The Game of Revolutionaries and Spies

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slides available on DBW preprint page

Joint work with
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The Model

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Invented by Beck, 1990s.
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$G$ is spy-good: $\sigma(G, m, r) = \lfloor r/m \rfloor$ for all $r, m$.

$G$ is spy-bad for particular $(r, m)$: $\sigma(G, m, r) = r - m + 1$. 
The Plan

• Some graphs are spy-good: \([r/m]\)
Trees, dominated graphs, webbed trees, unicyclic graphs (almost: \([r/m]\) spies suffice).
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  Chordal or bipartite examples, hypercubes (and product graphs), random graphs, king’s-move grids (almost?) (Howard–Smyth [2012])
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- Some graphs are **spy-good**: $[r/m]$
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- Some graphs are **in between**: $cr/m$
  Complete multipartite (good upper and lower bounds). Complete bipartite (exact answers for $m \in \{2, 3\}$).
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But what can one say **in general**?
Spy-Good Graphs

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**Def.** webbed tree - a graph $G$ with a rooted spanning tree $T$ where every edge of $G - E(T)$ joins siblings in $T$. 

![Diagram of a webbed tree](image-url)
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**Thm.** Webbed trees are spy-good.
Dominating vertex $z$

**Thm.** Dominating Vertex $\implies \sigma(G,m,r) = \lfloor r/m \rfloor$. 
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**Pf.** stable position - each vertex $\nu$ other than $z$ has $\lfloor r(\nu)/m \rfloor$ spies, where $r(\nu) = \#\text{revs at } \nu$. 

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Spies end initial round stable.

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\[\begin{array}{ccc}
S & R & R \\
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\end{array}\]
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**Dominating vertex** \( z \)

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**Idea:** Restore stability after each round using matching in a bipartite graph.
Restoring Stability

$m = 2$

$X = \text{previous } m\text{-sets of revs at vertices other than } z.$

$X' = \text{previous excess spies at } z.$
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\[
\begin{align*}
X & \quad x_1 \quad x_2 \quad x_3 \quad x_4 \\
X' & \quad x_3 & \quad x_4 \\
\text{rev moves} & \quad \text{all edges} \\
Y & \quad y_1 \quad y_2 \quad y_3 \quad y_4 \\
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\end{align*}
\]
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Hall’s Condition: For \( T \subseteq Y \), the \( m|T| \) revs in these meetings came from \( N(T) \cap X \) or from no meeting in \( X \), so \( m|T| \leq m|N(T) \cap X| + r - m|X| \).
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Hall’s Condition: For \( T \subseteq Y \), the \( m|S| \) revs in these meetings came from \( N(T) \cap X \) or from no meeting in \( X \), so \( m|T| \leq m|N(T) \cap X| + r - m|X| \). We compute

\[
|N(T)| = |N(T) \cap X| + |X'| \geq |T| - (\lfloor r/m \rfloor - |X|) + |X'| = |T|.
\]
Webbed Tree

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**Pf.**  $r(\nu) = \#\text{revs now at } \nu$.  $C(\nu) = \text{children of } \nu$.  $s(\nu) = \#\text{spies now at } \nu$.  $D(\nu) = \nu$ plus descendants.
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\[ w(\nu) = \sum_{u \in D(\nu)} r(u). \]
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$$w(\nu) = \sum_{u \in D(\nu)} r(u).$$

**Spy Rule:**

$$s(\nu) = \left\lfloor \frac{w(\nu)}{m} \right\rfloor - \sum_{x \in C(\nu)} \left\lfloor \frac{w(x)}{m} \right\rfloor \text{ at each vertex } \nu.$$
**Thm.**  $G$ is a webbed tree $\Rightarrow \sigma(G, m, r) = \lfloor r/m \rfloor$.

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**Rule** $\Rightarrow$ every meeting is guarded.
Webbed Tree

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**Rule** $\Rightarrow$ every meeting is guarded.

Since $\sum_{u \in D(\nu)} s(u) = \left\lfloor \frac{w(\nu)}{m} \right\rfloor$, **Rule** works in first round.
Theorem. $G$ is a webbed tree $\implies \sigma(G, m, r) = \lfloor r/m \rfloor$.

Proof. $r(\nu) = \#\text{revs now at } \nu$. $C(\nu) = \text{children of } \nu$. $s(\nu) = \#\text{spies now at } \nu$. $D(\nu) = \nu$ plus descendants.

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Idea: $\nu$ dominates the subgraph $G(\nu)$ induced by $\{\nu\} \cup C(\nu)$. Spies play on these subgraphs independently to reestablish the Rule.
Split into Subgames (sketch)
The $s(\nu)$ spies at $\nu$ split into $\hat{s}(\nu)$ playing in $G(\nu)$ and $\hat{\hat{s}}(\nu)$ playing in the parent’s graph. Let

$$\hat{s}(\nu) = \left[ \frac{w^*(\nu)}{m} \right] - \sum_{x \in C(\nu)} \left[ \frac{w(x)}{m} \right]$$

and

$$\hat{\hat{s}}(\nu) = \left[ \frac{w(\nu)}{m} \right] - \left[ \frac{w^*(\nu)}{m} \right].$$

Here $w^*(\nu) = w(\nu) - \#\text{revs counted by } w(\nu)$ that are in the parent’s graph after the revs next move.
Split into Subgames (sketch)

\[ \hat{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor \quad \text{and} \quad \hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor. \]

With an imagined split of revs into the subgames, \( \hat{s}(v) \) and \( \hat{s}(v) \) yield stable positions in the subgames.
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The spies can respond to those moves in each subgame to restore stability.
Split into Subgames (sketch)

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The actual moves by revs can be performed by the imagined distribution of revs.

The spies can respond to those moves in each subgame to restore stability.

The resulting new spy distributions restore the Rule:

\[ s'(\nu) = \hat{s}'(\nu) + \hat{\hat{s}}'(\nu) = \left[ \frac{w'(\nu)}{m} \right] - \sum_{x \in C(\nu)} \left[ \frac{w'(x)}{m} \right]. \]
Lem. If $G$ is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$. 
Cycles and Unicyclic Graphs

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**Pf.** Extra revs can’t help spies: may assume $r = sm$. 
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Positions of $m$th revs don’t move by more than one vertex; spies can follow to maintain the condition.
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**Thm.** If $G$ is unicyclic, then $\sigma(G, m, r) \leq \lceil r/m \rceil$. 
Cycles and Unicyclic Graphs

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**Pf. Idea:** Adding **one** spy and $m$ revs at any vertex of $C$ (or removing them) preserves the "cycle condition".
Lem. If $G$ is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

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**Pf. Idea:** Adding one spy and $m$ revs at any vertex of $C$ (or removing them) preserves the "cycle condition". May assume $r = sm$ and all revs start on the cycle. Maintain the cycle condition by keeping "fake" revs at a cycle vertex until an attached tree has enough revs to demand a spy according to the tree strategy.
Spy-Bad Graphs

**Def.** Given $r, m$, graph $G$ is spy-bad if $r - m$ spies lose.
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Spies occupy at most $r - m$ vertices of the clique.
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Revs initially occupy the vertices of the clique.
Spies occupy at most $r - m$ vertices of the clique.
Some $m$ uncovered revs can meet on the first round, unreachable by spies.
Domination Number

**Cor.** \( \sigma(G, m, r) \leq \gamma(G) \lfloor r/m \rfloor. \)
Domination Number

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$\therefore s \geq (r - t) + \frac{1}{3} \binom{t}{2}$. If $s \leq r - 2$, then $t \in \{3, 4\}$. 

![Diagram of hypercube with marked vertices and edges]
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If \( s \leq r - 2 \), then \( t \in \{3, 4\} \).

\( t = 4 \) leaves six threats at doubles, not reachable by two triples (two triangles don’t cover \( E(K_4) \)).
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A spy from some $j$ with $4 \leq j \leq r$ must move to guard $\emptyset$.
No spy can now reach a neighbor of $\{3, j\}$.
Next, revs at 3 and $j$ will move to $\{3, j\}$ and win.
Smaller dimensions

When $d \geq r$, revs beat $r - 2$ spies on $Q_d$ when $m = 2$. On smaller hypercubes, revs do almost as well.
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**Thm.** If $d < r < 2^{d/d^7}$, then $\sigma(Q_d, 2, r) \geq (d - 1) \lfloor r/d \rfloor$. 
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$\therefore$ revs win against fewer than $(d - 1)[r/d]$ spies.

Upper bounds on $\sigma(Q_d, 2, r)$ are unknown for $r > d$. (We think two spies beat four revs on $Q_3$.)
Larger Meeting Size on $Q_d$

**Idea:** Revs start at $r$ vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight $m$ after $m - 1$ steps. How many spies are needed to cover the threats?
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• Nowakowski-Winkler [1983] used retracts in the classical cop-and-robber game.
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For a spy move $u \rightarrow v$ in $G$, revs imagine $f(u) \rightarrow f(v)$ in $H$.

When revs win at $w$ in $H$, since no simulated spy is at $w$ and $f(w) = w$, the revs also win the real game then. ■
An Uncoverable Threat

**Lem.** If $u \in V(Q_d)$ has weight $m$, then a spy at $v$ is within distance $m - 1$ of $u$ if and only if $|u \cap v| \geq \frac{|v|+1}{2}$.
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True for $\alpha = .3247$ and $p = .0795$. 
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P[|I \cap v| < \frac{|v|+1}{2}] \geq (1 - p)^2(1 + 2p) = q.
\]

By FKG Inequality, \( P[\text{all } v \in S \text{ fail}] \geq q^t = e^{t \ln q} \).

For \( m \leq \alpha tp \) with \( \alpha < 1 \), Chernoff Bound yields

\[
P[|I| < m] = P[|I| - tp < m - tp] \leq e^{-(1-\alpha)^2 tp/2}.
\]

The desired property \( P[\text{all } v \text{ fail}] > P[|I| < m] \) holds when \( \ln[(1 - p)^2(1 + 2p)] > -(1 - \alpha)^2 p/2 \).

True for \( \alpha = .3247 \) and \( p = .0795 \).

Now \( t \geq \frac{m}{\alpha p} \geq 38.73m \) yields the conclusion. \( \blacksquare \)
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**Cor.** If $G$ is a cartesian product of $d$ nontrivial graphs, and revs win $RS(Q_d, m, r, s)$, then revs win $RS(G, m, r, s)$. 
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Better for \( m \leq 52 \). Perhaps \( \sigma(Q_d, m, r) \approx r - 2m \).
King’s-move Grid (Howard–Smyth)
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\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
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**Pf.** A group of 8 revs can beat 5 spies (clever!).
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On the first round, the revs from $T$ meet at the special vertex $x$ and win.
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The Usefulness of Stable Positions

**Def.** A position is **stable** if (1) all meetings are covered, and (2) $\hat{s}_{N[\nu]} \geq \hat{r}/m$ for all $\nu \in V(G)$.
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We seek a matching in \( H \) to cover \( Y \).
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If \( T \subseteq Y \), then \( |T| \leq b + \frac{\hat{r}}{m} \), where \( b = \# \text{vertices in } N_G[T] \) hosting meetings before the round.
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For any $y \in T$, we get $|N_H(T)| \geq b + \hat{s}_{N[y]} \geq b + \frac{\hat{r}}{m} \geq |T|$.
Restoring Stability

**Lem.** If $G$ is $q$-common with $n$ vertices, $\varepsilon > 0$, and a position in $RS(G, m, r, s)$ has (1) all meetings covered, (2) $\hat{s} \geq \frac{1+\varepsilon}{q} \frac{\hat{r}}{m}$, and (3) $\hat{s} \geq \frac{\ln n}{2[1-(1+\varepsilon)^{-1}]^2 q^2}$, then the free spies can move to create a stable position.
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**Pf.** Each free spy moves to a uniformly random nbr. With positive probability, the resulting position is stable. Let $X_\nu$ be the resulting number of free spies in $N[\nu]$. $G$ is $q$-common $\Rightarrow$ $X_\nu$ is sum of $\hat{s}$ trials $P[\text{success}] \geq q$. Chernoff Bound $\Rightarrow P[X_\nu - E[X_\nu] < -\alpha] < e^{-2\alpha^2/\hat{s}}$. 
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**Pf.** Each free spy moves to a uniformly random nbr. With positive probability, the resulting position is stable. Let $X_\nu$ be the resulting number of free spies in $N[\nu]$. $G$ is $q$-common $\Rightarrow$ $X_\nu$ is sum of $\hat{s}$ trials $\mathbb{P}[\text{success}] \geq q$. Chernoff Bound $\Rightarrow$ $\mathbb{P}[X_\nu - \mathbb{E}[X_\nu] < -a] < e^{-2a^2/\hat{s}}$. With $\mathbb{E}[X_\nu] \geq q \hat{s}$ and $a = (1 - \frac{1}{1+\epsilon})q \hat{s}$, we obtain $\mathbb{P}[X_\nu < \frac{1}{1+\epsilon} q \hat{s}] < e^{-2(1-\frac{1}{1+\epsilon})^2 q^2 \hat{s}} \leq e^{-\ln n} = \frac{1}{n}$. 
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$\therefore$, with positive prob. each $N[\nu]$ receives at least $\frac{1}{1+\epsilon} q\hat{s}$ free spies, which by (2) is at least $\hat{r}/m$. 

$\blacksquare$
Random Conclusions

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Mitsche–Prałat [2012+]: (Using more intricate structure of random graphs and more complicated spy strategy:) If $r \geq \Omega(\frac{\log n}{p})$, then $\sigma(G, m, r) \leq \frac{r}{m} + 7 \log_{1/(1-p)} n$. 


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Let $G_k = K_{n,\ldots,n}$ with $k$ parts and $n \geq r$. 
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Lower Bound (Rev strategy)

Case 1: \( s_i > t \) for some \( i \); revs swarm to part \( i \). New meetings use \( m \) incoming revs, not guardable by spies from part \( i \). At least \( \lfloor (k - 1)t/m \rfloor \) additional spies must come from other parts, so

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s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t \left[ 1 + \frac{k-1}{m} \right] = \frac{k-1+m}{k} \frac{r}{m}.
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Hence spies from other parts must guard $\left\lfloor \frac{(r-s_i)/m}{m} \right\rfloor$ new meetings. Summing $s - s_i \geq \frac{r-s_i-m+1}{m}$ yields

$$(k-1 + \frac{1}{m})s > k \frac{r-m+1}{m}, \text{ so } s > \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1} \frac{r}{m+c} - k.$$
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The requirement from Case 2 is weaker (better for spies) than from Case 1.
Thm. For $k, m \in \mathbb{N}$, spies win on $G_k$ if $s \geq \frac{k}{k-1} \frac{r}{m} + k.$
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**Pf.** Hall’s Theorem yields a matching that covers new meetings with free spies who can move there.
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Spy Strategy:
(1) After revs have moved, cover all newly created meetings, moving the fewest possible spies to do so.
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1. After revs have moved, cover all newly created meetings, moving the fewest possible spies to do so.

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**Lem.** Equal distribution in (2) guarantees that the round ends stable. (meaning $\hat{s} - s_i \geq \frac{\hat{r}}{m}$)
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Complete Bipartite Graphs

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Complete Bipartite Graphs

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**Lower bd:** Strategy for revs to win quickly (small \( s \)).

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Complete Bipartite Graphs

\( m \geq k = 2 \). Proofs more difficult, but same approach.

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**Thm.** \((m = 2)\) Spies win if and only if \( s \geq \frac{7r}{10} \approx \frac{7}{5} \frac{r}{m} \).
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**Thm.** ($m \geq 4$, fixed) Spies win only if $s > \frac{3-o(1)}{2} \frac{r}{m}$. 
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- For large fixed \( m \), the threshold \( t \) for the number of spies needed to win satisfies \( 1.5 \frac{r}{m} < t < 1.58 \frac{r}{m} \).
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- For large fixed $m$, the threshold $t$ for the number of spies needed to win satisfies $1.5 \frac{r}{m} < t < 1.58 \frac{r}{m}$.

** Conj.** For fixed $m$, the threshold for the number of spies needed to win is asymptotic to $1.5 \frac{r}{m}$. 
Rev Strategy for $m = 3$ when $4 \mid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$
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**Ex.** If $r = 4k$, then revs win against $2k - 1$ spies.
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Start with $2k$ revs each on part $X_1$ and part $X_2$. Start with $s_i$ spies in $X_i$; may assume $s_1 \leq k - 1 < s_2$. 

\[
\begin{array}{c}
S & R & R & S \\
R & R & R & S \\
R & R & R & S \\
X_1 & X_2
\end{array}
\]
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Start with $s_i$ spies in $X_i$; may assume $s_1 \leq k - 1 < s_2$.
With only $2k - 1 - s_1$ spies, $X_2$ has $s_1 + 1$ uncovered revs.
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With only $2k – 1 – s_1$ spies, $X_2$ has $s_1 + 1$ uncovered revs.
Move $2(s_1 + 1)$ revs from $X_1$; make $s_1 + 1$ meetings in $X_2$.
Not coverable by the $s_1$ spies from $X_1$; spies lose.

\[\begin{array}{c}
\text{S}
\end{array}\]

\[\begin{array}{c}
\text{R}
\end{array}\]

\[\begin{array}{c}
\text{R}
\end{array}\]

\[\begin{array}{c}
\text{R}
\end{array}\]

\[\begin{array}{c}
\text{X}_1
\end{array}\]

\[\begin{array}{c}
\text{S}
\end{array}\]

\[\begin{array}{c}
\text{R}
\end{array}\]

\[\begin{array}{c}
\text{R}
\end{array}\]

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\text{R}
\end{array}\]

\[\begin{array}{c}
\text{S}
\end{array}\]

\[\begin{array}{c}
\text{X}_2
\end{array}\]
Rev Strategy for $m = 3$ when $4 \nmid r$

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To finish lower bound, we may assume $r = 4k + 2$. 
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The symmetric strategy fails to defeat $2k$ spies!
Rev Strategy for $m = 3$ when $4 \nmid r$

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To finish lower bound, we may assume $r = 4k + 2$.

The symmetric strategy fails to defeat $2k$ spies!

By starting all $4k + 2$ revs in $X_1$ (forcing $\geq \lfloor r/3 \rfloor$ spies to start in $X_1$), revs can defeat $2k$ spies in two rounds. (How many revs move to $X_2$ in round 1 depends on how many spies start in $X_2$.)
Larger $m$ and Upper Bound

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$. 
Larger $m$ and Upper Bound

**Cor.** $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\left\lfloor m/3 \right\rfloor} \right\rfloor \right\rfloor$.

**Pf.** Let $m' = \left\lfloor m/3 \right\rfloor$. Group revs into cells of size $m'$. 
Larger $m$ and Upper Bound

**Cor.** $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lceil \frac{r}{\lfloor m/3 \rfloor} \right\rceil \right\rfloor$.

**Pf.** Let $m' = \lfloor m/3 \rfloor$. Group revs into cells of size $m'$.

A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs.
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**Pf.** Let $m' = \lfloor m/3 \rfloor$. Group revs into cells of size $m'$. A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs. $\therefore \sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \lfloor r/m' \rfloor \right\rfloor$. ■
Larger $m$ and Upper Bound

**Cor.** $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lceil \frac{r}{m/3} \right\rceil \right\rfloor$.

**Pf.** Let $m' = \left\lceil m/3 \right\rceil$. Group revs into cells of size $m'$.

A cell moves as one player in a game with meeting size 3 and $\left\lfloor r/m' \right\rfloor$ revs. $\therefore \sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \right\rfloor \left\lceil r/m' \right\rceil \right\rfloor$.

**Upper Bounds:** $\sigma(G_2, 2, r) = \left\lfloor \frac{\left\lfloor 7r/2 \right\rfloor - 3}{5} \right\rfloor$, $\sigma(G_2, 3, r) = \left\lfloor r/2 \right\rfloor$, $\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3}) \frac{r}{m} + 1$. 
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**Cor.** $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{m/3} \right\rfloor \right\rfloor$.

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$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$, $\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3}) \frac{r}{m} + 1$.

Spies play greedy migration strategy.
Larger \( m \) and Upper Bound

**Cor.** \( \sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{m/3} \right\rfloor \right\rfloor. \)

**Pf.** Let \( m' = \lfloor m/3 \rfloor \). Group revs into cells of size \( m' \).

A cell moves as one player in a game with meeting size 3 and \( \lfloor r/m' \rfloor \) revs. \( \therefore \sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \lfloor r/m' \rfloor \right\rfloor. \)

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Spies play greedy migration strategy.

In terms of \#revs and \#covered revs in each part, a desired number of spies in each part is computed.
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**Cor.** $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lceil \frac{r}{m/3} \right\rceil \right\rfloor$.

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Spies play greedy migration strategy. In terms of #revs and #covered revs in each part, a desired number of spies in each part is computed. Spies achieve that "greedily", leaving vertices with few revs and moving to vertices with many revs.
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Spies play greedy migration strategy. In terms of #revs and #covered revs in each part, a desired number of spies in each part is computed. Spies achieve that "greedily", leaving vertices with few revs and moving to vertices with many revs. The computed values prevent the revs from winning by swarming a part, and that is shown to be sufficient for a greedy migration strategy to be a winning strategy.
Open Problems

Ques. Is every interval graph spy-good?
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**Ques.** For random graphs with constant $p$,

$r < \ln 2 \ln m \Rightarrow \sigma(G, m, r) = r - m + 1$, but

$r > (4 + \varepsilon)m \ln n \Rightarrow \sigma(G, m, r) < 4r/m$.

For various $p(n)$, how sharp is the threshold in $r$ between spy-bad and spy-pretty-good?
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\end{align*}
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For various $p(n)$, how sharp is the threshold in $r$ between spy-bad and spy-pretty-good?

**Ques.** For each $m$, what is $\lim_{r \to \infty} \frac{\sigma(G_2, m, r)}{r/m}$?
References


[middle two papers (and these slides) available at DBW preprint page (under homepage)]