SHORT PROOFS FOR INTERVAL DIGRAPHS

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Abstract. We give short proofs of the adjacency matrix characterizations of interval digraphs and unit interval digraphs.

1. INTRODUCTION

An intersection representation of a graph assigns each vertex a set so that vertices are adjacent if and only if the corresponding sets intersect. Beineke and Zamfirescu [1] introduced the analogous concept of intersection digraph, under the name "connection digraph". Let $\{(S_v, T_v)\}$ be a collection of ordered pairs of sets indexed by a set V; we call S_v the *source set* and T_v the *sink set* for v. The *intersection digraph* of this collection is the digraph with vertex set V having an edge from u to v if and only if $S_u \cap T_v \neq \emptyset$. The pairs of sets form an *intersection representation.*

Harary, Kabell, and McMorris [3] defined an equivalent intersection model for bipartite graphs. Treating the partite sets as source vertices and sink vertices, we represent each vertex by one set and take the intersection graph, but we ignore intersections between source sets or between sink sets to obtain a bipartite graph. Intersection digraphs correspond to intersection bigraphs by splitting each vertex v into a source copy x_v represented by S_v and a sink copy y_v represented by T_v, and optionally deleting source or sink vertices when the corresponding set in the representation is empty.

When the source sets and sink sets are all intervals, we obtain an *interval digraph* or *interval bigraph.* When they are intervals of the same length, we obtain a *unit interval digraph* or *unit interval bigraph.* Interval digraphs were characterized in [8] and in [9], and unit interval digraphs were characterized in [10]. This note presents shorter proofs of the adjacency matrix characterizations of these classes.

We use $N^+(v)$ and $N^-(v)$ to denote the successor set (out-neighbors) and predecessor set (in-neighbors) of a vertex v in a digraph. We use $A(D)$ for the adjacency matrix of a digraph D. For a bipartite graph with source vertex set X and sink vertex set Y, the *biadjacency matrix* is the submatrix of the adjacency matrix consisting of the rows for X and the columns for Y.

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2. FERRERS DIGRAPHS

A digraph is a *Ferrers digraph* if its successor sets (or its predecessor sets) form a chain under inclusion. We use a characterization of this class in our proof for interval digraphs. Riguet [6] introduced Ferrers digraphs as "Ferrers relations" and proved the equivalence of A,B,C,D below. Ducamp and Falmagne [2] called them *biorders* and proved E.

In an arbitrary matrix, we define a *stair* to be a walk from the upper left corner to the lower right corner that moves rightward or downward between rows and between columns, crossing each row and column once. The *understair* consists of the positions below or to the left of the stair, and the *overstair* consists of the positions above or to the right of it.

THEOREM 1 [6,2]. For a digraph D, the following conditions are equivalent.

A) $A(D)$ has no 2 by 2 submatrix that is a permutation matrix.

B) The successor sets of D are linearly ordered by inclusion.

C) The predecessor sets of D are linearly ordered by inclusion.

D) The rows and columns of $A(D)$ can be permuted independently so that some stair in the resulting matrix separates the 0's from the 1's.

E) (*Biorder representation*) There exist two real-valued functions f, g on $V(D)$ such that $uv \in E(D)$ if and only if $f(u) > g(v)$.

Proof: B⇔A⇔C. The successor sets fail to form an inclusion chain if and only if there exist u, v such that $x \in N^+(u) - N^+(v)$ and $y \in N^+(v) - N^+(u)$, which holds if and only if rows u, v and columns x, y form the forbidden submatrix. The analogous argument applies for predecessor sets.

B,C⇒D. It suffices to permute the rows and the columns so that every entry below or leftward of a 1 is a 1. Place the rows in increasing order of out-degree and the columns in decreasing order of in-degree, breaking ties arbitrarily. If $A_{rs} = 1$, then the inclusion orders yield $v_s \in N^+(u_i)$ for all $i \geq r$ and $u_r \in N^-(v_j)$ for all $j \leq s$, as desired.

D⇒E. Consider such a permutation of $A(D)$. The stair takes $2n$ moves, crossing row u after its last 1 and column v above its first 1. Let $f(v) = r$ if row v is crossed on step r, and let $g(v) = r$ if column v is crossed on step r. Now $f(u) > g(v)$ corresponds to crossing row u after column v, meaning that row u is below the stair in column v, which holds if and only if $uv \in E(D)$.

E⇒A. If D has a biorder representation f, g, and rows u, v and columns x, y of $A(D)$ form a permutation matrix with $A_{u,x} = A_{v,y} = 1$, then $f(u) > g(x)$ and $f(v) > g(y)$, but $f(u) \leq g(y)$ and $f(v) \leq g(x)$. Summing yields two contradictory inequalities. □

3. INTERVAL DIGRAPHS

A 0,1-matrix has a *zero-partition* if its 0's admit a partition into sets C and R such that every entry to the right of an R is an R and every entry below a C is a C. A 0,1-matrix has the *partitionable zeros property* if its rows and columns can be permuted independently to obtain a matrix having a zero-partition. The interval digraphs are those whose adjacency matrices have the partitionable zeros property [8,9]. The addition of rows or columns of 0's does not affect this property, so the same statement characterizes biadjacency matrices of interval bigraphs. The *complement* $\bar{D}$ of a digraph D has adjacency matrix obtained by converting 0's to 1's and 1's to 0's in $A(D)$.

When the sets in an intersection representation are required to be arcs on a circle, we obtain *circular-arc graphs* and *circular-arc digraphs*. Tucker's characterization [12,13] of circular-arc graphs generalizes to a characterization of circular-arc digraphs [9], which in turn specializes to another characterization of interval digraphs [9]. Given a stair in a matrix, let V_i be the maximal set of consecutive positions in row i, beginning immediately to the right of the stair, such that every position in V_i has a 1. Similarly, let W_j be the maximal set of consecutive positions in column j, beginning immediately below the stair, such that every position in W_j has a 1. We say that a matrix has the *stair-linear ones property* if and only if its rows and columns can be permuted independently to admit a stair such that every 1 in the matrix is covered by the union of the V_i's and W_j's.

Among the properties below, [8] gave short proofs of A⇒C and C⇒B and a quite lengthy proof of B⇒A. Subsequently, [9] gave a short proof of C⇔D and a moderately lengthy proof of D⇔A. We provide a short simultaneous proof by using biorder representations of Ferrers digraphs for B⇒C and by presenting D⇒A more simply.

THEOREM 2 [8,9]. For a digraph D, the following conditions are equivalent.
A) D is an interval digraph.
B) $\bar{D}$ is the edge-disjoint union of two Ferrers digraphs.
C) $A(D)$ has the partitionable zeros property.
D) $A(D)$ has the stair-linear ones property.

Proof: A⇒B. Let $S_v = [a(v), b(v)]$ and $T_v = [c(v), d(v)]$ in an interval representation of D. When $uv \in E(\bar{D})$, we have $S_u \cap T_v = \emptyset$. We put $uv \in E(D_1)$ if $b(u) < c(v)$ and $uv \in E(D_2)$ if $d(v) < a(u)$; this expresses $\bar{D}$ as the edge-disjoint union of D_1 and D_2. Each satisfies the biorder characterization of Ferrers digraphs.

B⇒C. Suppose $\bar{D}$ is the edge-disjoint union of Ferrers digraphs D_1, D_2. By the biorder characterization of Ferrers digraphs, there exist functions a, b, c, d such that $(b(u) < c(v) \Leftrightarrow uv \in E(D_1))$ and $(d(v) < a(u) \Leftrightarrow uv \in E(D_2))$. Place the rows of $A(D)$ in increasing order of $a(u)$, and place the columns in increasing order of $c(v)$. Let R and C be the set of 0's in $A(D)$ corresponding to edges of D_1 and D_2, respectively; this partitions the 0's. Since $b(u) < c(v)$ when $uv \in E(D_1)$, the column ordering guarantees that every position to the right of an R is in R. Similarly, since $d(v) < a(u)$ when $uv \in E(D_2)$, the row ordering guarantees that every position below a C is in C.

C⇒D [9]. Permute the rows and columns of $A(D)$ to exhibit a zero-partition. Let S be the set of positions that contain an R or lie somewhere above an R. By the definition of zero-partition, S is an overstair that contains no C. Every 0 in the overstair is an R, and hence the positions to its right are all 0. Every 0 in the understair is in C, so the positions below it are 0. Hence the 1's are covered as required for the stair-linear ones property.

D⇒A. Consider a permutation and stair exhibiting the stair-linear ones property. Let $u_1, \ldots, u_n$ be the vertex ordering by rows, and let $v_1, \ldots, v_n$ be the ordering by columns. We produce an interval representation. Let $a(u_i) = r$ if the stair crosses row i on move r, and let $c(v_j) = r$ if the stair crosses column j on move r. Let $b(u_i) = a(u_i)$ when V_i is empty, and otherwise let $b(u_i) = c(v_j)$, where j is the column of the rightmost position in V_i. Similarly, let $d(v_j) = c(v_j)$ when W_j is empty, and otherwise let $d(v_j) = a(u_i)$, where i is the row of the the lowest position in W_j. Now let $S_u = [a(u), b(u)]$ and $T_v = [c(v), d(v)]$. If position (i, j) is in the overstair, then $S_u \cap T_v \neq \emptyset$ if and only if j is small enough that $(i, j) \in V_i$. Similarly, if (i, j) is in the understair, then $S_u \cap T_v \neq \emptyset$ if and only if i is small enough that $(i, j) \in W_j$. Thus $S_u \cap T_v \neq \emptyset$ if and only if $uv \in E(D)$. □

These characterizations do not provide a polynomial-time recognition algorithm for interval digraphs. Müller [5] has found such an algorithm, using the fact that interval bigraphs belong to the class of "chordal bipartite graphs".

4. UNIT INTERVAL DIGRAPHS

Sen and Sanyal [10] gave several characterizations of unit interval digraphs, generalizing results of Roberts [7] on unit interval graphs. We present a short proof of the adjacency matrix characterization and the equivalence between unit interval digraphs and *proper interval digraphs*, which are the interval digraphs having representations in which none of the intervals properly contains another. We use the bigraph model.

Since interval bigraphs form a subclass of the interval bigraphs, the desired condition is a restriction of the partitionable zeros property. A 0,1-matrix is a *monotone consecutive arrangement* if the 0's can be partitioned into two sets C and R such that every entry above or to the right of an R is an R and every entry below or to the left of a C is a C. A 0,1-matrix *has* a monotone consecutive arrangement (MCA) if its rows and columns can be permuted independently to obtain an MCA. The non-square matrices allowed by the bigraph model permit a simpler inductive proof for C⇒A.

THEOREM 3 [10]. For a 0,1-matrix A, the following are equivalent.

A) A is the bipartite adjacency matrix of a unit interval bigraph.

B) A is the bipartite adjacency matrix of a proper interval bigraph.

C) A has a monotone consecutive arrangement.

Proof: A⇒B. No interval properly contains another of the same length.

B⇒C. A proper interval representation of the bigraph with biadjacency matrix A assigns an interval $f(v)$ for each vertex v. Index the source vertices $x_1, \ldots, x_m$ in the order of the left endpoints of $\{f(x_i)\}$. Index the sink vertices $y_1, \ldots, y_n$ in the order of the left endpoints of $\{f(y_j)\}$. Note that in a proper interval representation the right endpoint order is the same as the left endpoint order.

Consider (i, j) such that $A_{ij} = 0$, meaning that $f(x_i) \cap f(y_j) = \emptyset$. Put $(i, j) \in R$ if $f(x_i)$ is to the left of $f(y_j)$, and put $(i, j) \in C$ if $f(x_i)$ is to the right of $f(y_j)$. If $(i, j) \in R$

and $i' < i$, then $f(x_{i'})$ ends before $f(x_i)$ ends. Thus if $(i,j) \in R$, then $f(x_{i'})$ is to the left of $f(y_j)$ and $(i',j) \in R$. Similarly, if $j' > j$, then $f(y_{j'})$ starts after $f(y_j)$ starts, and $(i,j) \in R$ implies $(i,j') \in R$. The argument for C is similar, and thus A has an MCA.

C⇒A. Given an m by n MCA A, we construct a unit interval representation of the corresponding bigraph G with source vertex set $X = \{x_1, \ldots, x_m\}$ and sink vertex set $Y = \{y_1, \ldots, y_n\}$. The representation assigns intervals $f(x_i) = [a_i, a_i + 1]$ and $f(y_j) = [d_j - 1, d_j]$, with $a_1 < \cdots < a_m$ and $d_1 < \cdots < d_n$ and all endpoints distinct. We use induction on $m + n$; the construction is immediate when $\min\{m, n\} = 1$.

Suppose that $\min\{m, n\} > 1$. If $A_{mn} = 0$, then the last row or column of A is all zero, and we add add one rightmost interval to the representation for the matrix obtained by deleting that row or column. If $A_{mn} = 1$ but $A_{m,n-1} = A_{m-1,n} = 0$, then we add a rightmost intersecting pair of intervals to the representation for the matrix obtained by deleting the last row and last column.

Hence we may assume that $A_{mn} = 1$ and also (by symmetry) that $A_{m-1,n} = 1$. Let column k be the column of the first 1 in row m. Apply the induction hypothesis to the matrix obtained by deleting row m; this yields the endpoints $a_1, \ldots, a_{m-1}$ and $d_1, \ldots, d_n$. Let $\alpha = \max\{a_{m-1}, d_{k-1}\}$ (with $\alpha = a_{m-1}$ if $k = 1$). Since $A_{mk} = A_{m-1,n} = 1$ and A is an MCA, x_{m-1} is adjacent to $y_k, \ldots, y_n$. Thus $a_{m-1} \le d_k$; since all endpoints are distinct, the inequality is strict. We complete the representation for G by choosing a_m from the open interval (α, d_k), distinct from all other endpoints. Since $d_{k-1} < a_m$, we have $[a_m, a_m + 1]$ disjoint from $f(y_1), \ldots, f(y_{k-1})$. Since $a_m < d_k < \cdots < d_n < a_{m-1} + 2 < a_m + 2$, the new interval intersects $f(y_k), \ldots, f(y_n)$. All required properties hold. □

Lin and West [4] gave a forbidden submatrix characterization of the zero-partitionable matrices having MCA's. When given the initial 1's in the rows and columns of an MCA, the proof above produces a unit interval representation in time linear in $m+n$. Steiner [11] found a direct recognition algorithm for unit interval bigraphs using known results about bipartite permutation graphs; the running time is linear in the size of the matrix.

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