition into consecutive chains which each intersects the rank of largest Whitney number. Such a partition is called a Dilworth decomposition of the order. Symmetric and skew chain decompositions are clearly Dilworth decompositions.

One direction in which Sperner's theorem has been generalized is to direct products of partial orders. If $P = A \times B$, we define a semi-antichain in P to be a subset in which no two elements are equal in one component and ordered in the other. Note that any antichain in P is a semi-antichain. We wish to find the maximum-sized semi-antichain. Katona [5], Kleitman [6], and Schonheim [7] proved such "two-part Sperner theorems", so-called because the answer is still

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[1] Non-increasing Whitney numbers is a necessary, but not sufficient, condition for a partial order to be a skew chain order.

the largest Whitney number (of the product), though the condition is weakened from antichains to semi-antichains. Griggs [3] unifies their results: if P is a direct product of two symmetric chain orders, then the size of the largest semi-antichain in P is the maximum Whitney number. We obtain an analogous result for direct products of skew chain orders. The proof is like that of Griggs for symmetric chain orders.

The motivation for this investigation is a problem concerning certain collections of integer rectangles. Given an integer-cornered rectangle, we wish to find the maximum collection of integer-cornered subrectangles where no two have projections equal in one direction and ordered by inclusion in the other. The inclusion ordering of the subrectangles is a direct product of two skew chain orders, and the condition on the collection is that it be a semi-antichain. Our main result on skew chain orders gives an immediate and very simple answer to this question.

2. The main result

Although skew chain orders satisfy the Sperner property, their direct products do not satisfy the usual two-part Sperner theorem.

Theorem 1. *If P is a direct product of two skew chain orders S and T, then the largest semi-antichain in P is obtained by taking all those elements for which the ranks of the two components in their original orders equal each other.*

Proof. Two elements of equal rank in a poset are identical or unordered, so if we have two such elements equal in one component, they are either completely identical or are unordered in the other component. So, the set we describe is a semi-antichain for any direct product, and we must show that for skew chain orders there is none larger. We denote the size of this set by $\bar{W}_0$, the zero-th "anti-Whitney number". $\bar{W}_k$ is the number of elements for which the rank of the first component in its original order minus the rank of the second component in its original order is k.

Any partition of S, T into skew chains $\{C_s\}$, $\{D_t\}$ induces a partition of P into "skew" rectangles $\{C_s \times D_t\}$. The elements of such a rectangle can be represented by pairs (i, j), where $0 \leq i \leq r(C_s)$, $0 \leq j \leq r(D_t)$. ($r(P)$ denotes the (maximum) rank of a partial order.) $(i, j)_{s,t}$ represents the element of P whose first component is of rank i in C_s and second component is of rank j in D_t. Any semi-antichain F contains at most one element from any row or any column of such a rectangle. Therefore the most elements F can contain in any rectangle equals the "length" of the shorter side. That is,

$$|F| \leq \sum_{s,t} \min\{r(C_s), r(D_t)\}.$$

We claim the right-hand side equals $\bar{W}_0$. From each induced rectangle, choose the elements $\{(i, i)_{s,t} : 0 \leq i \leq \min\{r(C_s), r(D_t)\}\}$. In this manner we obtain all elements whose components have equal ranks, and we glean the maximum from each rectangle. □

As Griggs [3] does for symmetric chain orders, we can weaken the hypothesis of the theorem to conclude by the same proof that for skew chain orders S and T, $F \subset S \times T$ is bounded in size by $\bar{W}_0$ as long as for *some* specified skew chain decompositions of S and T no two elements of F are equal in one component and lie on the same chain in the decomposition of the other component.

The same result holds of course for the product of two antiskew chain orders. for the product of a skew chain order with an antiskew chain order, this reasoning yields the customary Sperner result: the largest semi-antichain (or antichain) in such a product consists of the elements in the rank where the Whitney number is largest. In this case that is W_n, where n is the rank of the antiskew chain order. We take all the elements where the rank is i in the skew component and $n-i$ in the antiskew component.

For the product of a skew chain order with a symmetric chain order, we must be slightly cleverer to obtain the bound given by the partition into rectangles. If the symmetric chain order has two middle ranks, let n be the lower, or else let n be the unique middle. Suppose $\{C_s\}$ partitions the skew order into skew chains and $\{D_t\}$ partitions the symmetric order in symmetric chains. Using the same notation as before, we obtain the maximum semi-antichain by taking

$$\{(i, n-\tfrac{1}{2}i)_{s,t} : i \text{ even},\ 0 \leq i \leq \min\{r(C_s), |D_t|-1\}\}$$
$$\cup\{(i, n+\tfrac{1}{2}(i+1))_{s,t} : i \text{ odd},\ 1 \leq i \leq \min\{r(C_s), |D_t|-1\}\}.$$

In other words, we pair all the elements of ranks 0 and n, 1 and $n+1$, 2 and $n-1$, 3 and $n+2$, and so on. By the same arguments as before this is a semi-antichain and the maximum such.

3. Application to sets of rectangles

Consider the set of rectangles with integer corners, ordered by inclusion, where the x-coordinates run from 0 up to N and the y-coordinates from 0 up to M. We ask for the largest collection of such rectangles such that no two are ordered if they match exactly in one coordinate. We have the following result.

Theorem 2. *In a rectilinear region, the largest collection of rectangles with integer coordinates where no two match in one dimension and are ordered by inclusion in the other is obtained by taking all the squares.*

Let P be the partial order in question. It is in fact the direct product of two interval orders; any rectangle can be identified uniquely by the intervals it covers on the x and y axes. We claim the interval order is a skew chain order. In the partially ordered set N of intervals in $[0, N]$, there is one interval $[0, N]$ of rank N, $N+1[i, i]$ of rank 0, indeed $N+1-k$ of rank k. In between, the element $[i, j]$, $i<j$ is covered by $\{[i, j+1), [i-1, j]\}$ and covers $\{[i+1, j], [i, j-1]\}$. Let $C_i = \{[i, j]: i \leq j \leq N\}$. Then $\{C_i\}$ is a skew chain decomposition of N. Similarly define D_j for M, and $P = N \times M$ is our poset of interest.

The conditions of the problem require that the collection be a semi-antichain F. Theorem 1 then implies that F is maximized by taking all the squares from ranks 0 through $\min\{2N, 2M\}$. These are the elements with equal component ranks.[2]

We note that this theorem does not generalize immediately to higher dimensions. In line with the usual conditions for three-part Sperner theorems, we state the problem as follows: Consider the rectangles with integer corners whose coordinates run from $(0, 0, 0)$ to (N, M, P). What is the largest collection F of such rectangles such that no two can be ordered if they agree exactly in two components? We would hope that F would be maximized by taking all the cubes, but this is not the case. Consider $N = M = P = 1$, the $0-1$ cube. There are 8 subcubes of rank 0 and 1 of rank 3, for a total of 9. However, the set of elements of rank 1, representing the 12 edges of the cube, also satisfies the conditions on F. The answer to this question remains open.

As with three-part Sperner theorems, additional conditions must be placed on F to get the desired answer of {all cubes}. Griggs [2, 4] finds fairly weak additional conditions for the Sperner result to hold in a direct product of three symmetric chain orders. He uses the decomposition into symmetric rectangles and places labels on them to accomodate the third dimension. The conditions on F translate to conditions on the labellings. We could similarly place labellings (for the third dimension) on our skew rectangles and obtain a three-part anti-Sperner-type theorem for triple products of skew orders. This would give us the additional conditions to place on F so the largest F would be {all cubes}.

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[2] If $M \geq N$, the number of these is $\sum_{i=0}^{N} (N+1-i)(M+1-i) = (N+1)(N+2)(3M-N+3)/6$.