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Size, Chromatic Number, and Connectivity

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Abstract. We consider the minimum number of edges in a k-edge-connected graph of order n with chromatic number at least c, obtaining the optimal bounds in most cases.

Extremal bounds on chromatic number of simple graphs have been given in terms of many other graph parameters, particularly in terms of diameter and various aspects of the degree sequence. It seems that the relationship between chromatic number and number of edges has not been fully explored when additional constraints are present. Certainly $\chi(G) \geq c$ requires at least $\binom{c}{2}$ edges, because in any optimal coloring there must be an edge between each pair of color classes. This bound is achieved by a c-clique plus isolated vertices. Tomescu [7] showed that this is the only c-chromatic graph on n vertices with $\binom{c}{2}$ edges.

The situation becomes more complicated when we add a connectivity requirement. Let $g(n, k, c)$ denote the minimum number of edges in a k-edge-connected graph having order n and chromatic number at least c (we assume $n > k$ nd $n \geq c$). We determine $g(n, k, c)$ for most values of the arguments. Many of our extremal examples are also k-connected; when this occurs the same result holds with connectivity in place of edge-connectivity.

We use the notations $n(G)$, $e(G)$, $\chi(G)$ for the order, size, and chromatic number of G. A graph G is vertex-c-critical if $\chi(G) = c$ and $\chi(G - v) = c - 1$ for every $v \in V(G)$, and G is c-critical if G is vertex-c-critical and also $\chi(G - e) = c - 1$ for every $e \in E(G)$.

Let (*) denote the condition that there exists a c-critical graph with order n and size $\binom{c}{2} + k(n + 1 - c)/2 - 1$. We prove the following results.

$$g(n,k,c) = \begin{cases} \binom{c}{2} + n - c & \text{if } k = 1 \\ \binom{c}{2} + \lceil k(n+1-c)/2 \rceil & \text{if } c > k \geq 2 \text{ and } n \geq c + k, \text{except:} \\ \binom{c}{2} + k(n+1-c)/2 - 1 & \text{if } n > c = k + 1 \text{ and } (*) \\ \lceil kn/2 \rceil & \text{if } c \leq \lceil (k+1)/2 \rceil \\ \lceil kn/2 \rceil & \text{if } c \leq k \text{ and } n > k + c \end{cases}$$

If $c + k > n > c > k + 1$, then the lower bound $g(n,k,c) \geq \binom{c}{2} + \lceil k(n+1-c)/2 \rceil$ holds but may not be best possible. Similarly, for $c + k > n > k > c$, the lower bound of $\lceil kn/2 \rceil$ holds, but constructions with this value are known only for isolated cases.

The result for $k = 1$ follows from elementary arguments.

Theorem 1. *Every connected c-chromatic n-vertex graph has at least* $\binom{c}{2} + n - c$ *edges, and this is best possible.*

Proof. Equality holds for the graph obtained by identifying a vertex of a c-clique with an endpoint of a path having $n - c$ edges. The lower bound is trivial for $c = 2$, since a connected graph has at least $n - 1$ edges. Henceforth we assume $c \geq 3$ and apply induction on n for fixed c.

For $n = c$, the bound again is trivial. For $n > c$, let G be a connected c-chromatic n-vertex graph with minimum number of edges m. If G is vertex-critical, then every vertex has degree at least $c - 1$. In this case, $m \geq n(c-1)/2 = n + n(c-3)/2 > n + c(c-3)/3 = n - c + \binom{c}{2}$. Otherwise, there exists $x \in V(G)$ such that $G - x$ has at least $\binom{c}{2} + n - 1 - c$ edges, and there is at least one more edge involving x. $\square$

Examination of the proof shows inductively that equality holds only for graphs consisting of a c-clique plus pendant trees.

Theorem 2. *If* $n > c > k \geq 3$, *then* $g(n,k,c) \geq \binom{c}{2} + \lceil k(n+1-c)/2 \rceil$, *except* $g(n,k,c) = \binom{c}{2} + k(n+1-c)/2 - 1$ *if* $c = k + 1$ *and there is a c-critical n-vertex graph with* $(c-1)(n+1)/2 - 1$ *edges.*

Proof. Suppose G is k-edge-connected and has order n and chromatic number at least c. Dirac [2] proved that every c-critical graph is $c - 1$-edge-connected; since $c > k$, we may assume that G is c-critical or that every c-critical subgraph of G has fewer vertices.

If G is not c-critical, let H be a c-critical subgroup of G. Since $\delta(H) \geq c - 1$, we have $e(H) \geq (c-1)n(H)/2$. Let G' be the multigraph obtained from G by contracting every edge of H, keeping multiple edges as they arise. Since edge-disjoint paths remain between pairs of vertices, G' is k-edge-connected. Hence $e(G') \geq kn(G')/2$. Furthermore $n(H) + n(G') = n + 1$, $n(H) \geq c$, and $c - k - 1 \geq 0$. Hence

$$e(G) \geq \frac{1}{2}[(c-1)n(H) + k(n - n(H) + 1)]$$
$$\geq \frac{1}{2}[(c-k-1)c + k(n+1)] = \binom{c}{2} + k(n+1-c)/2$$

Now suppose G is c-critical. Since a c-critical graph is $c-1$-edge-connected, we may assume $k = c - 1$, since the desired bound in this case is larger than the desired bound for smaller k. For $c \geq 3$, Dirac [3] proved that a c-critical graph of order n that is not a clique has at least $(c-1)(n+1)/2 - 1$ edges. When $k = c - 1$, we have $\binom{c}{2} + k(n+1-c)/2 = (c-1)(n+1)/2$, and hence the exception has the claimed value. □

Dirac [4] subsequently characterized the c-critical graphs having $(c-1)(n+1)/2 - 1$ edges. For example, when $c = 3$ this occurs for the odd cycles. Toft [6] presents a survey of results on color-critical graphs.

A simple construction establishes the desired upper bound when $n \geq c + k$. Harary [5] (see [1, p48]) constructed k-connected n-vertex graphs with $\lceil kn/2 \rceil$ edges whenever $n > k$. In addition to this, we use graph operations that preserve k-connectedness. The simplest is adding a vertex with at least k neighbors. Also, if G, H are k-connected, and u, v have degree k in G, H respectively, then forming the disjoint union $(G - u) + (H - v)$ and adding a matching between $N_G(u)$ and $N_H(v)$ yields a k-connected graph.

Theorem 3. *If* $n \geq c + k$ *and* $c \geq k \geq 2$, *then* $g(n,k,c) \leq \binom{c}{2} + \lceil k(n+1-c)/2 \rceil$.

Proof. Let H be a k-connected graph with $n - c + 1$ vertices and $\lceil k(n-c+1)/2 \rceil$ edges. Let v be a vertex of degree k in H. Let Q be a clique of order c, disjoint from H. Form G by adding to the disjoint union $(H - v) + Q$ a matching from $N_H(v)$ to k vertices in Q. Now G is k-connected, has chromatic number at least c, and has exactly $\binom{c}{2} + \lceil k(n+1-c)/2 \rceil$ edges. □

If $n < c + k$, this construction can be applied to a k-edge-connected multigraph with $n - c + 1$ vertices, and the resulting G will have the desired edge-connectivity and clique number, but it will be a multigraph.

For even k, the Harary graph $H_{n,k}$ has $kn/2$ edges obtained by placing n vertices around a circle and joining each to its $k/2$ nearest neighbors on each side. When k is odd, we join each vertex to its $k/2 - 1$ nearest neighbors on each side and to one "opposite vertex", except that if n is also odd one vertex has an extra "opposite vertex". In all cases, there exist cliques consisting of $\lceil (k+1)/2 \rceil$ consecutive vertices on the circle.

When $c \le k$, the lower bound of $\lceil kn/2 \rceil$ is immediate from the degree requirements on connectivity. If $c = k$, this bound equals $\binom{c}{2} + \lceil k(n+1-c)/2 \rceil$, and the construction of Theorem 3 is valid if $n \ge 2c$. If $c \le \lceil (k+1)/2 \rceil$, then the Harary graph $H_{n,k}$ has sufficiently large cliques to provide the upper bound (actually, $\chi(H_{n,k}) = \lceil (k+1)/2 \rceil + 1$ for most large n). For the case $\lceil (k+1)/2 \rceil < c \le k$ and $n \ge c + k$, adding edges to $H_{n,k}$ to create a c-clique does not yield a good construction, but we will use these graphs in another way to obtain an optimal construction.

Theorem 4. *If $c \le \lceil (k+1)/2 \rceil$, or if $c \le k$ and $n > k + c$, then $g(n,k,c) = \lceil kn/2 \rceil$.*

Proof. As noted above, we need only provide a construction for the case $k/2 + 1 < c \le k$ and $n > k + c$. We begin with the disjoint union $H_{n-c,k} + K_c$. Let the vertices of the two components be $v_1, \ldots, v_{n-c}$ and $u_1, \ldots, u_c$, respectively. We add edges between the components and then delete edges within the copy of $H_{n-c,k}$ to bring the vertices of K_c to degree k and then restore the degree of the vertices in $H_{n-c,k}$. This yields the desired size and a c-clique, and we do this so the result is k-connected. We describe the construction and argument in detail for even k; they are slightly more complicated for odd k.

For $1 \le i \le c$, we add edges from u_i to $\{v_i, \ldots, v_{i+k-c}\}$ to increase the degree of u_i to k. For $1 \le i \le k-c+1$, the degree added to v_i and to v_{k+1-i} is i. For $k-c+1 \le i \le c$, the degree added to v_i is $k-c+1$. To restore the degrees, we delete the edge from v_i to v_j for all $1 \le i \le k/2$ and $\max\{k/2+1, k/2-c+i\} \le j \le k/2+i$. The upper bound in the second condition ensures that $v_iv_j \in E(H_{n,k})$, and the lower bound gets the degrees correct. The numerical details are illustrated in Fig. 1, which plots the added degree on the left and the deleted edges on the right.

For $H_{n-c,k}$, the essence of the proof of k-connectedness is that for any pair x, y of nonadjacent vertices, there are $k/2$ internally-disjoint x, y-paths in each direction

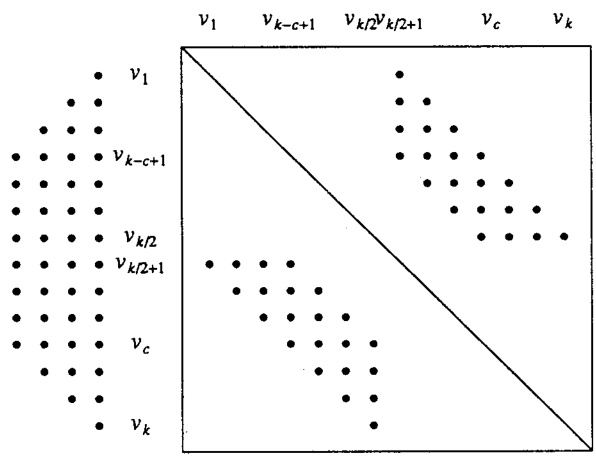

Fig. 1. Changes in degree

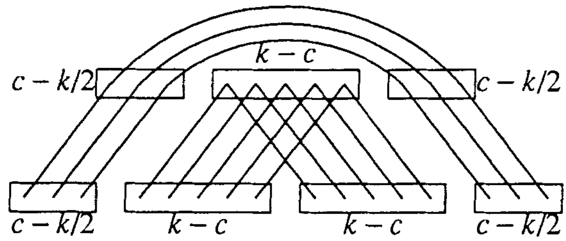

Fig. 2. Disjoint paths

around the circle. These paths begin with neighbors of one vertex and take steps of length $k/2$ to reach the neighborhood of the other vertex. To restore x, y-paths in our graph, we provide disjoint paths from $\{v_1, \ldots, v_{k/2}\}$ to $\{v_{k/2+1}, \ldots, v_k\}$ (we have deleted many direct edges between these sets – all but one if $c = k/2 + 2$). The initial edges of the $k/2$ paths are $v_1 u_1, \ldots, v_{k/2} u_{k/2}$, and the final edges are $u_{c-k/2+1} v_{k/2+1}, \ldots, u_c v_k$. This completes $k - c$ paths of length 2, and the remaining $c - k/2$ paths are completed by a matching between $\{u_1, \ldots, u_{c-k/2}\}$ and $\{u_{k/2+1}, \ldots, u_c\}$ (see Fig. 2).

Finally, we consider paths from our special vertices. For u_i, it suffices to provide disjoint paths to $\{v_1, \ldots, v_k\}$. We have $v_i, \ldots, v_{i+k-c} \in N(u_i)$. We also use (u_i, u_j, v_j) for $j < i$ and (u_i, u_j, v_{j+k-c}) for $i < j \leq c$. Finally, we consider v_i; by symmetry, we may assume $i \leq k/2$. We have direct edges to the $k/2$ vertices of $H_{n-c,k}$ immediately before v_i, so we need only construct disjoint paths to $v_{k/2+1}, \ldots, v_k$ not using those vertices. If $i \leq 1 + k - c$, then $u_1, \ldots, u_i, v_{i+1}, \ldots, v_{k/2} \in N(v_i)$, and we can continue along the paths described in Fig. 2. If $i > 1 + k - c$, we have $v_{k/2+1}, \ldots, v_{k/2+i-(k-c+1)} \in N(v_i)$. As we pick up $v_{k/2+j}$ and lose u_j from $N(v_i)$, we can use $u_{j+c-k/2}$ in place of u_j in the matching from the clique in Fig. 2. □

When $c < k$ and $n < k + c$, there are a variety of special cases in which special constructions achieve the bound $\lceil kn/2 \rceil$. We have no general construction for this situation and shall refrain from describing the special constructions.

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