Duality for Semiantichains and Unichain Coverings in Products of Special Posets

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Origin of the Problem

Def. In a poset P (we only consider finite posets),
chain = a family of pairwise comparable elements;
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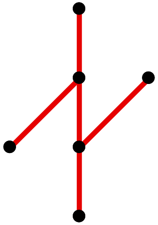

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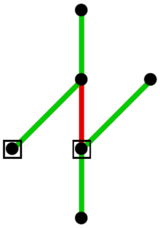

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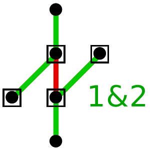
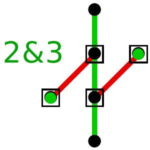
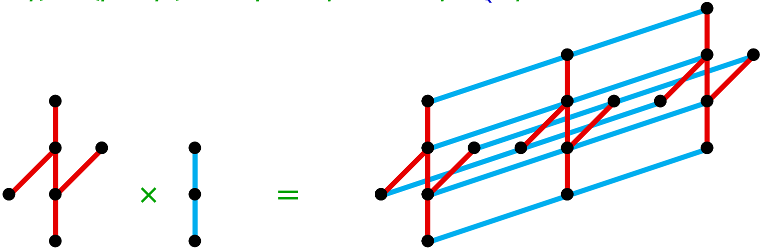




Product posets

Def. product of P and Q is the poset $P \times Q$ on $\{(p, q) : p \in P, q \in Q\}$ such that
$(p, q) \le (p', q') \iff p \le_P p'$ and $q \le_Q q'$.



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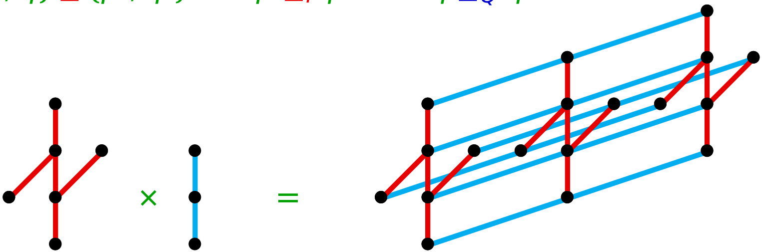

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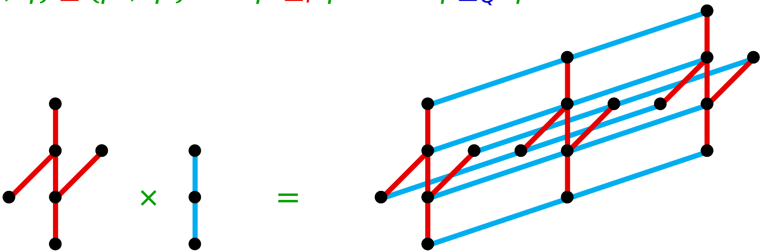

Def. For a product $P \times Q$,
unichain = a chain that is constant in one coordinate;
semiantichain = a family in which pairs are comparable only if they differ in both coordinates.

- A semiantichain and a unichain meet at most once.

The Semiantichain Conjecture

Conj. (Saks–West ∼1978) Every product poset $P \times Q$ has a semiantichain A and a unichain covering $\mathcal{C}$ with

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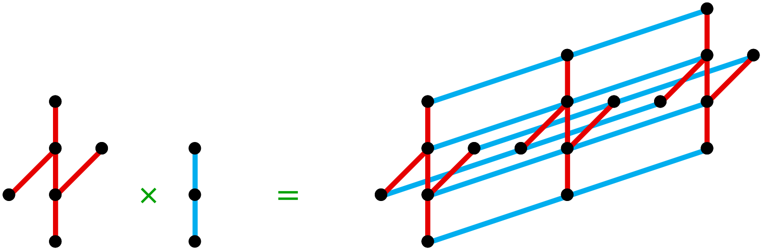

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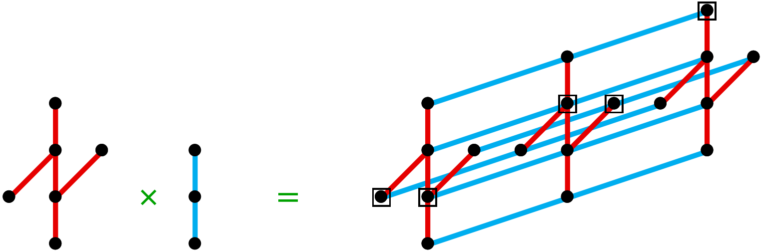

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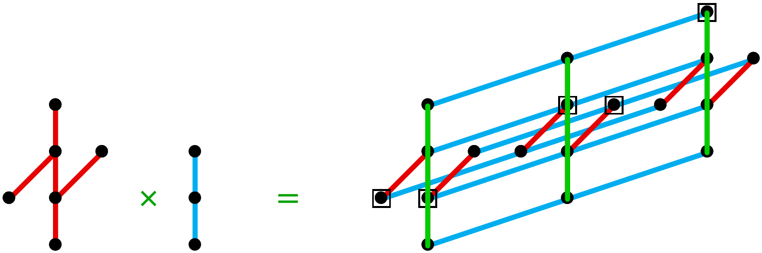

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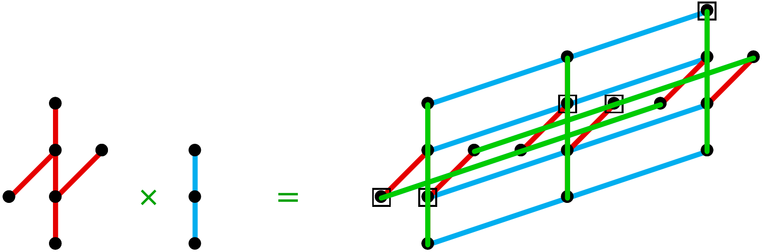

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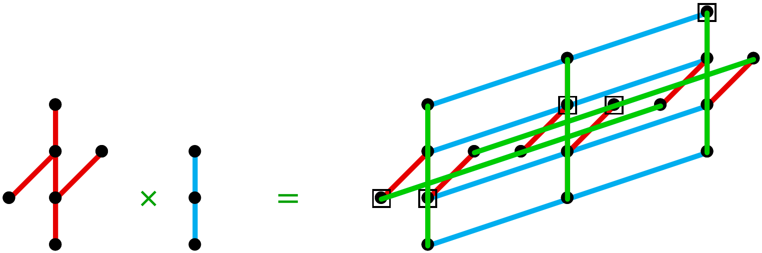

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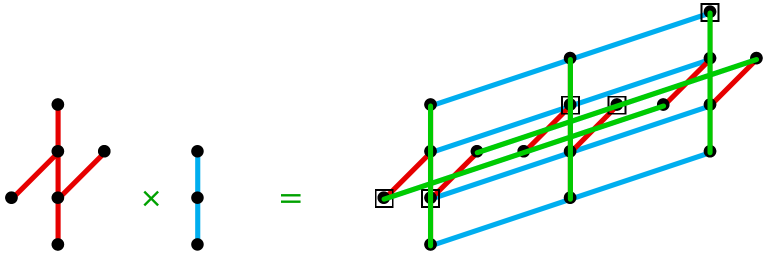

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- Other cases: Tovey–West [1981, 1985], West [1987]

New Result

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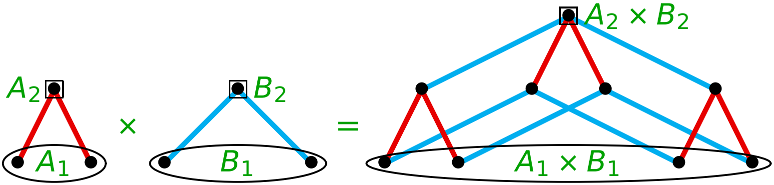
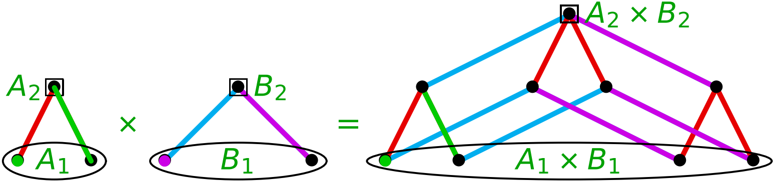
Def. Given antichain partitions $A_1, \ldots, A_h$ of P and $B_1, \ldots, B_h$ of Q, pairing A_i with $B_{\sigma(i)}$ for $1 \le i \le h$ yields a decomposable semiantichain $\bigcup_{i=1}^{h} A_i \times B_{\sigma(i)}$. Indexing in size order, the biggest has size $g(\mathcal{A}, \mathcal{B}) = \sum_{i=1}^{h} |A_i|\,|B_i|$.



Decomposable Covering

Def. Given chain decompositions $C_1, \ldots, C_r$ of P and $D_1, \ldots, D_s$ of Q, covering each $C_i \times D_j$ with $\min\{|C_i| D_j|\}$ copies of the longer chain yields a decomposable unichain covering of size $m(\mathcal{C}, \mathcal{D}) = \sum_{i,j} \min\{|C_i||D_j|\}$.



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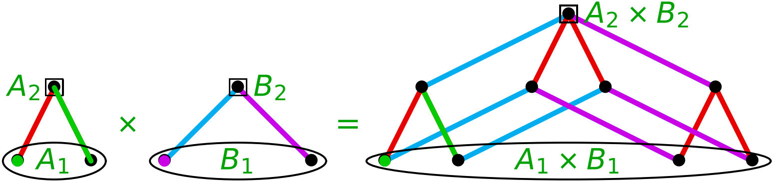

Always $m(\mathcal{C}, \mathcal{D}) \geq g(\mathcal{A}, \mathcal{B})$. West–Tovey [1981] gave some sufficient conditions for equality.

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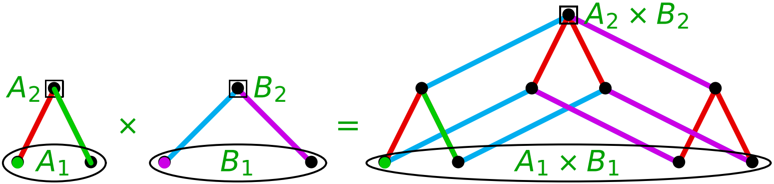

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Idea: Products of posets with width 2 always have a largest semiantichain that is decomposable.

Decomposable Semiantichains

Lem. A poset of width 2 and height $n - h$ has an antichain decomp. with sizes $2^{(h)}1^{(n-2h)}$ (multiset).

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Cor. Given posets P and P' of width 2 with $(r \leq r')$

	size	height	antichains	sizes
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Pf.

	r	$n-2r$	computations
$\mathcal{A}$:	$2\cdots2$	$1\ \cdots\ 1$	$4r+2(r'-r)+n'-2r' = 2r+n'$
$\mathcal{A}'$:	$2\ \cdots\ 2$	$1\cdots1$	$4r+(n-2r-r')+(r'-r) = n+r+r'$
	r'	$n'-2r'$	$4r+2(n-2r) = 2n$

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Similarly, P' has chain decompositions
$\mathcal{C}'$ with sizes (a', b') and
$\mathcal{D}'$ with sizes $(n' - r', k', l')$.

Equality

Case 1: $n - r \geq n' - r'$. Use $\mathcal{D}$ and $\mathcal{C}'$:

$$\begin{aligned} m(n-r, k, l; a', b') &= m(n-r; a', b') + m(k, l; a', b') \\ &\leq m(\infty; a', b') + m(k, l; \infty, \infty) \\ &= n' + 2r = g(\mathcal{A}, \mathcal{A}'). \end{aligned}$$

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Case 2: $r' \leq n - r \leq n' - r'$.

Partition $P \times P'$ into $L \times P'$ and $(P - L) \times P'$.

Cover $L \times P'$ using L and $\mathcal{D}'$.

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Case 3: $n - r \leq r'$. Use $\mathcal{C}$ and $\mathcal{C}'$.

$$m(a,b;a',b') \leq m(a,b;\infty,\infty) = 2(a+b) = 2n = g(\mathcal{A},\mathcal{A}').$$

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- The Semiantichain Conjecture is the statement that $\alpha(G \square H) = \theta(G \square H)$ when G and H are comparability graphs. (Equality only for the full graph.)

Products, Perfection, and Semiantichains

Thm. (Ravindra–Parthasarathy ['77]; de Werra–Hertz ['99])
For G, H connected, $G \square H$ is perfect exactly when:
a) both are bipartite;
b) one is a TDF graph and the other is complete;
c) one is a BFF graph with diamond, the other is a tree.

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TDF graph = triangulated and "diamond-free".
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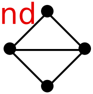
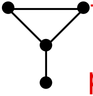
diamond or kite





flag or paw

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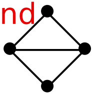

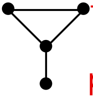

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paw

Cor. The Semiantichain Conjecture holds when
a) Both posets have height 2;
b) (a special case of the Greene–Kleitman Theorem);
c) One is a ranking, the other height 2 and acyclic.

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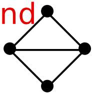
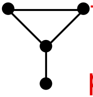
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(Olariu ['88]: paw-free $\Rightarrow$ tri-free or $K_{n_1,\ldots,n_r}$.)