3-Reconstructibility of Rooted Trees

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Abstract

A rooted tree is *ℓ-reconstructible* if it is determined by its multiset of rooted subtrees (with the same root) obtained by deleting ℓ vertices. We determine which rooted trees are ℓ-reconstructible for $\ell \le 3$ and show how this can be used to study reconstructibility of unrooted trees.

1 Introduction

A graph is ℓ-reconstructible if it is determined by its multiset of (unlabeled) subgraphs obtained by deleting ℓ vertices. The famous Reconstruction Conjecture of Kelly [4, 5] and Ulam [16] is that every graph with at least three vertices is 1-reconstructible. Manvel [9, 10] refined the conjecture.

Conjecture 1.1 (Manvel [9, 10]). *For $\ell \in \mathbb{N}$, there exists a threshold M_ℓ such that every graph with at least M_ℓ vertices is ℓ-reconstructible.*

The original Reconstruction Conjecture is $M_1 = 3$. Manvel named the generalization "Kelly's Conjecture" in honor of the final sentence in Kelly [5], which suggested that one can study reconstruction from the $(n-2)$-deck. Manvel noted that Kelly may have expected the statement for $n = 2$ to be false.

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When discussing ℓ-reconstructibility, the multiset of subgraphs of an n-vertex graph G obtained by deleting ℓ vertices is called the $(n-\ell)$-*deck* or simply the *deck* of G, denoted $\mathcal{D}_{n-\ell}(G)$. We are given only the isomorphism class of each subgraph and do not know which vertices were deleted. The resulting $\binom{n}{\ell}$ subgraphs are called the *cards* of the deck. For an n-vertex graph G, being ℓ-reconstructible means that no other graph has the same $(n-\ell)$-deck as G, or $\mathcal{D}_{n-\ell}(G) = \mathcal{D}_{n-\ell}(H)$ implies $G \cong H$.

The elementary observation that motivates the extension from 1-reconstructibility to ℓ-reconstructibility is the following.

Observation 1.2. *For any graph G, the k-deck $\mathcal{D}_k(G)$ determines the $(k-1)$-deck $\mathcal{D}_{k-1}(G)$.*

Proof. Each card in $\mathcal{D}_{k-1}(G)$ appears in exactly $|V(G)| - k + 1$ cards in $\mathcal{D}_k(G)$. $\square$

Thus every graph that is ℓ-reconstructible is also $(\ell-1)$-reconstructible, and the goal is to determine for each graph G the maximum ℓ such that G is ℓ-reconstructible, which we may call its *maximum reconstructibility*.

Nýdl [13] proved that M_ℓ in Conjecture 1.1 grows superlinearly in ℓ, if it exists. For some restricted families, existence of a threshold is known. According to Nýdl [14], Bondy and Hemminger [1] mentioned a preprint of Giles proving that sufficiently large trees are ℓ-reconstructible, but this was apparently never published. Recently, Groenland, Johnston, Scott, and Tan [3] proved that n-vertex trees are reconstructible from their k-decks when $k \ge \frac{8}{9}n + \frac{4}{9}\sqrt{8n+5} + 1$. This implies that n-vertex trees are ℓ-reconstructible when $n \ge 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell})$. For $k = n-3$, the general result reduces to n-vertex trees being reconstructible from their $(n-3)$-decks when $n \ge 194$.

A smaller threshold for ℓ-reconstructibility of trees is conjectured. Nýdl [12] conjectured that trees with at least $2\ell+1$ vertices are *weakly ℓ-reconstructible*, meaning that no two such n-vertex trees have the same $(n-\ell)$-deck. Groenland et al. [3] found one counterexample by computer, consisting of two 13-vertex trees with the same 7-deck. Giles [2] proved that trees with at least six vertices are 2-reconstructible, and this is sharp (on five vertices, the tree obtained by subdividing one edge of the star with three edges has the same 3-deck as the disjoint union of a 4-cycle and an isolated vertex).

In [7], the present authors proved that n-vertex acyclic graphs form an *ℓ-recognizable* family when $n \ge 2\ell + 1$ (and $(n,\ell) \ne (5,2)$), meaning that membership in the family is determined by the $(n-\ell)$-deck. (The proof for $n \ge 2\ell + 2$ is simpler.) The combination of ℓ-recognizability and weak ℓ-reconstructibility is ℓ-reconstructibility, so the modification of Nýdl's conjecture with the counterexample from [3] is now the following.

Conjecture 1.3. *For $n \ge 2\ell + 1$ (except $(n,\ell) \in \{(5,2),(13,6)\}$), every n-vertex tree is ℓ-reconstructible, and this threshold is sharp.*

For sharpness in general, Nýdl found two trees with $2k$ vertices having the same k-deck. A *spider* is a tree having at most one vertex with degree at least 3. Let $S_{a_1,\dots,a_d}$ denote the spider consisting of a vertex of degree d that is the endpoint of paths with lengths $a_1,\dots,a_d$; the tree has $1+\sum_{i=1}^d a_i$ vertices. Nýdl proved that $S_{k-1,k-1,1}$ and $S_{k,k-2,1}$ with $2k$ vertices have the same k-deck. A short proof of this was given by Kostochka and West [8] using the tools developed by Spinoza and West [15] for graphs with maximum degree 2.

The threshold $n \ge 2\ell+1$ works for several other restricted families. Spinoza and West [15] proved that all graphs having maximum degree at most 2 and at least $2\ell+1$ vertices are ℓ-reconstructible. Furthermore, this is sharp, since the path $P_{2\ell}$ and the disjoint union $C_{\ell+1}+P_{\ell-1}$ of a cycle and a path have the same ℓ-deck. In fact, [15] determined the maximum reconstructibility for all graphs with maximum degree at most 2.

Also, in [7] the present authors proved that if $n \ge 2\ell+1$, then n-vertex graphs having no component with more than $n-\ell$ vertices are ℓ-reconstructible. The result is sharp, since two graphs with 2ℓ vertices that are disjoint unions of paths ($P_\ell+P_\ell$ and $P_{\ell+1}+P_{\ell-1}$) have the same ℓ-deck. This follows from the result of Spinoza and West [15] that any two graphs with the same number of vertices and edges whose components are all cycles with at least $k+1$ vertices or paths with at least $k-1$ vertices have the same k-deck. Furthermore, as noted by Kostochka and West [8], ℓ-reconstructibility of graphs consisting of $\ell-1$ isolated vertices plus one component with $n-\ell+1$ vertices is equivalent to the original Reconstruction Conjecture.

Results on ℓ-reconstructibility for small ℓ are also known for degree lists, connectedness, random graphs, disconnected graphs, complete multipartite graphs, 3-regular graphs, and r-regular graphs that are not 2-connected. Kostochka and West [8] surveyed these results. In addition to their general result on trees, Groenland et al. [3] also extended some of the other earlier results. They proved that the degree list of an n-vertex graph is reconstructible from the k-deck when $k \ge \sqrt{2n\log(2n)}$ and that the family of connected n-vertex graphs is ℓ-recognizable when $n \ge 10\ell$.

In this paper, motivated by the problem of ℓ-reconstructibility of trees, we study ℓ-reconstructibility of rooted trees from rooted connected subtrees.

Definition 1.4. A *rooted tree* is a tree with one vertex distinguished as a *root*; other vertices are undistinguished, with no order specified among them. The *rooted pieces* or *r-pieces* of a rooted tree are the rooted subtrees that are the components obtained by deleting the root, with the original neighbors of the root designated as the roots in the r-pieces. The *size* of an r-piece is its number of vertices.

A *rooted connected card* or *rc1-card* of a rooted tree T with root z is a rooted tree T' with root z obtained by deleting a leaf of T other than z. More generally, the *rcℓ-cards* of a rooted tree with n vertices and root z are the rooted subtrees with $n-\ell$ vertices that have root z. The *rcℓ-deck* of a rooted tree is the multiset of its rcℓ-cards; the root vertex is known in each card, but otherwise the vertices are unlabeled.

As suggested earlier, proving reconstructibility for the graphs in a particular family $\mathcal{G}$ often is done in two steps. First we show that every graph having the same deck as a graph in $\mathcal{G}$ is also in $\mathcal{G}$; this is defined as showing that the family is *recognizable*. We can then restrict our attention to reconstructions in $\mathcal{G}$ to show that only one graph (in $\mathcal{G}$) has this deck; this is showing that the family is *weakly reconstructible*. *Weakly ℓ-reconstructible* means doing this with the $(n-\ell)$-deck.

In the context of rooted trees, "weakly ℓ-reconstructible" acknowledges the stipulation that the full structure is a rooted tree. However, the very notion of rcℓ-deck is unclear for rooted graphs that are not rooted trees, so the model makes sense primarily for rooted trees. Hence it would not be confusing to use "ℓ-reconstructible" in this context. Nevertheless, we will maintain the precise language in stating results, saying that a rooted tree is *weakly ℓ-reconstructible* if no other rooted tree has the same rcℓ-deck.

Our main result is that rooted trees are weakly 3-reconstructible, with certain exceptions, and that the exceptions can be distinguished by knowing certain additional information such as the number of leaves and/or the number of copies of the spider $S_{2,1,1}$.

Along the way, we need to prove the analogous results for reconstruction from rc1-decks and rc2-decks. Giles [2] used this tool in proving that trees with at least six vertices are 2-reconstructible. However, he did not include all the details of the proof that rooted trees (with certain exceptions) are weakly 2-reconstructible. Since we need that result, for completeness we include a full proof.

Our results for rooted trees are proved in the next section. In the final two sections, we show how they apply to 3-reconstructibility of trees. We first recognize that the deck comes only from a tree. The basic idea is then to identify a particular vertex of the tree in a subset of the cards in the deck so that the resulting rooted trees form the rc3-deck of the original tree when viewed as a tree rooted at that vertex. The results on weak 3-reconstructability of rooted trees then determine the tree.

Using this approach, we showed that trees with at least 25 vertices are 3-reconstructible. We present only a portion of the proof, showing that the trees in a restricted class are 3-reconstructible, because the details of the omitted cases are quite long (another 23 manuscript pages), and because the result of Groenland et al. [3] yields 3-reconstructibility for trees with at least 194 vertices, which omits only finitely many cases. The full details of our proof can be found at https://faculty.math.illinois.edu/∼west/pubs/tree3rec.pdf. Our aim here is to illustrate applicability of weak 3-reconstructibility of rooted trees. It is likely that the theorem and its application generalize to larger ℓ, but that might require long proofs.

2 Weak reconstructibility of rooted trees

In discussing weak reconstructibility of rooted trees, we know that we have the rcℓ-deck of a rooted tree. We consider only rooted trees as reconstructions and ignore the problem of showing that the deck comes from a rooted tree. Our proof of the first result is similar to Kelly's original proof that trees are reconstructible, with the r-pieces here being analogous to arms from the centers of trees in his proof.

Theorem 2.1. *Rooted trees are weakly* 1*-reconstructible.*

Proof. We assume that we are given the rc1-cards of a rooted n-vertex tree T. We use induction on n. When $n \leq 2$, there is only one n-vertex rooted tree. When $n = 3$, there are two n-vertex rooted trees, and they are distinguished by the number of rc1-cards. For the induction step, suppose $n > 3$, and let z be the root of T.

Since $n \geq 4$ and z is given in each card, T has only one r-piece if and only if every rc1-card has only one r-piece. Hence we know whether T has one r-piece or more than one.

If T has only one r-piece, then let T' be the rooted tree obtained from $T-z$ by designating the neighbor of z in T as the root z'. The rc1-cards of T' are obtained from those of T by deleting z and designating its neighbor as the root z'. By the induction hypothesis, we can reconstruct T' from these, and we reconstruct T by adding z to T', adjacent to z'.

Hence we may assume that T has more than one r-piece. Over all the rc1-cards of T, the r-pieces include the r-pieces of T plus some rooted trees that are not r-pieces of T, obtained by deleting a leaf of an r-piece of T. In particular, all the largest r-pieces that arise are actual r-pieces of T, since they cannot arise from a larger r-piece by deleting a vertex.

If all r-pieces in all cards have only one vertex, then T is a star rooted at the center. We can recognize this, so in the remaining case some r-piece of T has more than one vertex.

Among all the largest r-pieces of the rc1-cards, let M be one that occurs most often. Cards with fewer than the maximum number of r-pieces isomorphic to M arise by deleting a leaf of an r-piece isomorphic to M. Let v be a leaf of M, and let $L = M - v$, with the same root. Among the rc1-cards of T with fewer than the maximum number of r-pieces isomorphic to M, find a card T' with the maximum number of r-pieces isomorphic to L. Reconstruct T by replacing an r-piece isomorphic to L in T' with an r-piece isomorphic to M. $\square$

Before we discuss weak 2-reconstructibility of rooted trees, we note some exceptions.

Example 2.2. *Rooted trees with common rc2-decks.* For $n \geq 3$, there are two n-vertex rooted trees having the same rc2-deck with one rc2-card. Let $\hat{P}_n$ denote the path P_n as a rooted tree with an endpoint as the root. The only rc2-card of $\hat{P}_n$ is $\hat{P}_{n-2}$. Let $\hat{P}'_n$ denote the rooted tree consisting of $\hat{P}_{n-2}$ with two children added at its leaf. Again $\hat{P}_{n-2}$ is the only rc2-card.

For $n \geq 5$, there are two n-vertex rooted trees having the same rc2-deck with three rc2-cards. Let $\hat{Q}_n$ be the n-vertex rooted tree obtained from $\hat{P}_{n-4}$ by appending two copies of $\hat{P}_2$ at the leaf. Let $\hat{Q}'_n$ be the n-vertex rooted tree obtained from $\hat{P}_{n-4}$ by appending $\hat{P}_1$ and $\hat{P}'_3$ at the leaf. Both $\hat{Q}_n$ and $\hat{Q}'_n$ have rc2-deck consisting of two copies of $\hat{P}_{n-2}$ and one copy of $\hat{P}'_{n-2}$. These rooted trees are shown (with others) in Figure 1.

Definition 2.3. A rooted tree is *trivial* if it has only one vertex. The *root-degree* of a rooted tree is the degree of the root. We write a rooted tree with r-pieces $T_1, \ldots, T_d$ as $U(T_1, \ldots, T_d)$. An rcℓ-card of a tree T with root degree d^* is j*-slim* if it has root-degree $d^* - j$.

Theorem 2.4. *Except for $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$, rooted n-vertex trees are weakly 2-reconstructible. Knowing also the number of leaves distinguishes the exceptions having the same rc2-deck.*

Proof. We have the rc2-deck of an n-vertex rooted tree T, with root z specified in each rc2-card. If $n = 2$, then $T = \hat{P}_2$. If $n = 3$, then $T \in \{\hat{P}_3, \hat{P}'_3\}$.

If T has no vertices other than z and its children, then T has $\binom{n-1}{2}$ rc2-cards, all stars, and no other n-vertex rooted tree has this many rc2-cards. If T has exactly one vertex other than z and its children, then T has $\binom{n-2}{2} + 1$ rc2-cards; again T is determined. For $n = 4$, this leaves only $\hat{P}_4$ and $\hat{P}'_4$, which have the same rc2-deck but different numbers of leaves. This completes the proof for $n \leq 4$; we proceed inductively with $n \geq 5$.

Let $d^* = d_T(z)$. In the remaining cases, T has at least two vertices that are not children of z. Now d^* is the maximum of the root-degrees in the rc2-cards, and $d^* \leq n - 3$. The root-degree is the number of r-pieces of T, which we now know.

If $d^* = 1$, then T has only one r-piece; let z' be the child of z. Let T' be the r-piece of T (that is, $T' = T - z$ with z' as root). The rc2-cards of T' arise from those of T by deleting z and are rooted at z'. By the induction hypothesis, we can reconstruct T' from its rc2-deck unless $T' \in \{\hat{P}_{n-1}, \hat{P}'_{n-1}, \hat{Q}_{n-1}, \hat{Q}'_{n-1}\}$, which holds if and only if T is the corresponding member of $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$ (the cases $T \in \{\hat{Q}_n, \hat{Q}'_n\}$ do not arise when $d^* = 1$ until $n \geq 6$). We reconstruct T by adding z adjacent to z' in T'; in the exceptional cases the number of leaves distinguishes between the two members of a pair with the same rc2-deck.

Hence we may assume $2 \leq d^* \leq n - 3$. When $n = 5$, we thus consider only instances with $d^* = 2$; they are $\hat{Q}_5$ and $\hat{Q}'_5$ (from Example 2.2) and $U(\hat{P}_1, \hat{P}_3)$. Since $U(\hat{P}_1, \hat{P}_3)$ has two rc2-cards while $\hat{Q}_5$ and $\hat{Q}'_5$ have three, $U(\hat{P}_1, \hat{P}_3)$ is reconstructible, while $\hat{Q}_5$ and $\hat{Q}'_5$ have the same rc2-deck but have different numbers of leaves. Hence we may assume $n \geq 6$.

Note that T has at least two trivial r-pieces if and only if T has an rc2-card with root-degree $d^* - 2$. In this case, we reconstruct T by adding two trivial r-pieces to such a card.

Hence we may assume that all rc2-cards have root-degree at least $d^* - 1$, and T has at most one trivial r-piece. In the remainder of this proof we use "slim card" to denote a 1-slim rc2-card. There are at least $d^* - 1$ slim cards when T has a trivial r-piece, with equality only when all other r-pieces are paths with at least three vertices. When T has no trivial r-piece, the number of slim cards is the number of r-pieces isomorphic to $\hat{P}_2$, which is at most d^*. Hence we know whether T has a trivial r-piece unless the deck has $d^* - 1$ or d^* slim cards.

Suppose first that T has exactly d^* slim cards and has reconstructions both with and without a trivial r-piece. The reconstruction without a trivial r-piece is $U(\hat{P}_2, \ldots, \hat{P}_2)$, in which no rc2-card has an r-piece of size at least 3. This also requires $n = 2d^* + 1$, which yields $d^* \geq 3$ since $n > 5$. In a reconstruction having a trivial r-piece, having exactly d^* slim cards requires that $d^* - 2$ r-pieces are paths with size at least 3. With $d^* \geq 3$, some rc2-card does have an r-piece with size at least 3, which is not true for $U(\hat{P}_2, \ldots, \hat{P}_2)$. Hence when T has d^* slim cards we can recognize whether T has a trivial r-piece.

Now suppose that T has exactly $d^* - 1$ slim cards. If T has a trivial r-piece, then all other r-pieces are paths with size at least 3. Here rc2-cards with two trivial r-pieces arise only by deleting two vertices from an r-piece of size 3, so there are at most $d^* - 1$ of them, with equality only when all nontrivial r-pieces are copies of $\hat{P}_3$. If T has no trivial r-piece, then having $d^* - 1$ slim cards requires that every r-piece except one is $\hat{P}_2$, and the other r-piece has size more than 2. Hence exactly $\binom{d^*-1}{2}$ rc2-cards have two trivial r-pieces. Since $\binom{d^*-1}{2} > d^* - 1$ when $d^* \geq 5$, we may assume $d^* \leq 4$. If confusion remains, then the instances with one trivial r-piece or no trivial r-pieces are as listed below, where $\hat{T}_m$ denotes any rooted tree with m vertices. The possibilities require $n \geq 9$ when $d^* = 3$ and $n \geq 5$ when $d^* = 2$. In each case, the two possibilities are distinguished by the number of rc2-cards.

	one trivial r-piece	#rc2-cards	no trivial r-piece	#rc2-cards
$d^* = 4$	$U(\hat{P}_1, \hat{P}_3, \hat{P}_3, \hat{P}_3)$	9	$U(\hat{P}_2, \hat{P}_2, \hat{P}_2, \hat{T}_4)$	≥ 10
$d^* = 3$	$U(\hat{P}_1, \hat{P}_3, \hat{P}_{n-5})$	5	$U(\hat{P}_2, \hat{P}_2, \hat{T}_{n-5})$	≥ 6
$d^* = 2$	$U(\hat{P}_1, \hat{P}_{n-2})$	2	$U(\hat{P}_2, \hat{T}_{n-3})$	≥ 3

Therefore, the rc2-deck determines whether T has a trivial r-piece. Now consider the case where the rc2-deck of T has a slim card. If either T has no trivial r-piece or T has both a trivial r-piece and a slim card with a trivial r-piece, reconstruct T from a slim card by adding $\hat{P}_2$ as an r-piece.

In the remaining case with a slim card, T has a trivial r-piece and the slim cards are the rc1-cards for the rooted tree T' obtained by deleting the trivial r-piece from T. By Theorem 2.1, we can reconstruct T' from these and obtain T by adding a trivial r-piece.

Hence we may assume that T has no slim cards. Thus every r-piece of T has size at least 3, and every rc2-card has d^* pieces. Since $d^* \geq 2$ and every r-piece has size at least 3, every actual r-piece of T appears as an r-piece in some rc2-card. As in Theorem 2.1, the largest r-pieces that appear in rc2-cards are actual r-pieces of T, and we see them. Let b be the maximum size of an r-piece of T.

Suppose first that every r-piece of T has size b. We recognize this case from rc2-cards whose r-pieces all have size b except for one with size $b-2$, which must have lost two vertices because the others could not have lost any. Call such rc2-cards “pure” cards. Among all the pure cards, let C be one in which the multiplicity of some r-piece R in the list of r-pieces of size b is as small as possible (possibly 0). Reconstruct T by replacing the small r-piece in C with a copy of R.

Hence we may assume that some r-piece of T has size less than b. Now an rc2-card C' having a smallest r-piece over all the rc2-cards arises by deleting two vertices from a smallest r-piece of T. In C' we see all r-pieces of T of size b, with their multiplicities.

Let M be a largest r-piece of T, and let d' be the number of r-pieces of T isomorphic to M. Let T' be obtained from T by deleting the d' r-pieces of T isomorphic to M. The rc2-cards of T' are obtained from the rc2-cards of T having d' r-pieces isomorphic to M by deleting the copies of M. If $d' < d^* - 1$, then T' has at least two r-pieces, each of size at least 3. Hence T' is not any of the exceptional rooted trees, and by the induction hypothesis we can reconstruct T' and replace the copies of M to obtain T.

In the final case, T consists of $d^* - 1$ pieces isomorphic to M and one smaller piece R', and we know this. Let L be a rooted tree obtained from M by deleting one leaf; let a be the number of leaves whose deletion from M yields L. In every rc2-card of T having $d^* - 2$ r-pieces isomorphic to M and one piece isomorphic to L, the remaining r-piece is an rc1-card of R' (smaller than L). Each rc1-card of R' arises this way on exactly $a(d^* - 1)$ cards. Hence we obtain the rc1-deck of R'. By Theorem 2.1, we can reconstruct R' and thus T. □

Definition 2.5. A *broom* is a tree obtained from a star by growing a path from the center of the star. A *rooted broom* is a broom with root chosen as the endpoint of the path grown from the center of the star. A rooted path and a star rooted at its center are both degenerate examples of rooted brooms. A *branch vertex* is a vertex with degree at least 3.

Example 2.6. In the statement of the next lemma, when $\ell = 2$ the class (b) includes $\hat{P}_n$ and $\hat{P}'_n$, which are brooms, plus one rooted tree $\hat{P}''_n$ obtained by appending $\hat{P}_1$ and $\hat{P}_2$ at the leaf of $\hat{P}_{n-3}$. Note that $\hat{P}''_n$ has two leaves and two rc2-cards, while $\hat{P}_n$ and $\hat{P}'_n$ each have only one rc2-card. All rc2-cards for these three rooted trees are copies of $\hat{P}_{n-2}$, but the three examples are distinguished by knowing the number of leaves and the number of rc2-cards.

Lemma 2.7. *For $\ell \geq 2$ and $n \geq \ell+2$, the rcℓ-cards of an n-vertex rooted tree T are pairwise isomorphic if and only if (a) T is a rooted broom or (b) T is formed by merging the leaf of $\hat{P}_{n-\ell-1}$ with the root of a rooted tree having $\ell + 2$ vertices.*

Proof. The rcℓ-cards of a rooted broom are isomorphic rooted brooms. In class (b), every rcℓ-card is $\hat{P}_{n-\ell}$.

Let T be a rooted tree with pairwise isomorphic rcℓ-cards. We may assume $T \neq \hat{P}_n$. Let v be the branch vertex of T nearest to the root. If T is not in class (b), then at least $\ell + 2$ vertices lie below v in the tree. Also, the subtree rooted at v has at least two r-pieces, since v is a branch vertex.

It suffices to prove that a rooted tree $\widetilde{T}$ with at least two r-pieces and at least $\ell+3$ vertices has distinct rcℓ-cards if it is not a rooted star. If there are at most ℓ vertices below the root outside the largest piece, then there is an rcℓ-card with one r-piece and an rcℓ-card with more than one r-piece. If there are more than ℓ vertices below the root outside a largest r-piece, then iteratively deleting a leaf from a smallest remaining r-piece until ℓ have been removed yields an rcℓ-card whose list of sizes of r-pieces differs from that of an rcℓ-card obtained by iteratively deleting a leaf from a largest remaining r-piece until ℓ have been removed. $\square$

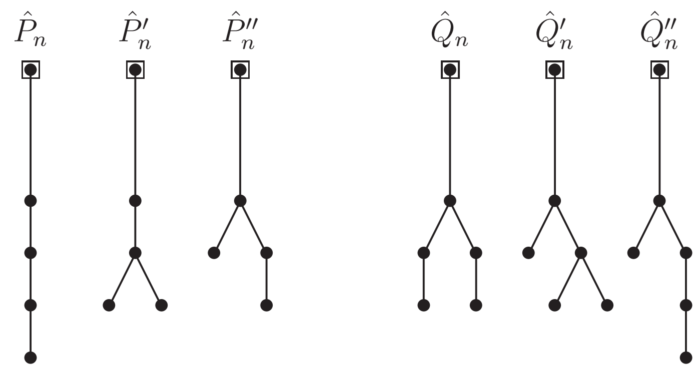

Figure 1: Special rooted trees.

In studying weak 3-reconstructibility of rooted trees, more exceptions arise.

Example 2.8. *Rooted trees with common rc3-decks.* We describe specific rooted trees by attaching a rooted forest below the leaf of a rooted path. Several appear in Figure 1, including $\hat{P}''_n$ as defined in Example 2.6. With $\hat{Q}_n$ and $\hat{Q}'_n$ defined as in Example 2.2, let $\hat{Q}''_n$ be the rooted tree obtained by putting $\hat{P}_1$ and $\hat{P}_3$ below the leaf of $\hat{P}_{n-4}$. Let $\hat{B}_{n,t}$ be the n-vertex rooted broom with t leaves (thus $\hat{B}_{n,1} = \hat{P}_n$ and $\hat{B}_{n,2} = \hat{P}'_n$).

Every way of putting a rooted forest having ℓ vertices below the leaf of $\hat{P}_{n-\ell}$ yields an n-vertex rooted tree whose rcℓ-deck has a single card, $\hat{P}_{n-\ell}$. It is more helpful to describe this as putting a rooted tree with $\ell+1$ vertices under the leaf of $\hat{P}_{n-\ell-1}$. When $\ell = 2$, the resulting trees are $\hat{P}_n$ and $\hat{P}'_n$. When $\ell = 3$, the tree whose root is put under the leaf of $\hat{P}_{n-4}$ is one of $\{\hat{P}_4, \hat{P}'_4, \hat{P}''_4, \hat{B}_{4,3}\}$.

More generally, every way of adding $\ell+1$ vertices in d nonempty rooted trees under the leaf of $\hat{P}_{n-\ell-1}$ yields an n-vertex rooted tree whose rcℓ-deck consists of d copies of $\hat{P}_{n-\ell}$. For $(\ell, d) = (3, 2)$ and $n \geq 5$, we obtain three trees: $\hat{Q}_n$, $\hat{Q}'_n$, and $\hat{Q}''_n$.

For $n \geq 6$, several pairs of n-vertex rooted trees obtained from $\hat{P}_{n-6}$ by adding one of two 6-vertex rooted trees under the leaf have the same rc3-deck as listed below. Within each pair, the two rooted trees are distinguished by the number of leaves.

under leaf of $\hat{P}_{n-6}$	#rc3-cards	rc3-deck	#leaves
$U(\hat{P}_1, \hat{P}_4)$ or $U(\hat{P}_1, \hat{P}'_4)$	2	one $\hat{P}_{n-3}$, one $\hat{P}'_{n-3}$	2 or 3
$U(\hat{P}_2, \hat{P}_3)$ or $U(\hat{P}_1, \hat{P}''_4)$	3	two $\hat{P}_{n-3}$, one $\hat{P}'_{n-3}$	2 or 3
$U(\hat{P}_2, \hat{P}'_3)$ or $U(\hat{P}_1, \hat{B}_{4,3})$	4	three $\hat{P}_{n-3}$, one $\hat{P}'_{n-3}$	3 or 4

Finally, let $\hat{T}^+$ denote the rooted tree obtained from a rooted tree $\hat{T}$ by adding a trivial r-piece as an extra child of the root. If $\hat{T}$ and $\tilde{T}$ are rooted trees that have the same rcℓ-deck and also have the same rc$(\ell-1)$-deck, then $\hat{T}^+$ and $\tilde{T}^+$ also have the same rcℓ-deck. Their deck consists of the common rc$(\ell-1)$-deck of $\hat{T}$ and $\tilde{T}$ together with the cards of their common rcℓ-deck extended by adding a trivial r-piece. When $\ell = 3$, this occurs just when $\{\hat{T}, \tilde{T}\}$ is $\{\hat{P}_{n-1}, \hat{P}'_{n-1}\}$ (for $n \geq 4$) or $\{\hat{Q}_{n-1}, \hat{Q}'_{n-1}\}$ (for $n \geq 6$), since those are the only pairs of $(n-1)$-vertex trees having the same rc2-deck.

When $\ell = 3$, all these examples have root-degree at most 2, except for root-degree 3 when $n = 6$ in the pair obtained by adding a trivial r-piece to $\hat{Q}_5$ or $\hat{Q}'_5$, and when $n = 4$ for $\hat{B}_{4,3}$ in the set $\{\hat{P}_4, \hat{P}'_4, \hat{P}''_4, \hat{B}_{4,3}\}$. Besides $\hat{P}'_3$ in the pair $\{\hat{P}_3, \hat{P}'_3\}$ the examples with root-degree 2 are of those of the form $\{\hat{T}^+, \tilde{T}^+\}$ in the preceding paragraph where $\hat{T}$ and $\tilde{T}$ are exceptions with root-degree 1 that also have the same rc2-deck.

We mention two more pairs with seven vertices and root-degree 2: $\{U(\hat{P}_2, \hat{P}'_4), U(\hat{P}_3, \hat{P}_3)\}$ and $\{U(\hat{P}_2, \hat{P}''_4), U(\hat{P}_3, \hat{P}'_3)\}$. The trees in the first pair have the same rc3-deck consisting of two copies of $\hat{P}_4$ and two copies of $\hat{P}''_4$, and they are distinguished by the number of leaves. The trees in the second pair have the same rc3-deck consisting of one copy of $\hat{P}_4$, one copy of $\hat{P}'_4$, and three copies of $\hat{P}''_4$, but both of these rooted trees have three leaves.

Remark 2.9. *Rooted trees having the same rc2-deck also have the same rc3-deck.* We showed in Theorem 2.4 that a rooted tree is determined by its rc2-deck, except for $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$. If rooted trees with the same rc2-deck are the same, then they also have the same rc3-deck. Also, as noted in Example 2.8, $\hat{P}_n$ and $\hat{P}'_n$ have the same rc3-deck, as do $\hat{Q}_n$ and $\hat{Q}'_n$.

Theorem 2.10. *For $n \geq 4$, the n-vertex rooted trees not described in Example 2.8 are weakly 3-reconstructible. For the exceptions, it is sufficient to know also the number of leaves, except for the general pairs $\{\hat{P}'_n, \hat{P}''_n\}$ and $\{\hat{Q}_n, \hat{Q}''_n\}$ and the 7-vertex pair $\{U(\hat{P}_2, \hat{P}''_4), U(\hat{P}_3, \hat{P}'_3)\}$. Within each of these pairs, the two trees are distinguished by the number of copies of $S_{2,1,1}$ as an unrooted subtree.*

Proof. The behavior of the exceptions is verified by checking all the examples in Example 2.8. The examples having the same rc3-deck are distinguished by their number of leaves, except for the pairs listed in this theorem statement. For each rooted tree in $(\hat{P}'_n, \hat{Q}''_n, U(\hat{P}_3, \hat{P}'_3))$, there is one more copy of $S_{2,1,1}$ in $(\hat{P}''_n, \hat{Q}_n, U(\hat{P}_2, \hat{P}''_4))$, respectively.

Hence we may assume that we are given the rc3-deck of an n-vertex rooted tree T not in the list of exclusions. Let z be the root of T, specified in each rc3-card. Let $d^* = d_T(z)$. We use induction on n.

Case 1: *$n \le 6$ or $d^* \ge n-3$ or $d^* = 1$.* For $n \le 4$, all trees are exceptional.

Consider $n = 5$. The trees with $d^* = 1$ are in $\{\hat{P}_5, \hat{P}'_5, \hat{P}''_5, \hat{B}_{5,3}\}$, all exceptional. Those with $d^* = 2$ are in $\{\hat{Q}_5, \hat{Q}'_5, \hat{Q}''_5\}$, all exceptional. The only tree with $d^* = 3$ is $U(\hat{P}_1, \hat{P}_1, \hat{P}_2)$, whose rc3-deck is three copies of $\hat{P}_2$, and the only one with $d^* = 4$ is $U(\hat{P}_1, \hat{P}_1, \hat{P}_1, \hat{P}_1)$, whose rc3-deck is four copies of $\hat{P}_2$. These decks differ from those of the rooted trees in Example 2.8. Hence we may proceed inductively with $n \ge 6$.

Let s be the number of vertices of T other than z and its children. If $s \le 3$, then T has a rooted star with $n-4$ leaves as an rc3-card, while if $s \ge 4$ there is no such card. Furthermore, if $s = 3$, then there is exactly one such card, while if $s < 3$ there is more than one such card. Hence we can distinguish the cases $s < 3$, $s = 3$, and $s > 3$.

If $s = 0$, then $T = \hat{B}_{n,n-1}$ and T has $\binom{n-1}{3}$ rc3-cards (all stars). If $s = 1$, then T has $n-3$ trivial r-pieces and one 2-vertex r-piece, producing $\binom{n-2}{3} + (n-3)$ rc3-cards. If $s = 2$, then the r-pieces of T are trivial except for two copies of $\hat{P}_2$, or one copy of $\hat{P}_3$, or one copy of $\hat{P}'_3$. In these three cases the numbers of rc3-cards are $\binom{n-3}{3} + 2(n-4)$, or $\binom{n-3}{3} + (n-4) + 1$, or $\binom{n-2}{3} + 1$, respectively. These numbers and the count when $s = 1$ are distinct, except that $\binom{3}{3} + 4 = \binom{4}{3} + 1$ when $n = 6$, which occurs for the listed exceptions $\hat{Q}_5^+$ and $\hat{Q}_5'^+$. Hence the number of rc3-cards distinguishes all the nonexceptional rooted trees with $s \le 2$, which corresponds to $d^* \ge n-3$.

In the remaining cases, T has at least three vertices other than z and its children. Now d^* is the largest root-degree among the rc3-cards, and $d^* \le n-4$. We now know d^*, which is the number of r-pieces of T.

If $d^* = 1$, then T has only one r-piece; let z' be the child of z. Let T' be the r-piece of T, namely $T - z$ with z' as root. The rc3-cards of T' are obtained from those of T by deleting z and naming z' as the root. By the induction hypothesis, we can reconstruct T' from its rc3-deck unless T' is one of the exceptional trees on $n-1$ vertices, which holds if and only if T is one of the exceptional trees on n vertices. Outside the exceptional cases, we apply the induction hypothesis to T' and then reconstruct T by adding z above the root of T'.

Hence we may assume $2 \le d^* \le n-4$. When $n = 6$, this leaves only instances with $d^* = 2$. There are six such rooted trees, and they occur in three pairs of two rooted trees having the same rc3-deck, as listed in Example 2.8: $\{U(\hat{P}_1, \hat{P}_4), U(\hat{P}_1, \hat{P}'_4)\}$ (same as $\{\hat{P}_5^+, \hat{P}_5'^+\}$), $\{U(\hat{P}_2, \hat{P}_3), U(\hat{P}_1, \hat{P}''_4)\}$, and $\{U(\hat{P}_2, \hat{P}'_3), U(\hat{P}_1, \hat{B}_{4,3})\}$. Within each pair, the trees are distinguished by the number of leaves.

Hence in all other cases we have $n \ge 7$ and $2 \le d^* \le n-4$, and we know d^*.

Case 2: *At least three trivial r-pieces.* Note that T has at least three trivial r-pieces if and only if T has an rc3-card with root-degree $d^* - 3$. In this case, we reconstruct T by adding three trivial r-pieces to such a card. Otherwise, all rc3-cards have root-degree at least $d^* - 2$, and T has at most two trivial r-pieces.

Case 3: *Two trivial r-pieces.* A 2-slim card can arise by deleting two trivial r-pieces and a third vertex or by deleting one trivial r-piece and one r-piece of size 2; there are no 2-slim cards when T has no trivial r-pieces. There are at least $d^* - 2$ 2-slim cards when T has two trivial r-pieces, with equality only when all other pieces are paths of size at least 3. There are at most $d^* - 1$ 2-slim cards when T has exactly one trivial r-piece, with equality if and only if all other pieces are $\hat{P}_2$. Hence the number of trivial r-pieces is known unless the number of 2-slim cards is $d^* - 2$ or $d^* - 1$.

Suppose first that T has exactly $d^* - 1$ 2-slim cards. If T has only one trivial r-piece, then $T = U(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_2)$ and $n = 2d^*$. Since $n > 6$, we have $d^* \geq 4$. No rc3-card has an r-piece of size at least 3. On the other hand, in a reconstruction with two trivial r-pieces and $n = 2d^*$, some r-piece must have size at least 3, and with $d^* \geq 4$ we can see such a piece in some rc3-card. Hence in this case we either reconstruct T or know that every reconstruction has two trivial r-pieces.

Now suppose T has exactly $d^* - 2$ 2-slim cards. Say that T has Type A or Type B depending on whether it has one or two trivial r-piece. If there is only one (Type A), then $T = U(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_2, \hat{T}')$, where $\hat{T}'$ has at least three vertices. If T has exactly two trivial r-pieces (Type B), then all the nontrivial r-pieces are paths of size at least 3.

In Type A, each 2-slim card has $d^* - 3$ pieces that are $\hat{P}_2$, plus one piece that is $\hat{T}'$, and $\hat{T}'$ must be a path since all pieces in Type B are paths. To obtain such a 2-slim card in Type B, we must delete the two trivial pieces and can reduce at most one piece to $\hat{P}_2$. Since there must be $d^* - 2$ such 2-slim cards, in Type B the nontrivial pieces must all be $\hat{P}_3$. Since also the 2-slim cards can only leave one piece with size 3, in Type B we have $T = U(\hat{P}_1, \hat{P}_1, \hat{P}_3, \hat{P}_3)$, and in Type A we have $T = U(\hat{P}_1, \hat{P}_2, \hat{P}_2, \hat{P}_3)$. All rc3-cards with three trivial r-pieces are $U(\hat{P}_1, \hat{P}_1, \hat{P}_1, \hat{P}_2)$; in Type A there are three of these, but in Type B there are only two. Hence Type A and Type B are distinguished by the rc3-deck, and we know how many trivial r-pieces T has.

Hence we can recognize whether T has exactly two trivial r-pieces. If so, then the 2-slim rc3-cards are the rc1-cards of the tree T' obtained by deleting the trivial r-pieces of T. By Theorem 2.1, we can reconstruct T', and we add two trivial r-pieces to obtain T.

Case 4: *One trivial r-piece.* In all remaining cases, T has at most one trivial piece. If T has a 2-slim rc3-card, then T has a trivial piece and a piece of size 2; reconstruct T from a 2-slim card by replacing these pieces.

When the rc3-deck has no 2-slim card, suppose that there exist both a reconstruction T with a trivial piece and a reconstruction T° with no trivial piece; we seek a contradiction.

If $d^* = 2$ and $n \geq 8$, then T° has an rc3-card with two pieces of size at least 2, but T does not. When $n = 7$, we consider cases to eliminate $d^* = 2$. All such possibilities for T° have $\hat{P}_4''$ as at least two rc3-cards, which eliminates $\hat{P}_5$, $\hat{P}_5'$, $\hat{P}_5''$, and $\hat{B}_{5,3}$ from being the nontrivial r-piece of T. The only remaining possibility for T that has $\hat{P}_4'$ as exactly one rc3-card is $U(\hat{P}_1, \hat{Q}_5')$, but the possibilities for T° that have $\hat{P}_4'$ as one rc3-card both have $\hat{P}_4''$ as three rc3-cards, while $U(\hat{P}_1, \hat{Q}'_5)$ has it only twice. The only remaining possibility for T° is then $U(\hat{P}_2, \hat{B}_{4,3})$, which has three copies of $\hat{P}'_4$ and no copy of $\hat{P}_4$. The only possibility for T with no copy of $\hat{P}_4$ is $U(\hat{P}_1, \hat{B}_{5,4})$, which has six copies of $\hat{P}'_4$. Hence we may assume $d^* \geq 3$.

Since there is no 2-slim card, r-pieces of T other than the trivial piece have size at least 3. The number of 1-slim cards with a trivial r-piece is then $2p$, where p is the number of r-pieces of T having size 3, since we can delete such a piece or delete two of its vertices (in only one way) and the trivial piece. In the rc3-deck of T°, to form a 1-slim card with a trivial piece it is necessary to eliminate one r-piece and reduce another to a single vertex; since each r-piece has at least two vertices, this requires $q \geq 2$, where q is the number of r-pieces of T° having size 2. Hence the number of 1-slim cards with a trivial piece is $q(q-1)$, by deleting both vertices from one piece of size 2 and one vertex from another. Hence $2p = q(q-1)$.

Now consider 1-slim cards with no trivial piece. From T, we must delete the trivial piece, so such a card has at most two r-pieces of size 2. From T°, we must delete one r-piece of size 2 or 3 and not create a trivial r-piece, so any such card will have at least $q-1$ r-pieces of size 2. Thus $q-1 \leq 2$. These two computations reduce the possibilities to $(p,q) \in \{(3,3),(0,1),(0,0),(1,2)\}$.

First consider $(p,q) = (3,3)$. Each of T and T° has six 1-slim cards with a trival piece. In each such card of T, the remaining pieces all have size at least 3. However, each such card of T° has a piece of size 2. Hence T and T° cannot both be reconstructions in this case.

Now consider $p = 0$. In this paragraph and the next, "card" means "0-slim rc3-card". Let s and t be the sizes of the second smallest r-pieces in T and T°, respectively. First consider $(p,q) = (0,1)$. Note that $p = 0$ implies $s \geq 4$, so no cards of T have two r-pieces of size 2. With $q = 1$ and $d^* \geq 3$, from T° there is such a card if $t \leq 5$, so we may assume $t \geq 6$. The minimum size of the second smallest r-piece in a card of T° having a trivial r-piece is $t-2$. In a card of T it is $s-3$. Hence $s-3 = t-2$, or $s = t+1 \geq 7$. The minimum size of the second smallest r-piece in a card of T° having an r-piece of size 2 is $t-3$. Since $s = t+1$, there is no such card of T.

Next consider $(p,q) = (0,0)$. Since T has a trivial piece, the rc3-deck has 1-slim rc3-cards. Since $q = 0$, every 1-slim rc3-card of T° is obtained by deleting an r-piece of size 3. Hence the lists of sizes of r-pieces in 1-slim rc3-cards are all the same. However, T has two pieces of size at least 4 (since $d^* \geq 3$) and generates two 1-slim rc3-cards with different lists of sizes of pieces by deleting the trivial piece and either two vertices from one other piece or one from each of two pieces.

Now consider $(p,q) = (1,2)$, and recall that $d^* \geq 3$. In both T and T°, the two smallest pieces together have four vertices. The two 1-slim cards with a trivial r-piece arise by deleting three of those four vertices. Those cards must be the same in both rc3-decks, so T and T° agree except for the two smallest r-pieces; let Y be the rooted tree obtained from either by deleting the two smallest r-pieces. Since $p = 1$ the smallest r-piece in Y has at least four vertices, so Y has at least five vertices. Let y_j be the number of rc$j$-cards of Y. The two smallest pieces of T or T° may be $\{\hat{P}_1, \hat{P}_3\}$, $\{\hat{P}_1, \hat{P}_3'\}$, or $\{\hat{P}_2, \hat{P}_2\}$, In these three cases, the number of rc3-cards of the resulting full tree is $2y_0 + 2y_1 + 2y_2 + y_3$, or $2y_0 + 3y_1 + 3y_2 + y_3$, or $2y_0 + 3y_1 + 2y_2 + y_3$, respectively, where the coefficient of y_j is the number of ways of deleting $3 - j$ vertices from the two smallest r-pieces in forming cards. Since Y has at least five vertices, y_1 and y_2 are positive, so the numbers of rc3-cards differ in the three cases.

Hence we can recognize from the rc3-deck that we are in Case 4 and T has exactly one trivial r-piece. We already reconstructed T in the case that T has a 2-slim card, so we may assume that all nontrivial r-pieces of T have size at least 3. Let S be the set of 1-slim cards having a trivial r-piece. As discussed earlier, $|S| = 2p$, where p is the number of r-pieces of T with size 3. Let Y be the rooted tree obtained from T by deleting all the r-pieces of size 3. Any rc3-card in S shows Y and $p - 1$ of the 3-vertex pieces. Each such piece appears in $2p - 2$ members of S. Hence when $p \geq 2$ we can determine from S the number of pieces of size 3 that are $\hat{P}_3$ and the number that are $\hat{P}_3'$, thereby reconstructing all pieces of T. If $p = 1$, then as before the number of rc3-cards is $2y_0 + 2y_1 + 2y_2 + y_1$ or $2y_0 + 3y_1 + 3y_2 + y_3$ depending on whether the r-piece of T with size 3 is $\hat{P}_3$ or $\hat{P}_3'$. Hence we learn what that r-piece is to complete the reconstruction.

Hence we may assume $p = 0$; all r-pieces of T other than the one trivial piece have size at least 4. Let Y be the rooted tree obtained by deleting the trivial piece. The 1-slim cards of T are precisely the rc2-cards of Y. If Y is determined by its rc2-deck, then T is determined by adding a trivial piece to Y. If there is an alternative reconstruction $\tilde{Y}$ from the rc2-deck of Y, then Y and $\tilde{Y}$ also have the same rc3-deck, as observed in Remark 2.9. Now T and $\tilde{T}$ have the same rc3-deck, where $\tilde{T}$ arises from $\tilde{Y}$ by adding a trivial piece. The resulting T and $\tilde{T}$ are now Y^+ and $\tilde{Y}^+$, occuring as exceptions in Example 2.8. Otherwise Y is determined by its rc2-deck, which determines T.

Case 5: *T has no trivial r-piece but has an r-piece of size* 2. The only way to obtain a 1-slim rc3-card with a trivial r-piece when T has no trivial r-piece is to have two (or more) r-pieces of size 2 and delete both vertices from one of them and one from another. Hence if T has a 1-slim rc3-card with a trivial r-piece, then we reconstruct T from it by replacing the trivial r-piece with two copies of $\hat{P}_2$.

Hence we may assume that T has no 1-slim rc3-card with a trivial r-piece. Knowing that T has no trivial r-piece, we need to recognize that T has a piece of size 2. Suppose that the rc3-deck also has a reconstruction T° with no r-piece of size at most 2. A 1-slim rc3-card of T° arises only by deleting an r-piece of size 3 and hence has no r-piece of size 2. If T has an r-piece of size 3, then T has a 1-slim card with an r-piece of size 2. Hence in T every r-piece other than one of size 2 has size at least 4. Now a 1-slim rc3-card of T has at most one r-piece of size 3. This means that T° has at most two r-pieces of size 3 and therefore at most two 1-slim rc3-cards. Hence T has at most two leaves outside the r-piece of size 2.

If there is only one 1-slim card, then $T = U(\hat{P}_2, \hat{P}_{n-3})$, and there are only three rc3-

cards. However, there are at least four ways to delete at most three vertices from the r-piece of size 3 in T° (whether it is $\hat{P}_3$ or $\hat{P}_3'$), all leading to rc3-cards. Hence we may assume that there are two 1-slim cards, meaning that T has exactly two leaves outside the smallest piece. Now T has at most nine rc3-cards. If $d^* = 3$, then two r-pieces of size 3 and an additional piece of size at least 4 give T° more than nine rc3-cards. If $d^* = 2$, then T° (and hence also T) has exactly seven vertices. Here $T \in \{U(\hat{P}_2, \hat{P}_4'), U(\hat{P}_2, \hat{P}_4'')$ and $T^\circ \in \{U(\hat{P}_3, \hat{P}_3\}, U(\hat{P}_3, \hat{P}_3'), U(\hat{P}_3', \hat{P}_3')\}$. The possibilities for T have four or five rc3-cards, respectively, while those for T° have four, five, or six, respectively. This yields the pairs $\{U(\hat{P}_2, \hat{P}_4'), U(\hat{P}_3, \hat{P}_3)\}$ and $\{U(\hat{P}_2, \hat{P}_4''), U(\hat{P}_3, \hat{P}_3')\}$ which are exceptions having the same rc3-deck, as described in Example 2.8. In each pair the trees are distinguished by the number of unrooted copies of $S_{2,1,1}$.

Hence we recognize being in Case 5: T has one r-piece of size 2 (and no trivial r-piece). Say that a 1-slim card having an r-piece of size 2 is *defective*. If T has no defective card, then all pieces other than the one of size 2 have size at least 4, and the 1-slim rc3-cards of T are the rc1-cards of the rooted tree T' obtained by deleting the smallest r-piece from T. By Theorem 2.1, we can reconstruct T' and add an r-piece of size 2 to obtain T.

If T has a defective card, then T has a piece of size 3. A defective card contains as r-pieces all the r-pieces of T having size at least 4. It remains to determine the r-pieces of size 3; let there be p that are copies of $\hat{P}_3$ and q that are copies of $\hat{P}_3'$. Defective cards arise by deleting an r-piece of size 3 or by deleting the r-piece of size 2 and one leaf from an r-piece of size 3. Thus each copy of $\hat{P}_3$ loses at least one vertex in two defective cards, while each copy of $\hat{P}_3'$ is diminished in three defective cards. This yields $t = 2p+3q$, where t is the total number of defective cards. On the other hand $s = p+q-1$, where s is the number of pieces of size 3 in each defective card. Since we see s and t from the rc3-deck, we compute p and q.

Case 6: *Every r-piece of T has at least three vertices.* Note the exceptions $U(\hat{P}_3, \hat{P}_3)$ and $U(\hat{P}_3, \hat{P}_3')$, which share rc3-decks with $U(\hat{P}_2, \hat{P}_4')$ and $U(\hat{P}_2, \hat{P}_4'')$, respectively, as discussed in Case 5. If the rc3-deck of T is not from one of these exceptions, then in the remaining case we have determined that all r-pieces of T have size at least 3.

Since $d^* \ge 2$ and every r-piece has size at least 3, every r-piece of T appears as an r-piece of some rc3-card of T. Let b be the maximum size among all r-pieces in rc3-cards of T, and let c be the minimum such size, setting $c = 0$ if T has a 1-slim card. The smallest r-piece(s) of T have size $c+3$. Every card having an r-piece of size c (or 1-slim card if $c = 0$) has d^*-1 r-pieces that are all the r-pieces of T except one of size $c+3$. Call these *trim cards.* (Note that $b \ge c+3$, with equality possible.)

For every trim card, there is one r-piece of T with size $c+3$ that does not appear. If some trim card has an r-piece of size $c+3$, then T has more than one r-piece of size $c+3$, and over all the trim cards we see all the pieces, in particular all the r-pieces of size $c+3$. Choose a trim card C in which an r-piece R of size $c+3$ appears fewer than the maximum number of times, and reconstruct T from C by replacing the smallest r-piece with a copy of R (or adding R as an r-piece if $c = 0$).

In the remaining case, T has exactly one r-piece R of size $c+3$. Each trim card gives us the other r-pieces, with their multiplicities (hence we also know b). The set consisting of the smallest r-piece in each trim card is the rc3-deck of R. Now we can reconstruct R by the induction hypothesis and hence reconstruct T unless R is in the set of exceptions.

Let M be an r-piece of T with b vertices, and let d' be the number of r-pieces of T isomorphic to M. Obtain T' from T by deleting the r-pieces isomorphic to M. The rc3-cards of T' are obtained from the rc3-cards of T with d' r-pieces isomorphic to M by deleting the copies of M. If $d' < d^* - 1$, then T' has at least two pieces, one of which is the exception R and the others of which have at least four vertices. No exceptions in Example 2.8 fit this description, so by the induction hypothesis T' is reconstructible from its rc3-deck, which we have obtained. After reconstructing T', we add the copies of M to obtain T.

Hence we may assume that T consists of $d^* - 1$ r-pieces isomorphic to M and one smaller piece R with s vertices (here $s = c+3$). We know M, s, and b. First suppose that $s \le b-3$. Let L be an rc2-card of M, and let a be the number of copies of L as an rc2-card of M. In every rc3-card of T having $d^* - 2$ r-pieces isomorphic to M and one piece isomorphic to L, the remaining r-piece is an rc1-card of R (smaller than L). Each rc1-card of R arises this way on exactly $a(d^* - 1)$ cards. Hence we obtain the rc1-deck of R. By Theorem 2.1, we can reconstruct R and thus reconstruct T. If $s = b-2$ and $d^* \ge 3$, then we can similarly use an rc1-card L' of M and obtain the rc1-deck of R from cards with $d^* - 3$ r-pieces isomorphic to M and two r-pieces isomorphic to L.

Next suppose $s = b-2$ and $d^* = 2$. Let L be an rc1-card of M, and let a be the number of copies of L as an rc1-card of M. In every rc3-card of T having an r-piece isomorphic to L, the other r-piece is an rc2-card of R (smaller than L). Each rc2-card of R arises this way on exactly a cards. Hence we obtain the rc2-deck of R. By Theorem 2.4, we can reconstruct R and hence T unless we have the common rc2-deck of $\hat{P}_s$ and $\hat{P}'_s$ or of $\hat{Q}_s$ and $\hat{Q}'_s$. To distinguish these, note that an rc3-card of T having pieces of sizes $b-3$ and $b-2$ arises by deleting three vertices from M (in which case the other piece is R) or by deleting two vertices from M and one vertex from R. Hence the number of these cards is $ij + j'$, where i is the number of leaves of R, j is the number of rc2-cards of M, and j' is the number of rc3-cards of M. Since we know M, we know j and j' and can compute i. This distinguishes between $\hat{P}_s$ and $\hat{P}'_s$ and between $\hat{Q}_s$ and $\hat{Q}'_s$ for R.

In the final case, $s = b-1$. Since we know M, we know the rc1-deck $L_1, \ldots, L_m$ of M, where m is the number of leaves of M. Let S be the multiset of rc3-cards of T in which one piece has $b-3$ vertices, one has $b-1$ vertices, and the rest are isomorphic to M. Such cards arise by deleting three vertices from one copy of M or by deleting one vertex from a copy of M and two vertices from R. If for some member of S the piece with $b-1$ vertices is not in the rc1-deck of M, then that piece is R and we complete the reconstruction.

Otherwise, R is an rc1-card of M. Let p be the number of rc3-cards of M, and let q be the number of rc2-cards of R; at present q is unknown. Let k be the multiplicity of a given rc1-card L of M. If $L \neq R$, then the number of members of S in which the piece with $b-1$ vertices is L is $(d^*-1)kq$. If $L = R$, then that number is $(d^*-1)(kq+p)$. Since we know $|S|$ from the deck, and the multiplicities of the various rc1-cards of M sum to m, we can compute $q = \frac{|S|-(d^*-1)p}{(d^*-1)m}$. Since we know the multiplicity of each rc1-card of M, we can now determine which one occurs too often as a piece of a member of S, and that is R. □

In applying Theorem 2.10 to 3-reconstructibility of tree, knowing the number of leaves of a tree and the number of copies of $S_{2,1,1}$ distinguishes the exceptions when a rooted tree is not weakly 3-reconstructible. Fortunately, we will be able to determine these quantities. In the unrooted setting, we proved in [6] that when $n \geq 7$, the $(n-3)$-deck of an n-vertex graph determines its degree list. Also, the number of copies of $S_{2,1,1}$ is determined by the $(n-3)$-deck when $n \geq 8$, by Observation 1.2.

3 Trees with Cost at Most $(n-4)/2$

In this section we use Theorem 2.10 to show that trees in a large class are 3-reconstructible. We consider a "cost" parameter related to the notion of centroids in trees. The term "centroid" has been used in various ways in the literature, but this definition seems most common.

Definition 3.1. The *cost* of a vertex v in a tree T is the maximum number of vertices in a component of $T - v$. A *centroid* of a tree is a vertex with smallest cost. The *cost* of T, which we write as $c(T)$, is the minimum cost among the vertices of T, that is, the cost of a centroid of T.

It is well known that a tree has a unique centroid or has two adjacent centroids, yielding *unicentroidal* or *bicentroidal* trees, respectively. We need this fact in a stronger form. Myrvold [11] heavily used centroids and a more detailed version of this lemma in proving that trees with at least five vertices are 1-reconstructible from only three cards; that is, every such tree has "reconstruction number" 3.

Lemma 3.2. *Every n-vertex tree has either a unique centroid, with cost less than $n/2$, or two adjacent centroids, with cost exactly $n/2$.*

Proof. If a vertex v in a tree T has cost greater than $n/2$, then its neighbor in the large component of $T - v$ has smaller cost. Hence a vertex with smallest cost has cost at most $n/2$. For such a vertex v and any neighbor u, the forest $T - u$ has a component consisting of v plus all the other components of $T - v$, thereby yielding cost at least $n/2$. Furthermore, if v has cost less than $n/2$, then u has cost more than $n/2$. Moving further away from v increases the cost more.

Hence there are at most two centroids, and the cost is at most $n/2$. Furthermore, we have also shown that if the cost is less than $n/2$, then the centroid is unique. □

Definition 3.3. The *pieces* of a unicentroidal tree T having centroid z are the components of $T - z$; when we know T and z, the neighbors of z in the pieces are the *roots* of the pieces. In a bicentroidal tree, the two subtrees obtained by deleting the edge joining the centroids are the *branches* of the tree, and the roots of the branches are the centroids. The *size* of a piece or branch is the number of vertices.

We begin with a short self-contained argument that we can recognize the $(n-3)$-decks of trees.

Lemma 3.4. *If T is an n-vertex tree with $n \ge 7$, then every graph having the same $(n-3)$-deck as T is a tree.*

Proof. The $(n-3)$-deck provides the 2-deck, so every reconstruction has $n-1$ edges. All cards are acyclic, so reconstructions have no cycles of length at most $n-3$. Therefore, a non-tree reconstruction G must be $C_{n-1}+P_1$ or $C_{n-2}+P_2$ or the graph C' consisting of C_{n-2} plus a pendant edge and an isolated vertex.

The numbers of copies of P_{n-3} and $K_{1,3}$ in these alternatives are $(n-1,0)$ or $(n-2,0)$ or $(n,1)$, respectively; we know these values, since $n-3 \ge 4$. If T is a path, then its $(n-3)$-deck has only four copies of P_{n-3}, but $4 < n-2$. Hence we may assume that T has a branch vertex, which puts a copy of $K_{1,3}$ into the 4-deck and requires $G = C'$. However, a tree with only one copy of $K_{1,3}$ contains at most five copies of P_{n-3} (achieved by $S_{n-3,1,1}$), and $5 < n$. □

In studying connected cards in the $(n-\ell)$-deck of an n-vertex tree, it is helpful to know which vertices of the original tree can be centroids of the card. When ℓ is fixed, the centroid of a connected card cannot be very far from the original centroid. Our primary interest will be in the two largest pieces of a tree. The remaining vertices form appendages from the centroid, so we use a botanical term describing appendages on trees.

Definition 3.5. A *j-burl* in a tree T is a vertex v such that there are exactly j vertices in $T - v$ outside the two largest components. The *burl* of the tree T is the set of these outside vertices when v is the centroid of T. Recall that $c(T)$ denotes the cost of T, which is the number of vertices in a largest component of the forest obtained by deleting a centroid of T.

Lemma 3.6. *Let z be a centroid of an n-vertex tree T. Let u be a centroid of a connected card C in the $(n-\ell)$-deck of T. If $n > 2\ell$, then $z \in V(C)$. If $c(T) \le (n-\ell+1)/2$, then u is*

z or a neighbor of z. If $c(T) = (n - \ell + 2)/2$, then u can have distance 2 from z only if their common neighbor has degree 2 in T. If $c(T) = n/2$, then u can be outside the neighborhoods of the centroids of T only if the closer centroid is a j-burl with $j \le (\ell - 4)/2$.

Proof. Omitting z from a connected card requires the vertices of the card to lie in one piece of T, which requires $(n - \ell) \le c(T) \le n/2$ and hence cannot happen when $n > 2\ell$.

Every component of $T - z$ has at most $c(T)$ vertices. Therefore, if u is a vertex outside the closed neighborhood of z, in $T - u$ there is a component with at least $n - c(T) + 1$ vertices. In order to make u a centroid of C, this component must be cut down to at most $(n - \ell)/2$ vertices. Hence $n - c(T) + 1 - \ell \le (n - \ell)/2$, which simplifies to $c(T) \ge (n - \ell + 2)/2$.

When this inequality is violated, there is no such centroid. When it holds with equality, the large component of $T - u$ must have exactly $n - c(T) + 1$ vertices, so u and z must have a common neighbor with degree 2.

Suppose that T is bicentroidal and the centroid u of C has distance at least 2 from the closer centroid z of T, and let z be a j-burl in T. To become a piece of $C - u$, the component of $T - u$ containing z must be trimmed to at most $(n - \ell)/2$ vertices. That component of $T - u$ contains z, the large component of $T - z$, the j-burl at z, and at least one vertex between u and z, so it has at least $2 + j + n/2$ vertices and can only lose ℓ in the card C. Hence $2 + j + n/2 - \ell \le (n - \ell)/2$, which simplifies to $j \le (\ell - 4)/2$. □

Corollary 3.7. *Let T be an n-vertex tree. If $c(T) \ne (n - 1)/2$, then every centroid in every connected $(n-3)$-card C of T is or has a neighbor that is a centroid of T. If $c(T) = (n-1)/2$ and C has a centroid u that is not a neighbor of the centroid z of T, then C is bicentroidal and the common neighbor of u and z has degree 2 in T and is the other centroid of C.*

Proof. Let z be a centroid of T, and set $\ell = 3$ in Lemma 3.6. Now $c(T) \le (n - 2)/2$ keeps the centroid of C within distance 1 of z, while $c(T) = (n - 1)/2$ allows it to move one step farther when the common neighbor has degree 2 (the card is then bicentroidal).

When T is bicentroidal, there is no j-burl when j is negative, so the centroid of C must be a neighbor of a centroid of T. □

Our arguments for reconstruction of a tree T from the $(n-3)$-deck are based on the value of the cost $c(T)$. Therefore, we need lemmas that enable us to recognize this value from the deck. For a deck $\mathcal{D}$, let $c(\mathcal{D})$ denote the maximum cost among connected cards in $\mathcal{D}$.

Lemma 3.8. *The $(n - \ell)$-deck $\mathcal{D}$ of an n-vertex tree T satisfies*

$$c(\mathcal{D}) = \begin{cases} c(T) & \text{if } c(T) \le (n - \ell)/2, \\ \lfloor (n - \ell)/2 \rfloor & \text{if } c(T) > (n - \ell)/2. \end{cases}$$

Also, if $c(T) \le (n - \ell)/2$, then the centroid of T is a centroid in every connected card.

Proof. Let z be a centroid of T, and let X be a largest piece of T. By Lemma 3.6, z appears in every connected card C in $\mathcal{D}$.

First suppose $c(T) \le (n-\ell)/2$. Components of $C - z$ are contained in components of $T-z$ and hence have at most $c(T)$ vertices. Since $c(T) \le (n-\ell)/2 = |V(C)|/2$, we conclude that z is a centroid of C, and $c(C) \le (n-\ell)/2$. Furthermore, if C arises by deleting ℓ vertices outside X, then X is still a piece, so $c(C) = c(T)$. Hence $c(\mathcal{D}) = c(T)$.

Now suppose $c(T) > (n-\ell)/2$. Every connected card C satisfies $c(C) \le (n-\ell)/2$, so it suffices to construct a card C with cost $\lfloor (n-\ell)/2 \rfloor$. Delete successive leaves of X until exactly $\lfloor (n-\ell)/2 \rfloor$ vertices of X remain. Complete the card by successively deleting other leaves outside X. The number of vertices remaining outside X is $\lceil (n-\ell)/2 \rceil$, since the card has $n-\ell$ vertices. Thus z is the unique centroid if $n-\ell$ is odd, while both z and its neighbor x in X are centroids if $n-\ell$ is even. Hence $c(C) = \lfloor (n-\ell)/2 \rfloor$, and $c(\mathcal{D}) = \lfloor (n-\ell)2 \rfloor$. $\square$

Theorem 3.9. *For $n \ge 7$, trees with n vertices and cost at most $(n-5)/2$ are 3-reconstructible.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such a tree T. By Lemma 3.8, we recognize that T is in this family: every connected card has cost at most $(n-5)/2$. Lemma 3.8 also implies that every connected card has the centroid z of T as its unique centroid. With z distinguished in each connected card, the connected cards form the rc3-deck of T as a rooted tree with root z. In addition, $c(T) \le (n-5)/2$ requires $d_T(z) \ge 3$. The rooted trees in Example 2.8 that have root-degree 3 and are not reconstructible from their rc3-decks have six vertices. Hence by Theorem 2.10 we can reconstruct T from the deck. $\square$

For general ℓ, Lemma 3.8 will lead to ℓ-reconstructibility of n-vertex trees having cost less than $\lfloor (n-\ell)/2 \rfloor$ if rooted trees are proved to be weakly ℓ-reconstructible, since the centroid of T is can be determined in every connected card and used as a root. This also needs a suitable threshold for n and avoiding the exceptions to reconstruction of rooted trees.

Recognition of trees with cost at most $(n-5)/2$ means that we can also recognize trees with cost at least $(n-4)/2$. Our approach is to successively recognizing a subfamily of the remaining family of tree and show that the family is weakly reconstructible. For tree with cost $(n-4)/2$, the reconstruction is similar to Theorem 3.9 once we prove that the family is recognizable. We will consider not only the order of the largest piece, but also that of the next largest piece. We also consider special cards with two components.

Definition 3.10. For a unicentroidal tree T, let $c'(T)$ denote the size of the second largest piece in T, which may equal $c(T)$; we call $c'(T)$ the *subcost* of T. An $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$*-card* is a connected card such that for a centroid u the two largest components of $T-u$ have $(n-a)/2$ and $(n-b)/2$ vertices, with $a \le b$. For a deck $\mathcal{D}$, let $c'(\mathcal{D})$ denote the maximum subcost among the connected cards with maximum cost.

A card in the $(n-\ell)$-deck of an n-vertex graph is *balanced* if it consists of two components having $\lceil (n-\ell)/2 \rceil$ and $\lfloor (n-\ell)/2 \rfloor$ vertices, respectively.

When discussing a unicentroidal tree T, we let z denote the centroid, with neighbor x in a largest piece X and neighbor x' in a second largest piece X'.

By Lemma 3.8, the cost of a tree T is determined by the cost of its $(n-\ell)$-deck when $c(T) < \lfloor (n-\ell)/2 \rfloor$. When $\ell = 3$ and n is even, $c(\mathcal{D}) = (n-4)/2$ for $c(T) \in \{(n-4)/2, (n-2)/2, n/2\}$. Our next task is to recognize when $c(T) = (n-4)/2$.

Lemma 3.11. *If $\mathcal{D}$ is the $(n-3)$-deck of an n-vertex tree T, where n is even and $n \geq 20$, then $c(T) = (n-4)/2$ if and only if*

(a) $c(\mathcal{D}) = (n-4)/2$,

(b) $\mathcal{D}$ has no balanced cards, and

(c1) some connected card has cost $(n-10)/2$, or

(c2) $c'(\mathcal{D}) = (n-2j)/2$ for some $j \in \{2,3,4\}$, and T has a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card. Also, when $j = 4$ there is a $\left(\frac{n-6}{2}, \frac{n-8}{2}\right)$-card, and when $j = 3$ there is a $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card and a $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card.

Proof. By Lemma 3.8, we may assume $c(T) \in \{(n-4)/2, (n-2)/2, n/2\}$. In each case $c(\mathcal{D}) = (n-4)/2$, by Lemma 3.8.

Case 1: $c(T) = (n-4)/2$. Any balanced card has components with $(n-2)/2$ and $(n-4)/2$ vertices, since $n-3$ is odd. Thus deleting any single vertex from T leaves a component with at least $(n-2)/2$ vertices, contradicting $c(T) = (n-4)/2$. Hence (b) holds.

Define j by $c'(T) = (n-2j)/2$; note that $j \geq 2$. Every piece of T has at most $(n-4)/2$ vertices. By Lemma 3.8, the centroid of every connected card is z, the centroid of T. In any connected card obtained by deleting three vertices of X, there remain $(n-10)/2$ vertices of X and the entire second largest piece X'. If $j \geq 5$, then such a card has cost $(n-10)/2$.

Hence we may assume $j \in \{2,3,4\}$. The card described is a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card unless besides X and X' there is another piece in T with at least $(n-8)/2$ vertices. This requires $n \geq 1 + \frac{n-4}{2} + \frac{n-2j}{2} + \frac{n-8}{2}$, which simplifies to $2j \geq n-10$. Since $2j \leq 8$, we obtain the $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card unless $n \leq 18$. (When $n = 18$, for example, the spider $S_{7,5,5}$ has cost $(n-4)/2$, no connected card with cost 4 for (c1), and no $\left(\frac{n-8}{2}, \frac{n-10}{2}\right)$-card for (c2).)

For $c'(\mathcal{D}) = (n-2j)/2$, we need a $\left(\frac{n-4}{2}, \frac{n-2j}{2}\right)$-card. We keep the two largest pieces to make such a card by deleting three vertices from the burl. These are available, since there are $1 + \frac{n-2j}{2} + \frac{n-4}{2}$ vertices outside the burl, leaving $j+1$ vertices in the burl. Also $c'(\mathcal{D}) \leq c'(T)$ when each connected card has centroid z, so $c'(\mathcal{D}) = c'(T) = (n-2j)/2$.

Finally, deleting one vertex from X and two from the burl yields a $\left(\frac{n-6}{2}, \frac{n-2j}{2}\right)$-card. When $j = 3$, deleting two from X and one from X' yields an $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card.

Case 2: $c(T) = (n-2)/2$. Assume (a), (b), and (c). If $c'(T) \geq (n-4)/2$, then T has a balanced card, contradicting (b). Hence $c'(T) \leq (n-6)/2$.

Deleting ℓ vertices reduces the cost by at most ℓ. Thus no card of T has cost at most $(n-10)/2$, and (c1) fails. Hence (c2) holds, so $c'(\mathcal{D}) = (n-2j)/2$ for some $j \in \{2,3,4\}$, and with (a) there is a $\left(\frac{n-4}{2}, \frac{n-2j}{2}\right)$-card C.

The centroid of C is in $\{x, z, x'\}$, by Corollary 3.7. It cannot be x', since the component of $T - x'$ containing z has at least $(n+6)/2$ vertices, which cannot be cut to $(n-4)/2$ by deleting three vertices. If the centroid is z, then the second largest piece in T has at least $(n-2j)/2$ vertices. Since the largest piece in T has $(n-2)/2$ vertices, forming a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card with centroid z requires deleting at least four vertices.

Hence the centroid of C is x. The component of $T-x$ containing z has $(n+2)/2$ vertices. Since C is a $\left(\frac{n-4}{2}, \frac{n-2j}{2}\right)$-card, C must arise by deleting three vertices from that component of $T - x$. Thus X contains a piece Y of C with $(n-2j)/2$ vertices.

Now consider the required $\left(\frac{n-6}{2}, \frac{n-2j}{2}\right)$-card C' if $j \in \{3, 4\}$. The centroid of C' cannot be x, since the two biggest components of $T - x$ would together have to lose at least four vertices. Hence the centroid of C' is z. Now, as earlier when we studied C, there must be pieces as large as $(n-6)/2$ and $(n-2j)/2$ in T, so again $c'(T) \geq (n-2j)/2$. Again we must delete at least four vertices from the two largest pieces of T to obtain a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card.

Hence $j = 2$, and C is a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card. We have shown that the centroid of C is x, with a piece Y contained in X. Since Y uses all of X except the one vertex x, we have $d_T(x) = 2$. Now deleting x and two vertices from X' yields a balanced card, contradicting (b).

Case 3: $c(T) = n/2$. Here T is bicentroidal, with centroids z and z'. Let X be a second largest component of $T - z$, with x its neighbor of z; similarly define X' and x' from $T - z'$. By (b), T has no balanced cards, which requires $d_T(z), d_T(z') \geq 3$ and $|V(X)|, |V(X')| \leq (n-6)/2$, so $c'(T) \leq (n-6)/2$. Since $c(T) = n/2$, every connected card has cost at least $(n-6)/2$, so again (c1) cannot hold. Also, there is no $\left(\frac{n-8}{2}, \frac{n-10}{2}\right)$-card and no $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card, which are required when j in (c2) is 4 or 3, respectively. Hence $j = 2$ and $c'(\mathcal{D}) = (n-4)/2$.

Thus we have a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card C. Since $c'(T) \leq (n-6)/2$, moving away from z or z' to a vertex w allows only the component of $T - w$ containing z and z' to have more than $(n-6)/2$ vertices. Hence C cannot exist. $\square$

Theorem 3.12. *For even n with $n \geq 20$, trees with n vertices and cost $(n-4)/2$ are 3-reconstructible.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such a tree T. By Lemmas 3.8 and 3.11, we recognize from $\mathcal{D}$ that T is in this family. Lemma 3.8 also implies that every connected card has the centroid z of T as its unique centroid ($n-3$ is odd). Hence again the connected cards form the rc3-deck of T with root z, and again $d_T(z) \geq 3$. By the same proof as in Theorem 3.9, we can reconstruct T from the deck. $\square$

A tree T with higher cost may have connected cards whose centroids are not the centroid of T. Nevertheless, often we can find a subset of the connected cards whose centroids are identified as a particular vertex of T, such that appropriate subtrees in these cards form the rc3-deck of a fixed rooted subtree of T. We can then apply the technique used above.

Lemma 3.13. *Let T be an n-vertex tree, and let $\mathcal{D}$ be the $(n-3)$-deck of T. If a vertex v in T and a subset $\mathcal{D}'$ of $\mathcal{D}$ can be identified such that $\mathcal{D}'$ is the multiset of connected cards arising by deleting three vertices from one component H of $T-v$, and in each card of $\mathcal{D}'$ we know v and which neighbor of v is in H, then T is 3-reconstructible.*

Proof. In each card in $\mathcal{D}'$ we recognize the subgraph $T-V(H)$ by deleting the piece containing the neighbor of v in H. Over all cards in $\mathcal{D}'$, the vertices that belong to H provide the rc3-deck of H rooted at its vertex neighboring v.

Since we know the number of leaves of T and the number of leaves of T outside H, we know the number of leaves of T in H. Hence Theorem 2.10 allows us to reconstruct H unless $H \in \{\hat{P}'_k, \hat{P}''_k, \hat{Q}_k, \hat{Q}''_k\}$, where $k = |V(H)|$. By Observation 1.2, we know the number of copies of $S_{2,1,1}$ in T, and from the cards in $\mathcal{D}'$ we know the number of copies of $S_{2,1,1}$ using at least one vertex outside H. We therefore know the number of copies of $S_{2,1,1}$ contained in H. As noted in Theorem 2.10, we can reconstruct H and T. $\square$

Lemma 3.13 is the workhorse in proving the analogue of Theorem 3.12 for trees with higher cost. Most of the reconstruction arguments depend on knowing the cost and sometimes also the subcost of the reconstructions, so we must first recognize these families, which we do successively for increasing cost (sometimes also by subcost), ending with bicentroidal trees. Each step assumes that the earlier families are recognizable (and $n \geq 20$ suffices for all). Complete details are at https://faculty.math.illinois.edu/∼west/pubs/tree3rec.pdf.

The next section presents two pieces of the argument. These are relatively self-contained 3-reconstructibility arguments for high-cost trees under certain structural conditions. The second relies on the first and on Lemma 3.13.

4 High-Cost Trees with Special Structure

Here we consider two structural conditions that permit 3-reconstructibility. They hold in n-vertex trees only when the cost is at least $(n-2)/2$.

Definition 4.1. A *j-vertex* or *j^+-vertex* is a vertex with degree j or at least j, respectively. Similarly, a *j-neighbor* or *j^+-neighbor* is a neighbor that is a j-vertex or j^+-vertex, respectively. A *full vertex* is a 3^+-vertex that is not a 1-burl. A *caterpillar* is a tree having a single path incident with all edges; equivalently, it is a tree not containing $S_{2,2,2}$ as a subgraph.

Theorem 4.2. *For $n \geq 10$, every n-vertex caterpillar having maximum degree at most 3 is 3-reconstructible.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such an n-vertex tree. When $n \geq 2\ell + 1$, acyclic graphs are ℓ-recognizable ([7]), and the number of edges is known from the 2-deck, so all reconstructions are trees. Also we know the degree list, since $n - \ell \geq 5$ and these trees have no 4^+-vertex. Finally, since $n - \ell \geq 7$ we see that a tree with the given deck has no copy of $S_{2,2,2}$, and hence it is a caterpillar. Thus this family of n-vertex graphs is ℓ-recognizable.

Let T be a reconstruction of $\mathcal{D}$. We know the number s of 3-vertices in T and hence the number of vertices in a longest path in T; it is just $n - s$. Let the *end-distance* of a 3-vertex be its minimum distance from an endpoint of a longest path.

The short argument we give below handles all cases with $s \geq 5$. This argument is not valid when $s \geq 4$. However, those cases can also be handled by ad hoc arguments. *We omit the details of those arguments.*

Assume $s \geq 5$. Let $\mathcal{D}'$ be the set of connected cards having a path with $n - s$ vertices; such cards arise by deleting leaf neighbors of three distinct 3-vertices. Let v and w be the 3-vertices closest to the two ends of a longest path in T. The cards in $\mathcal{D}'$ having 3-vertices farthest apart have v and w as 3-vertices, telling us their end-distances. If they have distinct end-distances, then the various cards in $\mathcal{D}'$ having a 3-vertex closest to an endpoint of the path P with $n - s$ vertices give us the positions of all the other 3-vertices, reconstructing T.

If v and w have the same end-distance r, then consider the cards in $\mathcal{D}'$ where all 3-vertices have end-distance more than r. These arise by deleting the leaf neighbors of v, w, and one other 3-vertex; still at least two 3-vertices remain. Among these, the cards having 3-vertices farthest apart fix the end-distances of the second 3-vertex from each end.

Let q be the minimum end-distance among these two 3-vertices. If they both have end-distance q, then look at one card in $\mathcal{D}'$ having one 3-vertex with end-distance q and no 3-vertex with end-distance r. This shows us the remaining 3-vertices, and we know exactly where to add the three missing leaves. If only one of these two has end-distance q, then consider the cards in $\mathcal{D}'$ where the least end-distance of 3-vertices is q. Such cards are missing the leaf neighbors of v, w, and one other 3-vertex not having end-distance q. Since $s \geq 5$, over all such cards we obtain the positions of the other 3-vertices, since the 3-vertex with end-distance q distinguishes the two ends of P. $\square$

Note that the caterpillars in Theorem 4.2 and n-vertex trees having no full vertex as a centroid all have cost at least $(n-2)/2$.

Theorem 4.3. *Let $\mathcal{D}$ be the $(n-3)$-deck of an n-vertex tree T. If T is unicentroidal with no unicentroidal card having a full vertex as centroid, or if T is bicentroidal with both centroids having degree 2, then T is 3-reconstructible, without knowing in advance the cost or subcost.*

Proof. Let T be a tree in this family, with $(n-3)$-deck $\mathcal{D}$. If T has no full vertex at all, then T is a caterpillar with maximum degree at most 3, which by Theorem 4.2 is reconstructible from $\mathcal{D}$. Hence we may assume that T has a full vertex.

Any centroid z of T is a centroid with degree $d_T(z)$ in some connected card. Hence z must be a 1-burl or a 2-vertex. Thus T is a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree, a $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-tree, a $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-tree, or a bicentroidal tree (with no full vertex as centroid).

We consider first which vertices can be centroids of cards of T. When T is unicentroidal, label z, x, X, x', X' as usual. In a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree, deleting one leaf from each nontrivial piece plus any leaf from what remains yields a $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card or $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card with z as centroid. Deleting three vertices outside one of the nontrivial pieces yields a card with the root of that piece (x or x') as the centroid.

In a $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-tree, deleting two vertices from one piece and one from the other yields a bicentroidal card with z as a centroid. Deleting three outside X yields a unicentroidal card with centroid x or, if x is a 2-vertex, a bicentroidal tree with centroids x and its neighbor in X (symmetrically for X'). Similarly for a $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-tree, except that deleting one vertex from X' and two from X yields a $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-tree with centroid z, while deleting three vertices outside X' yields a bicentroidal tree with centroids z and x'.

Consider a bicentroidal tree with centroids z and z' as the roots of branches Y and Y', and $\langle x, z, z', x' \rangle$ being a path of nonleaf vertices. By symmetry, we describe only the cards where a majority of the deleted vertices are outside Y, since we are assuming $d_T(z) = d_T(z') = 2$. Deleting three vertices outside Y yields a card with centroid x, while deleting one vertex of Y and two outside Y yields a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card with centroid z.

Let v be a full vertex in T. In a card having v as a full vertex, we compute the distance from v to the centroid(s), taking the average of the two distances when the card is bicentroidal. In all types of trees above, this distance is minimized when all three deleted vertices are outside the piece or branch containing v. Otherwise, this distance is larger.

Let an *optimal card* be a connected card minimizing r, where in a unicentroidal tree r is the distance from the centroid to a closest full vertex, and in a bicentroidal card r is the minimum average distance from the two centroids to a single full vertex.

If no piece (or branch) occurs in all optimal cards, then each nontrivial piece of T has a full vertex at the same distance from the centroid in T. If these are v and v', then some optimal cards have v as the closest full vertex (distance r) from their centroid, while other cards have v' as this vertex (no card has both). Each optimal card contains one of the two resulting pieces, so over all optimal cards we obtain both. In each optimal card we see the centroid(s) of T and whether they have leaf neighbors, and in all cases we can determine T.

If in all optimal cards the piece containing a full vertex closest to the centroid is the same, then either it arises from the one full vertex in T or from two different full vertices in T. In the latter case, the non-constant pieces of the optimal cards provide two copies of the rc3-deck of the constant piece. In the former case, they give the rc3-deck of the piece of T

not containing the constant piece, and we reconstruct by Lemma 3.13.

When the neighbor x of a centroid z is the centroid in the optimal cards, the rc3-deck of the non-constant piece tells us whether z is a 1-burl or a 2-vertex, which distinguishes between $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-trees and bicentroidal trees in this family. The same notion distinguishes between $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-trees and $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-trees in this family. $\square$

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References

[1] J. A. Bondy and R. L. Hemminger, Graph reconstruction—a survey, *J. Graph Theory* 1 (1977), 227–268.

[2] W. B. Giles, Reconstructing trees from two-point deleted subtrees, *Discrete Math.* 15 (1976), 325–332.

[3] C. Groenland, T. Johnston, A. Scott, and J. Tan, Reconstructing trees from small cards, arXiv:2103.13359.

[4] P. J. Kelly, On isometric transformations, PhD Thesis, University of Wisconsin-Madison, 1942.

[5] P. J. Kelly, A congruence theorem for trees, *Pacific J. Math.* 7 (1957), 961–968.

[6] A. V. Kostochka, M. Nahvi, D. B. West, and D. Zirlin, Degree lists and connectedness are 3-reconstructible for graphs with at least seven vertices, *Graphs and Combinatorics* 36 (2020), 491–501.

[7] A. V. Kostochka, M. Nahvi, D. B. West, and D. Zirlin, Acyclic graphs with at least $2\ell + 1$ vertices are ℓ-recognizable, (submitted).

[8] A. V. Kostochka and D. B. West, On reconstruction of graphs from the multiset of subgraphs obtained by deleting ℓ vertices, *IEEE Transactions on Information Theory*, (published on-line March, 2020, DOI: 10.1109/TIT.2020.2983678).

[9] B. Manvel, On reconstruction of graphs, in *The Many Facets of Graph Theory* (Proc. Conf. Western Mich. Univ., Kalamazoo, Mich., 1968), (Springer, 1969), 207–214.

[10] B. Manvel, Some basic observations on Kelly's conjecture for graphs, *Discrete Math.* 8 (1974), 181–185.

[11] W. J. Myrvold, The ally-reconstruction number of a tree with five or more vertices is three, *J. Graph Theory* 14 (1990), 149–166.

[12] V. Nýdl, A note on reconstructing of finite trees from small subtrees, *Acta Univ. Carol. Math. Phys.* 31 (1990), 71–74.

[13] V. Nýdl, Finite undirected graphs which are not reconstructible from their large cardinality subgraphs, *Discrete Math.* 108 (1992), 373–377.

[14] V. Nýdl, Graph reconstruction from subgraphs, *Discrete Math.* 235 (2001), 335–341.

[15] H. Spinoza and D. B. West, Reconstruction from the deck of k-vertex induced subgraphs, *J. Graph Theory* 90 (2019), 497–522.

[16] S. M. Ulam, A collection of mathematical problems, *Interscience Tracts in Pure and Applied Mathematics* 8 (Interscience Publishers, 1960).