The Game of Revolutionaries and Spies

Douglas B. West

Zhejiang Normal University
and
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page

Joint work with
Jane V. Butterfield, Daniel W. Cranston,
Gregory J. Puleo, and Reza Zamani;
plus
Clifford Smyth and David Howard

The Model

Two teams: r revolutionaries and s spies on a graph G.

The Model

Two teams: r revolutionaries and s spies on a graph G.

Start: Each rev and then each spy occupies a vertex.

The Model

Two teams: r revolutionaries and s spies on a graph G.

Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

The Model

Two teams: r revolutionaries and s spies on a graph G.

Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

The Model

Two teams: r revolutionaries and s spies on a graph G.

Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

Def. $RS(G, m, r, s)$ is the resulting game; who wins?
Invented by Beck

The Model

Two teams: r revolutionaries and s spies on a graph G.

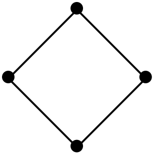
Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

Def. $RS(G, m, r, s)$ is the resulting game; who wins?

Ex. 3 revs beat 1 spy on C_4 (when $m = 2$).



The Model

Two teams: r revolutionaries and s spies on a graph G.

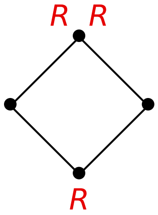
Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

Def. $RS(G, m, r, s)$ is the resulting game; who wins?

Ex. 3 revs beat 1 spy on C_4 (when $m = 2$).



The Model

Two teams: r revolutionaries and s spies on a graph G.

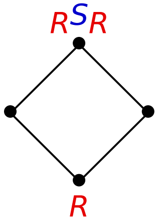
Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

Def. $RS(G, m, r, s)$ is the resulting game; who wins?

Ex. 3 revs beat 1 spy on C_4 (when $m = 2$).



The Model

Two teams: r revolutionaries and s spies on a graph G.

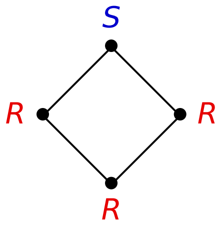
Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

Def. $RS(G, m, r, s)$ is the resulting game; who wins?

Ex. 3 revs beat 1 spy on C_4 (when $m = 2$).



The Model

Two teams: r revolutionaries and s spies on a graph G.

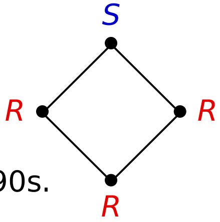
Start: Each rev and then each spy occupies a vertex.

Round: Each rev and then each spy moves or doesn't.

Goal: Revs want a meeting of size m unguarded by spies; spies want to prevent this.

Def. $RS(G, m, r, s)$ is the resulting game; who wins?

Ex. 3 revs beat 1 spy on C_4 (when $m = 2$).



Invented by Beck, 1990s.

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Obs. $s \geq r - m + 1 \;\Rightarrow\;$ spies win.

Spies follow all but $m - 1$ revs.

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Obs. $s \geq r - m + 1 \;\Rightarrow\;$ spies win.

Spies follow all but $m - 1$ revs.

Obs. $s < \lfloor r/m \rfloor \;\Rightarrow\;$ revs win.

Revs can make more meetings than spies can guard.

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Obs. $s \geq r - m + 1 \Rightarrow$ spies win.

Spies follow all but $m - 1$ revs.

Obs. $s < \lfloor r/m \rfloor \Rightarrow$ revs win.

Revs can make more meetings than spies can guard.

Ques. Fix G, m, r. How many spies needed to win?

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Obs. $s \geq r - m + 1 \;\Rightarrow\;$ spies win.

Spies follow all but $m - 1$ revs.

Obs. $s < \lfloor r/m \rfloor \;\Rightarrow\;$ revs win.

Revs can make more meetings than spies can guard.

Ques. Fix G, m, r. How many spies needed to win?

Def. $\sigma(G, m, r)$ = least s so spies win $RS(G, m, r, s)$.

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Obs. $s \geq r - m + 1 \;\Rightarrow\;$ spies win.

Spies follow all but $m - 1$ revs.

Obs. $s < \lfloor r/m \rfloor \;\Rightarrow\;$ revs win.

Revs can make more meetings than spies can guard.

Ques. Fix G, m, r. How many spies needed to win?

Def. $\sigma(G, m, r)$ = least s so spies win $RS(G, m, r, s)$.

G is spy-good: $\sigma(G, m, r) = \lfloor r/m \rfloor$ for all r, m.

Trivial Bounds

Obs. Assume always $s < |V(G)|$, else spies win.

Obs. $s \geq r - m + 1 \;\Rightarrow\;$ spies win.

Spies follow all but $m - 1$ revs.

Obs. $s < \lfloor r/m \rfloor \;\Rightarrow\;$ revs win.

Revs can make more meetings than spies can guard.

Ques. Fix G, m, r. How many spies needed to win?

Def. $\sigma(G, m, r)$ = least s so spies win $RS(G, m, r, s)$.

G is spy-good: $\sigma(G, m, r) = \lfloor r/m \rfloor$ for all r, m.

G is spy-bad for particular (r, m): $\sigma(G, m, r) = r - m + 1$.

The Plan

- Some graphs are spy-good: $\lfloor r/m \rfloor$

Trees, dominated graphs, webbed trees,
unicyclic graphs (almost: $\lceil r/m \rceil$ spies suffice).

The Plan

- Some graphs are spy-good: $\lfloor r/m \rfloor$

Trees, dominated graphs, webbed trees,
unicyclic graphs (almost: $\lceil r/m \rceil$ spies suffice).

- Some graphs are spy-bad: $r - m + 1$

Chordal or bipartite examples,
hypercubes (and more general product graphs),
king's-move grids (almost?) (Howard–Smyth [2012])

The Plan

- Some graphs are spy-good: $\lfloor r/m \rfloor$

Trees, dominated graphs, webbed trees,
unicyclic graphs (almost: $\lceil r/m \rceil$ spies suffice).

- Some graphs are spy-bad: $r - m + 1$

Chordal or bipartite examples,
hypercubes (and more general product graphs),
king's-move grids (almost?) (Howard–Smyth [2012])

- Some graphs are in between: cr/m

Complete multipartite (good upper and lower bounds).
Complete bipartite (exact answers for $m \in \{2, 3\}$).

The Plan

- Some graphs are spy-good: $\lfloor r/m \rfloor$

Trees, dominated graphs, webbed trees,
unicyclic graphs (almost: $\lceil r/m \rceil$ spies suffice).

- Some graphs are spy-bad: $r - m + 1$

Chordal or bipartite examples,
hypercubes (and more general product graphs),
king's-move grids (almost?) (Howard–Smyth [2012])

- Some graphs are in between: cr/m

Complete multipartite (good upper and lower bounds).
Complete bipartite (exact answers for $m \in \{2, 3\}$).

- Random graphs may be spy-good or spy-bad,
depending on edge probability (Mitsche–Prałat [2012]).

Spy-Good Graphs

Def. G is spy-good if $\lfloor r/m \rfloor$ spies win, for all r, m.

Spy-Good Graphs

Def. G is spy-good if $\lfloor r/m \rfloor$ spies win, for all r, m.

Thm. (Smyth [mid-90s]) Trees are spy-good.

Spy-Good Graphs

Def. G is spy-good if $\lfloor r/m \rfloor$ spies win, for all r, m.

Thm. (Smyth [mid-90s]) Trees are spy-good.

Thm. Graphs with a dominating vertex z are spy-good.
Spies wait at z except when guarding meetings elsewhere.

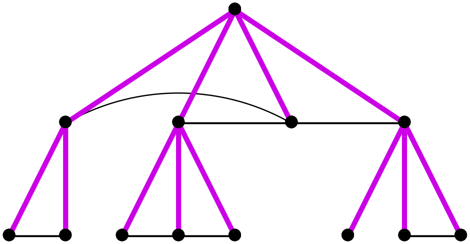
Spy-Good Graphs

Def. G is spy-good if $\lfloor r/m \rfloor$ spies win, for all r, m.

Thm. (Smyth [mid-90s]) Trees are spy-good.

Thm. Graphs with a dominating vertex z are spy-good.
Spies wait at z except when guarding meetings elsewhere.

Def. webbed tree - a graph G with a rooted spanning tree T where every edge of $G - E(T)$ joins siblings in T.



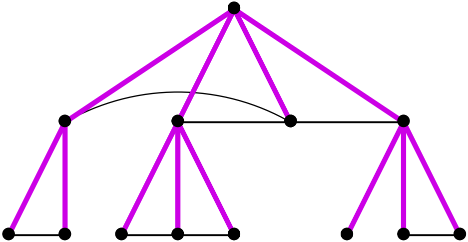
Spy-Good Graphs

Def. G is spy-good if $\lfloor r/m \rfloor$ spies win, for all r, m.

Thm. (Smyth [mid-90s]) Trees are spy-good.

Thm. Graphs with a dominating vertex z are spy-good.
Spies wait at z except when guarding meetings elsewhere.

Def. webbed tree - a graph G with a rooted spanning tree T where every edge of $G - E(T)$ joins siblings in T.



Thm. Webbed trees are spy-good.

Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

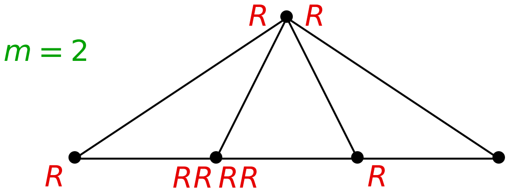
Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.



Dominating vertex z

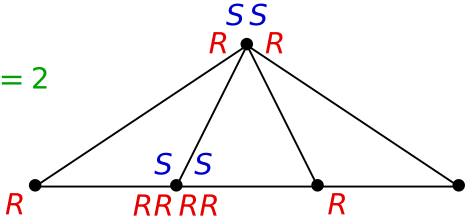
Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.

$m = 2$



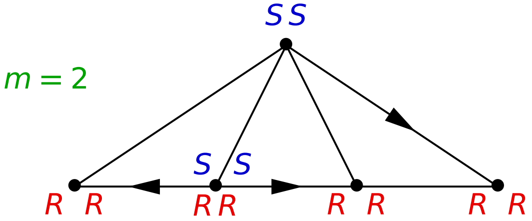
Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.



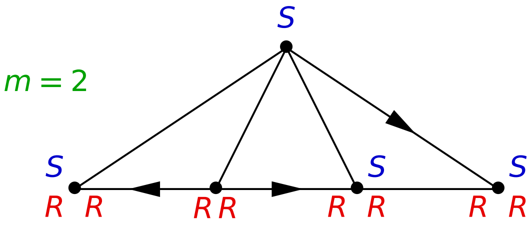
Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.



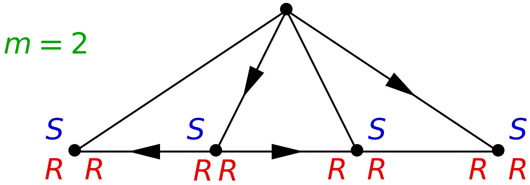
Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G,m,r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.



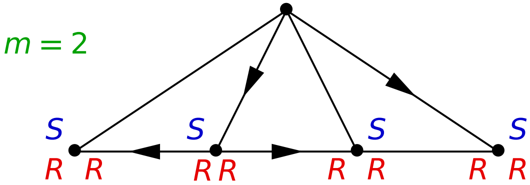
Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.



Idea: Restore stability via matching in bipartite graph.

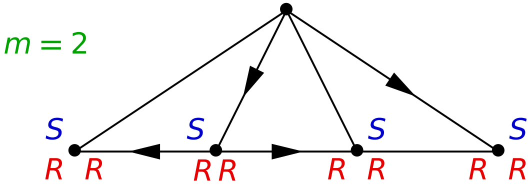
Dominating vertex z

Thm. Dominating Vertex $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. stable position - each vertex v other than z has $\lfloor r(v)/m \rfloor$ spies, where $r(v)$ = #revs at v.

stable position $\Rightarrow$ no unguarded meeting.

Spies make stable position in initial round.

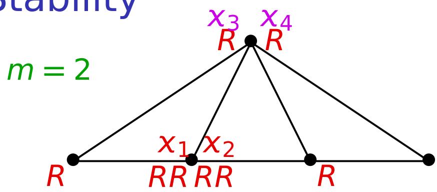


Idea: Restore stability via matching in bipartite graph.

Hall's Theorem: A bipartite graph with parts X and Y has a matching covering Y

$\Leftrightarrow$ each $T \subseteq Y$ has at least $|T|$ neighbors in X.

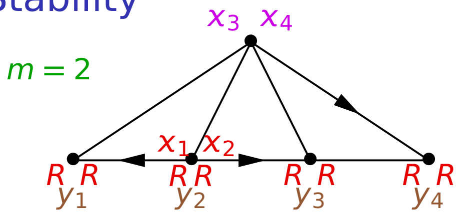
Restoring Stability



X = previous m-sets of revs at vertices other than z.

X' = previous excess spies at z.

Restoring Stability

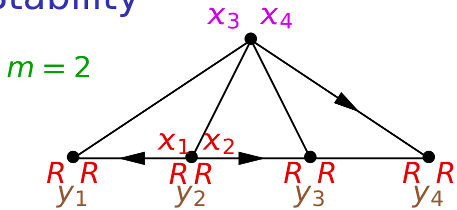


X = previous m-sets of revs at vertices other than z.

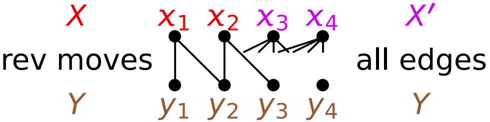
X' = previous excess spies at z.

Y = m-sets after revs move in this round.

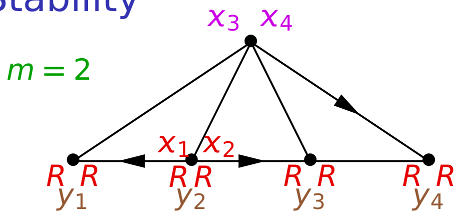
Restoring Stability



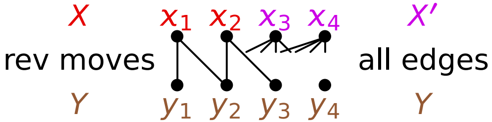
X = previous m-sets of revs at vertices other than z.
X' = previous excess spies at z.
Y = m-sets after revs move in this round.



Restoring Stability

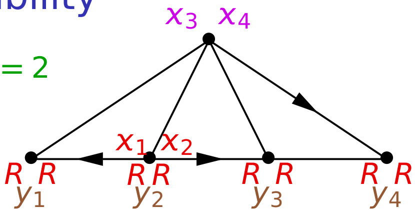


X = previous m-sets of revs at vertices other than z.
X' = previous excess spies at z.
Y = m-sets after revs move in this round.



Matching to cover Y $\Rightarrow$ Spies can restore stability.

Restoring Stability

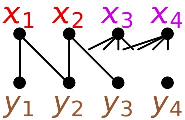


X = previous m-sets of revs at vertices other than z.
X' = previous excess spies at z.
Y = m-sets after revs move in this round.

X

rev moves

Y



X'

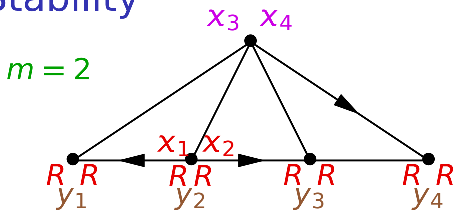
all edges

Y

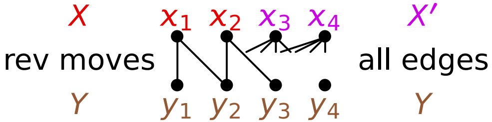
Matching to cover Y $\Rightarrow$ Spies can restore stability.

Hall's Condition: For $T \subseteq Y$, these $m|T|$ revs came from $N(T) \cap X$ or not from X, so $m|T| \leq m|N(T) \cap X| + r - m|X|$.

Restoring Stability



X = previous m-sets of revs at vertices other than z.
X' = previous excess spies at z.
Y = m-sets after revs move in this round.



Matching to cover Y $\Rightarrow$ Spies can restore stability.

Hall's Condition: For $T \subseteq Y$, these $m|T|$ revs came from $N(T) \cap X$ or not from X, so $m|T| \leq m|N(T) \cap X| + r - m|X|$.

$\therefore |N(T)| = |N(T) \cap X| + |X'| \geq |T| - (\lfloor \frac{r}{m} \rfloor - |X|) + |X'| = |T|$.

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. $r(v)$ = #revs now at v. $C(v)$ = children of v.
$s(v)$ = #spies now at v. $D(v)$ = v plus descendants.

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. $r(v)$ = #revs now at v. $C(v)$ = children of v.
$s(v)$ = #spies now at v. $D(v)$ = v plus descendants.

$$w(v) = \sum_{u \in D(v)} r(u).$$

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. $r(v)$ = #revs now at v. $C(v)$ = children of v.
$s(v)$ = #spies now at v. $D(v)$ = v plus descendants.

$$w(v) = \sum_{u \in D(v)} r(u).$$

Spy Rule:
$s(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor$ at each vertex v.

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G, m, r) = \lfloor r/m \rfloor$.

Pf. $r(v)$ = #revs now at v. $C(v)$ = children of v.
$s(v)$ = #spies now at v. $D(v)$ = v plus descendants.

$$w(v) = \sum_{u \in D(v)} r(u).$$

Spy Rule:
$$s(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor \geq \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w(v)-r(v)}{m} \right\rfloor.$$

Rule $\Rightarrow$ every meeting is guarded.

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G,m,r) = \lfloor r/m \rfloor$.

Pf. $r(v)$ = #revs now at v. $C(v)$ = children of v.
$s(v)$ = #spies now at v. $D(v)$ = v plus descendants.

$$w(v) = \sum_{u \in D(v)} r(u).$$

Spy Rule:
$$s(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor \geq \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w(v)-r(v)}{m} \right\rfloor.$$

Rule $\Rightarrow$ every meeting is guarded.

Since $\sum_{u \in D(v)} s(u) = \left\lfloor \frac{w(v)}{m} \right\rfloor$, **Rule** works in first round.

Webbed Tree

Thm. G is a webbed tree $\Rightarrow$ $\sigma(G,m,r) = \lfloor r/m \rfloor$.

Pf. $r(v)$ = #revs now at v. $C(v)$ = children of v.
$s(v)$ = #spies now at v. $D(v)$ = v plus descendants.

$$w(v) = \sum_{u \in D(v)} r(u).$$

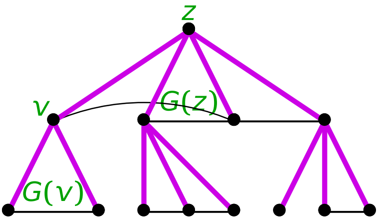
Spy Rule:
$$s(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor \geq \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w(v)-r(v)}{m} \right\rfloor.$$

Rule $\Rightarrow$ every meeting is guarded.

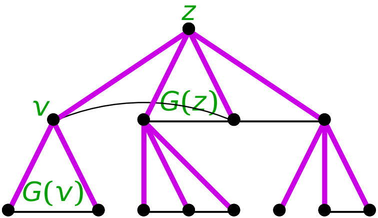
Since $\sum_{u \in D(v)} s(u) = \left\lfloor \frac{w(v)}{m} \right\rfloor$, **Rule** works in first round.

Idea: v dominates the subgraph $G(v)$ induced by $\{v\} \cup C(v)$. Spies play on these subgraphs independently to reestablish the **Rule**.

Split into Subgames (sketch)

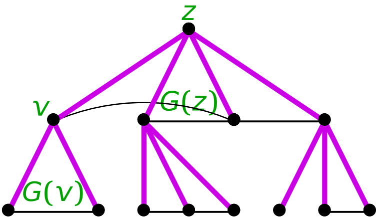


Split into Subgames (sketch)



The $s(v)$ spies at v split: $\check{s}(v)$ for $G(v)$ and $\hat{s}(v)$ for $G(z)$.

Split into Subgames (sketch)

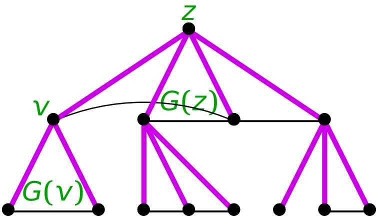


The $s(v)$ spies at v split: $\check{s}(v)$ for $G(v)$ and $\hat{s}(v)$ for $G(z)$.

$\check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor$ and $\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor$,

where $w^*(v)$ = #revs in $w(v)$ ending round in $D(v)$.

Split into Subgames (sketch)



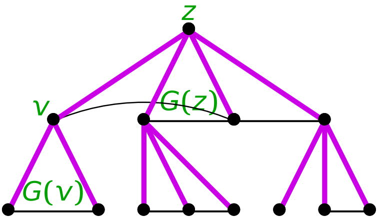
The $s(v)$ spies at v split: $\check{s}(v)$ for $G(v)$ and $\hat{s}(v)$ for $G(z)$.

$\check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor$ and $\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor$,

where $w^*(v)$ = #revs in $w(v)$ ending round in $D(v)$.

$\check{s}(v)$ and $\hat{s}(v)$ yield stable positions in the subgames.

Split into Subgames (sketch)



The $s(v)$ spies at v split: $\check{s}(v)$ for $G(v)$ and $\hat{s}(v)$ for $G(z)$.

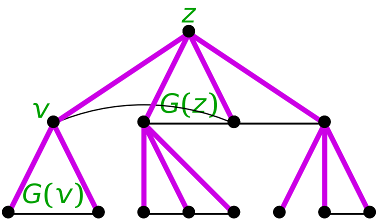
$\check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor$ and $\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor$,

where $w^*(v)$ = #revs in $w(v)$ ending round in $D(v)$.

$\check{s}(v)$ and $\hat{s}(v)$ yield stable positions in the subgames.

The spies can restore stability in each subgame.

Split into Subgames (sketch)



The $s(v)$ spies at v split: $\check{s}(v)$ for $G(v)$ and $\hat{s}(v)$ for $G(z)$.

$\check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor$ and $\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor$,

where $w^*(v)$ = #revs in $w(v)$ ending round in $D(v)$.

$\check{s}(v)$ and $\hat{s}(v)$ yield stable positions in the subgames.

The spies can restore stability in each subgame.

The resulting new spy distributions restore the **Rule**:

$$s'(v) = \check{s}'(v) + \hat{s}'(v) = \left\lfloor \frac{w'(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w'(x)}{m} \right\rfloor . \quad \blacksquare$$

Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

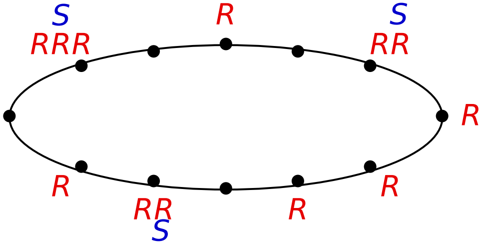
Pf. Extra revs can't help spies: may assume $r = sm$.

Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Pf. Extra revs can't help spies: may assume $r = sm$.

Spies follow every mth rev (here $m = 4$).

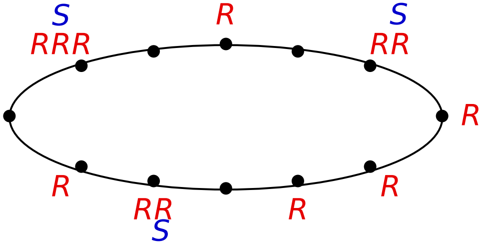


Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \le \lceil r/m \rceil$.

Pf. Extra revs can't help spies: may assume $r = sm$.

Spies follow every mth rev (here $m = 4$).



Positions of mth revs don't move by more than one vertex; spies can follow to maintain the condition. ∎

Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Thm. If G is unicyclic, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Thm. If G is unicyclic, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Pf. Idea: Adding one spy and m revs at any vertex of C (or removing them) preserves the "cycle condition".

Cycles and Unicyclic Graphs

Lem. If G is a cycle, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Thm. If G is unicyclic, then $\sigma(G, m, r) \leq \lceil r/m \rceil$.

Pf. Idea: Adding one spy and m revs at any vertex of C (or removing them) preserves the "cycle condition".

May assume $r = sm$ and all revs start on the cycle.

Cycles and Unicyclic Graphs

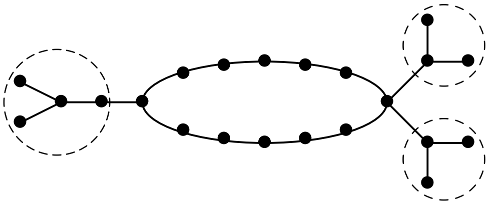
Lem. If G is a cycle, then $\sigma(G, m, r) \le \lceil r/m \rceil$.

Thm. If G is unicyclic, then $\sigma(G, m, r) \le \lceil r/m \rceil$.

Pf. Idea: Adding one spy and m revs at any vertex of C (or removing them) preserves the "cycle condition".

May assume $r = sm$ and all revs start on the cycle.

Maintain the cycle condition by keeping "fake" revs at a cycle vertex until an attached tree has enough revs to demand a spy according to the tree strategy. ∎



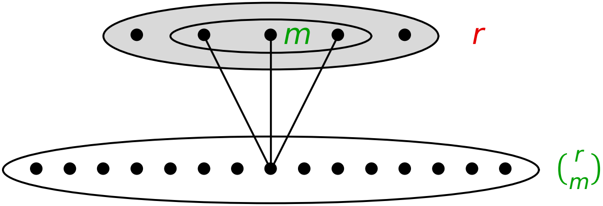
Spy-Bad Graphs

Def. Given r, m, graph G is spy-bad if $r - m$ spies lose.

Spy-Bad Graphs

Def. Given r, m, graph G is spy-bad if $r - m$ spies lose.

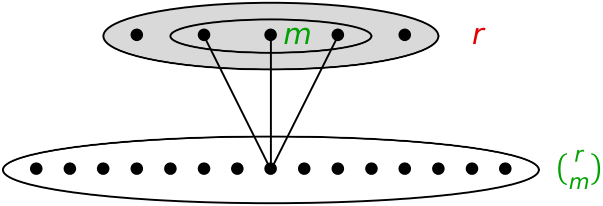
Prop. For $r, m \in \mathbb{N}$, some chordal graph is spy-bad.



Spy-Bad Graphs

Def. Given r, m, graph G is spy-bad if $r - m$ spies lose.

Prop. For $r, m \in \mathbb{N}$, some chordal graph is spy-bad.

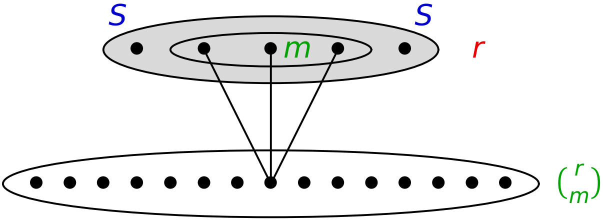


Revs initially occupy the vertices of the clique.

Spy-Bad Graphs

Def. Given r, m, graph G is spy-bad if $r - m$ spies lose.

Prop. For $r, m \in \mathbb{N}$, some chordal graph is spy-bad.



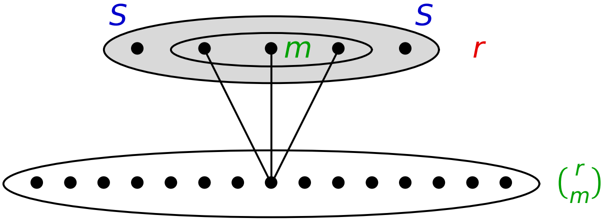
Revs initially occupy the vertices of the clique.

Spies occupy at most $r - m$ vertices of the clique.

Spy-Bad Graphs

Def. Given r, m, graph G is spy-bad if $r - m$ spies lose.

Prop. For $r, m \in \mathbb{N}$, some chordal graph is spy-bad.



Revs initially occupy the vertices of the clique.

Spies occupy at most $r - m$ vertices of the clique.

Some m uncovered revs can meet on the first round, unreachable by spies. ■

Domination Number

Cor. $\sigma(G, m, r) \leq \gamma(G) \lfloor r/m \rfloor$.

Domination Number

Cor. $\sigma(G, m, r) \le \gamma(G)\lfloor r/m \rfloor$.

Pf. With each vertex of a dominating set T, associate $\lfloor r/m \rfloor$ spies to play the "stabilization strategy" against any revs appearing in its neighborhood. ∎

Domination Number

Cor. $\sigma(G, m, r) \leq \gamma(G)\lfloor r/m \rfloor$.

Pf. With each vertex of a dominating set T, associate $\lfloor r/m \rfloor$ spies to play the "stabilization strategy" against any revs appearing in its neighborhood. ∎

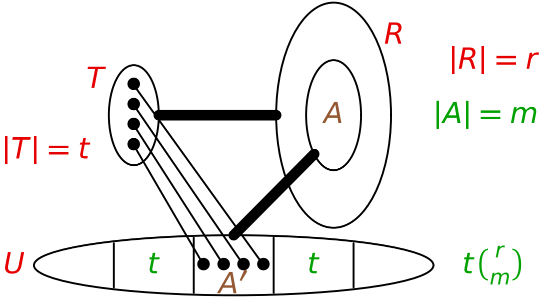
Thm. Given $t, m, r \in \mathbb{N}$ such that $t \leq m \leq r - m$, there is a graph G with $\gamma(G) = t$ and $\sigma(G, m, r) > t(r/m - 1)$.

Domination Number

Cor. $\sigma(G, m, r) \leq \gamma(G) \lfloor r/m \rfloor$.

Pf. With each vertex of a dominating set T, associate $\lfloor r/m \rfloor$ spies to play the "stabilization strategy" against any revs appearing in its neighborhood. ∎

Thm. Given $t, m, r \in \mathbb{N}$ such that $t \leq m \leq r - m$, there is a graph G with $\gamma(G) = t$ and $\sigma(G, m, r) > t(r/m - 1)$.



Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

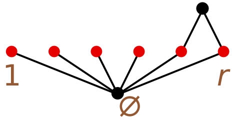
Pf. Claim $r - 2$ spies lose.

Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

Pf. Claim $r - 2$ spies lose.

Revs start at r singles, threatening $\binom{r}{2}$ doubles and $\varnothing$.

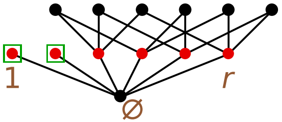


Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

Pf. Claim $r - 2$ spies lose.

Revs start at r singles, threatening $\binom{r}{2}$ doubles and $\varnothing$.



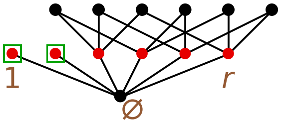
Let t = #revs left uncovered by spies initially.

Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

Pf. Claim $r - 2$ spies lose.

Revs start at r singles, threatening $\binom{r}{2}$ doubles and $\varnothing$.



Let t = #revs left uncovered by spies initially.

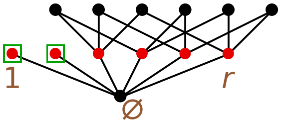
$\binom{t}{2}$ threats to watch. Spies at weight 2 can watch one; spies at weight 3 can watch three.

Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

Pf. Claim $r - 2$ spies lose.

Revs start at r singles, threatening $\binom{r}{2}$ doubles and $\varnothing$.



Let t = #revs left uncovered by spies initially.

$\binom{t}{2}$ threats to watch. Spies at weight 2 can watch one; spies at weight 3 can watch three.

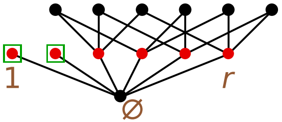
$\therefore$ $s \geq (r - t) + \frac{1}{3}\binom{t}{2}$. If $s \leq r - 2$, then $t \in \{3, 4\}$.

Hypercubes

Thm. For $m = 2$, the hypercube Q_d is spy-bad if $d \geq r$.

Pf. Claim $r - 2$ spies lose.

Revs start at r singles, threatening $\binom{r}{2}$ doubles and $\varnothing$.



Let t = #revs left uncovered by spies initially.

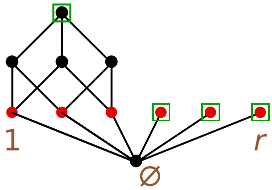
$\binom{t}{2}$ threats to watch. Spies at weight 2 can watch one; spies at weight 3 can watch three.

$\therefore\ s \geq (r - t) + \frac{1}{3}\binom{t}{2}$. If $s \leq r - 2$, then $t \in \{3, 4\}$.

$t = 4$ leaves six threats at doubles, not reachable by two triples (two triangles don't cover $E(K_4)$).

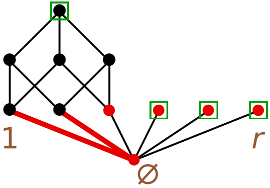
The case $t = 3$

$\therefore$ $r - 3$ spies occupy singles, plus one at a triple.
By symmetry, spy is at 123, with the others at $4, \ldots, r$.



The case $t = 3$

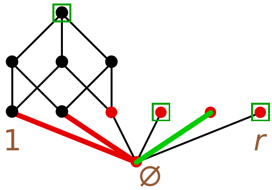
$\therefore$ $r - 3$ spies occupy singles, plus one at a triple.
By symmetry, spy is at 123, with the others at $4, \dots, r$.



Revs at 1 and 2 move to $\varnothing$.
For $3 \leq i \leq r$, the rev at i waits.

The case $t = 3$

$\therefore$ $r - 3$ spies occupy singles, plus one at a triple.
By symmetry, spy is at 123, with the others at $4, \ldots, r$.

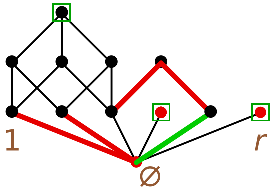


Revs at 1 and 2 move to $\varnothing$.
For $3 \leq i \leq r$, the rev at i waits.

A spy from some j with $4 \leq j \leq r$ must move to guard $\varnothing$.

The case $t = 3$

$\therefore$ $r - 3$ spies occupy singles, plus one at a triple.
By symmetry, spy is at 123, with the others at $4, \ldots, r$.



Revs at 1 and 2 move to $\emptyset$.
For $3 \leq i \leq r$, the rev at i waits.

A spy from some j with $4 \leq j \leq r$ must move to guard $\emptyset$.

No spy can now reach a neighbor of $\{3, j\}$.
Next, revs at 3 and j will move to $\{3, j\}$ and win. ∎

Smaller dimensions

When $d \geq r$, revs beat $r-2$ spies on Q_d when $m=2$.
On smaller hypercubes, revs do almost as well.

Smaller dimensions

When $d \geq r$, revs beat $r - 2$ spies on Q_d when $m = 2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Smaller dimensions

When $d \geq r$, revs beat $r-2$ spies on Q_d when $m=2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Pf. Let X be a code with distance 9 and $|X| \geq 2^d/d^8$.
Allocate d revolutionaries to each of $\lfloor r/d \rfloor$ points in X.

Smaller dimensions

When $d \geq r$, revs beat $r - 2$ spies on Q_d when $m = 2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Pf. Let X be a code with distance 9 and $|X| \geq 2^d/d^8$.
Allocate d revolutionaries to each of $\lfloor r/d \rfloor$ points in X.

Each group plays the earlier strategy at its point x.

Smaller dimensions

When $d \geq r$, revs beat $r-2$ spies on Q_d when $m=2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Pf. Let X be a code with distance 9 and $|X| \geq 2^d/d^8$.
Allocate d revolutionaries to each of $\lfloor r/d \rfloor$ points in X.

Each group plays the earlier strategy at its point x.

At least $d-1$ spies are needed to avoid losing near x.

Smaller dimensions

When $d \geq r$, revs beat $r-2$ spies on Q_d when $m=2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Pf. Let X be a code with distance 9 and $|X| \geq 2^d/d^8$.
Allocate d revolutionaries to each of $\lfloor r/d \rfloor$ points in X.

Each group plays the earlier strategy at its point x.

At least $d-1$ spies are needed to avoid losing near x.

Distance 9 is far enough to prevent spies working at x' from helping at x fast enough.

Smaller dimensions

When $d \geq r$, revs beat $r - 2$ spies on Q_d when $m = 2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Pf. Let X be a code with distance 9 and $|X| \geq 2^d/d^8$.
Allocate d revolutionaries to each of $\lfloor r/d \rfloor$ points in X.

Each group plays the earlier strategy at its point x.

At least $d - 1$ spies are needed to avoid losing near x.

Distance 9 is far enough to prevent spies working at x' from helping at x fast enough.

$\therefore$ revs win against fewer than $(d-1)\lfloor r/d \rfloor$ spies. ∎

Smaller dimensions

When $d \geq r$, revs beat $r-2$ spies on Q_d when $m=2$.
On smaller hypercubes, revs do almost as well.

Thm. If $d < r < 2^d/d^7$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.

Pf. Let X be a code with distance 9 and $|X| \geq 2^d/d^8$.
Allocate d revolutionaries to each of $\lfloor r/d \rfloor$ points in X.

Each group plays the earlier strategy at its point x.

At least $d-1$ spies are needed to avoid losing near x.

Distance 9 is far enough to prevent spies working at x' from helping at x fast enough.

$\therefore$ revs win against fewer than $(d-1)\lfloor r/d \rfloor$ spies. ■

Good upper bounds on $\sigma(Q_d, 2, r)$ (spy strategies) are not known for $r > d$.

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

A Tool: A retract of G is an induced subgraph H with a map $f\colon V(G) \to V(H)$ such that (1) $v \in V(H) \Rightarrow f(v) = v$ and (2) $uv \in E(G) \Rightarrow d_H(f(u), f(v)) \le 1$.

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

A Tool: A retract of G is an induced subgraph H with a map $f\colon V(G) \to V(H)$ such that (1) $v \in V(H) \Rightarrow f(v) = v$ and (2) $uv \in E(G) \Rightarrow d_H(f(u), f(v)) \le 1$.

• Nowakowski-Winkler [1983] used retracts in the classical cop-and-robber game.

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

A Tool: A retract of G is an induced subgraph H with a map $f\colon V(G) \to V(H)$ such that (1) $v \in V(H) \Rightarrow f(v) = v$ and (2) $uv \in E(G) \Rightarrow d_H(f(u), f(v)) \le 1$.

• Nowakowski-Winkler [1983] used retracts in the classical cop-and-robber game.

Lem. If H is a retract of G, then $\sigma(G,m,r) \ge \sigma(H,m,r)$.

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

A Tool: A retract of G is an induced subgraph H with a map $f\colon V(G) \to V(H)$ such that (1) $v \in V(H) \Rightarrow f(v) = v$ and (2) $uv \in E(G) \Rightarrow d_H(f(u), f(v)) \le 1$.

- Nowakowski-Winkler [1983] used retracts in the classical cop-and-robber game.

Lem. If H is a retract of G, then $\sigma(G,m,r) \ge \sigma(H,m,r)$.

Pf. The revs play $RS(G, m, r, s)$ by playing in H, using f to simulate spy moves for an imagined game in H.

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

A Tool: A retract of G is an induced subgraph H with a map $f\colon V(G)\to V(H)$ such that (1) $v\in V(H) \Rightarrow f(v)=v$ and (2) $uv\in E(G) \Rightarrow d_H(f(u),f(v))\le 1$.

• Nowakowski-Winkler [1983] used retracts in the classical cop-and-robber game.

Lem. If H is a retract of G, then $\sigma(G,m,r)\ge\sigma(H,m,r)$.

Pf. The revs play $RS(G,m,r,s)$ by playing in H, using f to simulate spy moves for an imagined game in H.

For a spy move $u\to v$ in G, revs imagine $f(u)\to f(v)$ in H.

Larger Meeting Size on Q_d

Idea: Revs start at r vertices of weight 1, threatening meetings at $\binom{r}{m}$ vertices of weight m after $m-1$ steps. How many spies are needed to cover the threats?

A Tool: A retract of G is an induced subgraph H with a map $f\colon V(G)\to V(H)$ such that (1) $v\in V(H) \Rightarrow f(v)=v$ and (2) $uv\in E(G) \Rightarrow d_H(f(u),f(v))\le 1$.

• Nowakowski-Winkler [1983] used retracts in the classical cop-and-robber game.

Lem. If H is a retract of G, then $\sigma(G,m,r)\ge\sigma(H,m,r)$.

Pf. The revs play $RS(G,m,r,s)$ by playing in H, using f to simulate spy moves for an imagined game in H.

For a spy move $u\to v$ in G, revs imagine $f(u)\to f(v)$ in H.

When revs win at w in H, since no simulated spy is at w and $f(w)=w$, the revs also win the real game then. ■

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \ge r - 38.73m$ when $d \ge r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \geq r - 38.73m$ when $d \geq r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

Retract Lemma allows assuming the spies not on revs are in Q_t. They must guard the $\binom{r}{t}$ threats at weight m.

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \geq r - 38.73m$ when $d \geq r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

Retract Lemma allows assuming the spies not on revs are in Q_t. They must guard the $\binom{r}{t}$ threats at weight m.

A probabilistic argument (generate a random set of indices) shows $\exists$ vertex of weight m having distance at least m from all these spies when $t \geq 38.73m$.

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \geq r - 38.73m$ when $d \geq r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

Retract Lemma allows assuming the spies not on revs are in Q_t. They must guard the $\binom{r}{t}$ threats at weight m.

A probabilistic argument (generate a random set of indices) shows $\exists$ vertex of weight m having distance at least m from all these spies when $t \geq 38.73m$.

When #spies $< r - 38.73m$, at least $38.73m$ revs are initially uncovered, and the probabilistic lemma guarantees existence of an unprotected threat. ∎

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \geq r - 38.73m$ when $d \geq r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

Retract Lemma allows assuming the spies not on revs are in Q_t. They must guard the $\binom{r}{t}$ threats at weight m.

A probabilistic argument (generate a random set of indices) shows $\exists$ vertex of weight m having distance at least m from all these spies when $t \geq 38.73m$.

When #spies $< r - 38.73m$, at least $38.73m$ revs are initially uncovered, and the probabilistic lemma guarantees existence of an unprotected threat. ∎

Cor. If G is a cartesian product of d nontrivial graphs, and revs win $RS(Q_d,m,r,s)$, then revs win $RS(G,m,r,s)$.

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \geq r - 38.73m$ when $d \geq r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

Retract Lemma allows assuming the spies not on revs are in Q_t. They must guard the $\binom{r}{t}$ threats at weight m.

A probabilistic argument (generate a random set of indices) shows $\exists$ vertex of weight m having distance at least m from all these spies when $t \geq 38.73m$.

When #spies $< r - 38.73m$, at least $38.73m$ revs are initially uncovered, and the probabilistic lemma guarantees existence of an unprotected threat. ■

Cor. If G is a cartesian product of d nontrivial graphs, and revs win $RS(Q_d,m,r,s)$, then revs win $RS(G,m,r,s)$.

- Earlier: If $d \geq r \geq m \geq 3$, then $\sigma(Q_d, m, r) > r - \frac{3}{4}m^2$.

An Uncoverable Threat

Thm. $\sigma(Q_d, m, r) \geq r - 38.73m$ when $d \geq r$.

Pf. Revs start at r singletons. Spies leave t uncovered.

Retract Lemma allows assuming the spies not on revs are in Q_t. They must guard the $\binom{r}{t}$ threats at weight m.

A probabilistic argument (generate a random set of indices) shows $\exists$ vertex of weight m having distance at least m from all these spies when $t \geq 38.73m$.

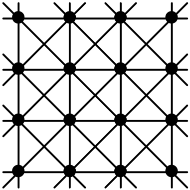
When #spies $< r - 38.73m$, at least $38.73m$ revs are initially uncovered, and the probabilistic lemma guarantees existence of an unprotected threat. ■

Cor. If G is a cartesian product of d nontrivial graphs, and revs win $RS(Q_d,m,r,s)$, then revs win $RS(G,m,r,s)$.

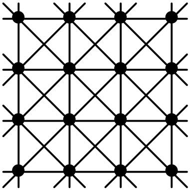
- Earlier: If $d \geq r \geq m \geq 3$, then $\sigma(Q_d, m, r) > r - \frac{3}{4}m^2$.

Better when $m \leq 52$. Perhaps $\sigma(Q_d, m, r) \approx r - 2m$.

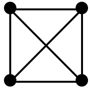
King’s-move Grid (Howard–Smyth)



King's-move Grid (Howard–Smyth)

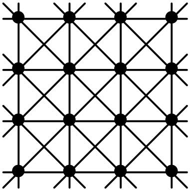


$RS(K_4, 2, 3, 1)$

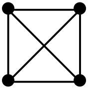


spy wins

King's-move Grid (Howard–Smyth)



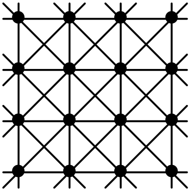
$RS(K_4, 2, 3, 1)$



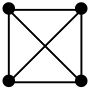
spy wins

Thm. One spy beats $2m - 1$ revs on $\mathbb{Z}^2$.

King’s-move Grid (Howard–Smyth)



$RS(K_4, 2, 3, 1)$

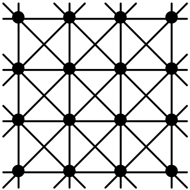


spy wins

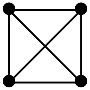
Thm. One spy beats $2m - 1$ revs on $\mathbb{Z}^2$.

Pf. The spy stays at the median position of the revs in each coordinate. ∎

King's-move Grid (Howard–Smyth)



$RS(K_4, 2, 3, 1)$



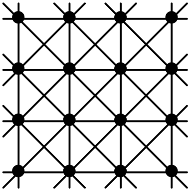
spy wins

Thm. One spy beats $2m - 1$ revs on $\mathbb{Z}^2$.

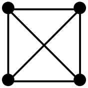
Pf. The spy stays at the median position of the revs in each coordinate. ∎

Thm. $\sigma(\mathbb{Z}^2, m, r) \leq r - 2m + 2$.

King's-move Grid (Howard–Smyth)



$RS(K_4, 2, 3, 1)$



spy wins

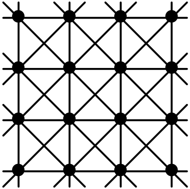
Thm. One spy beats $2m - 1$ revs on $\mathbb{Z}^2$.

Pf. The spy stays at the median position of the revs in each coordinate. ∎

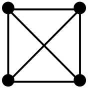
Thm. $\sigma(\mathbb{Z}^2, m, r) \leq r - 2m + 2$.

Pf. $r-2m+1$ spies follow revs; one guards $2m-1$ revs. ∎

King’s-move Grid (Howard–Smyth)



$RS(K_4, 2, 3, 1)$



spy wins

Thm. One spy beats $2m - 1$ revs on $\mathbb{Z}^2$.

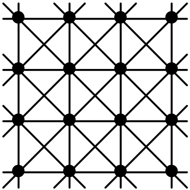
Pf. The spy stays at the median position of the revs in each coordinate. ∎

Thm. $\sigma(\mathbb{Z}^2, m, r) \le r - 2m + 2$.

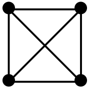
Pf. $r-2m+1$ spies follow revs; one guards $2m-1$ revs. ∎

Thm. If $m = 2$, then $6\lfloor r/8 \rfloor \le \sigma(\mathbb{Z}^2, m, r) \le r - 2$.

King's-move Grid (Howard–Smyth)



$RS(K_4, 2, 3, 1)$



spy wins

Thm. One spy beats $2m-1$ revs on $\mathbb{Z}^2$.

Pf. The spy stays at the median position of the revs in each coordinate. ■

Thm. $\sigma(\mathbb{Z}^2, m, r) \le r - 2m + 2$.

Pf. $r-2m+1$ spies follow revs; one guards $2m-1$ revs. ■

Thm. If $m = 2$, then $6\lfloor r/8 \rfloor \le \sigma(\mathbb{Z}^2, m, r) \le r - 2$.

Pf. A group of 8 revs can beat 5 spies (clever!). ■

Random Graphs can be Spy-Bad or Spy-Good

Let p, r, m depend on n, with $p \to 0$. For the random n-vertex graph G with edge probability p, how many spies are needed?

Random Graphs can be Spy-Bad or Spy-Good

Let p, r, m depend on n, with $p \to 0$. For the random n-vertex graph G with edge probability p, how many spies are needed?

Let $d = p(n-1)$ (expected average degree).

Let $L_n = \log_{1/(1-p)}(n) \sim \frac{\ln n}{p}$.

Random Graphs can be Spy-Bad or Spy-Good

Let p, r, m depend on n, with $p \to 0$. For the random n-vertex graph G with edge probability p, how many spies are needed?

Let $d = p(n-1)$ (expected average degree).

Let $L_n = \log_{1/(1-p)}(n) \sim \frac{\ln n}{p}$.

Thm. (Mitsche–Prałat [2012+]) If $d = n^{2/3+\gamma}$, where $0 \le \gamma \le 1/3 - O(1/\ln n)$, then

$$\sigma(G,m,r) = \begin{cases} r-m+1 & \text{if } r-m \le (3-\epsilon)\gamma L_n, \\ \Theta(L_n) & \text{if } r-m > (3-\epsilon)\gamma L_n \text{ and } \frac{r}{m} \in O(L_n), \\ (1+o(1))\frac{r}{m} & \text{if } \frac{r}{m} >> L_n. \end{cases}$$

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Let U be the vertices occupied by the $r - m$ spies.
Let T be the vertices with uncovered revs; note $|T| \geq m$.

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Let U be the vertices occupied by the $r - m$ spies.
Let T be the vertices with uncovered revs; note $|T| \geq m$.

On the first round, the revs from T meet at x and win.

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Let U be the vertices occupied by the $r - m$ spies.
Let T be the vertices with uncovered revs; note $|T| \geq m$.

On the first round, the revs from T meet at x and win.

More Clever: Almost surely for $|T| = 2$ and $|U| \leq r - m$, such x exists (easier to guarantee).

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Let U be the vertices occupied by the $r - m$ spies.
Let T be the vertices with uncovered revs; note $|T| \geq m$.

On the first round, the revs from T meet at x and win.

More Clever: Almost surely for $|T| = 2$ and $|U| \leq r - m$, such x exists (easier to guarantee).

With U as before, T = two other revs: they meet at x.

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Let U be the vertices occupied by the $r - m$ spies.
Let T be the vertices with uncovered revs; note $|T| \geq m$.

On the first round, the revs from T meet at x and win.

More Clever: Almost surely for $|T| = 2$ and $|U| \leq r - m$, such x exists (easier to guarantee).

With U as before, T = two other revs: they meet at x.

On round t, the uncovered meeting of size t and one uncovered rev meet at the new x to form uncovered meeting of size $t + 1$; revs win on round $m - 1$.

Spy-bad Random Graphs

For appropriate r (in terms of n and p), almost surely for all disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there exists $x \in V(G)$ adjacent to all of T and none of U.

The revs can then start at any r vertices.

Let U be the vertices occupied by the $r - m$ spies.
Let T be the vertices with uncovered revs; note $|T| \geq m$.

On the first round, the revs from T meet at x and win.

More Clever: Almost surely for $|T| = 2$ and $|U| \leq r - m$, such x exists (easier to guarantee).

With U as before, T = two other revs: they meet at x.

On round t, the uncovered meeting of size t and one uncovered rev meet at the new x to form uncovered meeting of size $t + 1$; revs win on round $m - 1$.

The needed condition holds when $r - m \leq (3 - \epsilon)\gamma L_n$.

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

Let $\hat{r}$ = total #free revs, $\hat{s}_U$ = #free spies on U.

Def. A position is stable if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

Let $\hat{r}$ = total #free revs, $\hat{s}_U$ = #free spies on U.

Def. A position is stable if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

Lem. If a round starts with a stable position, then the spies can cover all meetings at the end of the round.

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

Let $\hat{r}$ = total #free revs, $\hat{s}_U$ = #free spies on U.

Def. A position is stable if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

Lem. If a round starts with a stable position, then the spies can cover all meetings at the end of the round.

Pf. Let $Y = \{$vertices having meetings after revs move$\}$, $X = \{$spies$\}$, and $xy \in E(H)$ if spy x can reach vertex y.

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

Let $\hat{r}$ = total #free revs, $\hat{s}_U$ = #free spies on U.

Def. A position is stable if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

Lem. If a round starts with a stable position, then the spies can cover all meetings at the end of the round.

Pf. Let $Y = \{$vertices having meetings after revs move$\}$, $X = \{$spies$\}$, and $xy \in E(H)$ if spy x can reach vertex y.

Hall's Condition holds: If $T \subseteq Y$, then $|T| \leq \frac{bm+\hat{r}}{m}$, where b = #meetings in $N_G[T]$ at start of round.

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

Let $\hat{r}$ = total #free revs, $\hat{s}_U$ = #free spies on U.

Def. A position is stable if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

Lem. If a round starts with a stable position, then the spies can cover all meetings at the end of the round.

Pf. Let $Y = \{$vertices having meetings after revs move$\}$, $X = \{$spies$\}$, and $xy \in E(H)$ if spy x can reach vertex y.

Hall's Condition holds: If $T \subseteq Y$, then $|T| \leq \frac{bm+\hat{r}}{m}$, where b = #meetings in $N_G[T]$ at start of round.

Spies in $N_G[T]$ can reach T in one move, so for $y \in T$,

The Usefulness of Stable Positions

Def. m revs at a vertex v and one spy covering them (and no others at v) are bound. Other players are free.

Let $\hat{r}$ = total #free revs, $\hat{s}_U$ = #free spies on U.

Def. A position is stable if (1) all meetings are covered, and (2) $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

Lem. If a round starts with a stable position, then the spies can cover all meetings at the end of the round.

Pf. Let $Y = \{$vertices having meetings after revs move$\}$, $X = \{$spies$\}$, and $xy \in E(H)$ if spy x can reach vertex y.

Hall's Condition holds: If $T \subseteq Y$, then $|T| \leq \frac{bm+\hat{r}}{m}$, where b = #meetings in $N_G[T]$ at start of round.

Spies in $N_G[T]$ can reach T in one move, so for $y \in T$,

$$|N_H(T)| \geq b + \hat{s}_{N[y]} \geq b + \frac{\hat{r}}{m} \geq |T|. \quad \blacksquare$$

Spy-Good Random Graphs

Lem. Under appropriate conditions with all meetings covered, the free spies can move to a stable position.

Spy-Good Random Graphs

Lem. Under appropriate conditions with all meetings covered, the free spies can move to a stable position.

Pf. Each free spy moves to a uniformly random nbr. With probability > 0, the resulting position is stable. ∎

Spy-Good Random Graphs

Lem. Under appropriate conditions with all meetings covered, the free spies can move to a stable position.

Pf. Each free spy moves to a uniformly random nbr. With probability > 0, the resulting position is stable. ∎

Thm. Almost always the spies win $RS(G, m, r, s)$ when $0 < q < p < 1$, $s \geq \frac{1+\epsilon}{q}\frac{r}{m}$, and $s \geq \frac{r}{m} + \frac{\ln n}{2[1-(1+\epsilon)^{-1}]^2 q^2}$,

Spy-Good Random Graphs

Lem. Under appropriate conditions with all meetings covered, the free spies can move to a stable position.

Pf. Each free spy moves to a uniformly random nbr. With probability > 0, the resulting position is stable. ■

Thm. Almost always the spies win $RS(G, m, r, s)$ when $0 < q < p < 1$, $s \geq \frac{1+\epsilon}{q}\frac{r}{m}$, and $s \geq \frac{r}{m} + \frac{\ln n}{2[1-(1+\epsilon)^{-1}]^2 q^2}$,

Pf. In each round, the spies first cover all meetings by the first lemma, then restore stability by the second. ■

Spy-Good Random Graphs

Lem. Under appropriate conditions with all meetings covered, the free spies can move to a stable position.

Pf. Each free spy moves to a uniformly random nbr. With probability > 0, the resulting position is stable. ■

Thm. Almost always the spies win $RS(G,m,r,s)$ when $0<q<p<1$, $s \geq \frac{1+\epsilon}{q}\frac{r}{m}$, and $s \geq \frac{r}{m} + \frac{\ln n}{2[1-(1+\epsilon)^{-1}]^2 q^2}$,

Pf. In each round, the spies first cover all meetings by the first lemma, then restore stability by the second. ■

Cor. For $p = \frac{1}{2}$, $m \in \mathbb{N}$, and $c > 4$, almost always the spies win $RS(G,m,r,s)$ when $s \geq c\frac{r}{m}$ and $r \geq cm \ln n$.

Spy-Good Random Graphs

Lem. Under appropriate conditions with all meetings covered, the free spies can move to a stable position.

Pf. Each free spy moves to a uniformly random nbr. With probability > 0, the resulting position is stable. ■

Thm. Almost always the spies win $RS(G, m, r, s)$ when $0 < q < p < 1$, $s \geq \frac{1+\epsilon}{q}\frac{r}{m}$, and $s \geq \frac{r}{m} + \frac{\ln n}{2[1-(1+\epsilon)^{-1}]^2 q^2}$,

Pf. In each round, the spies first cover all meetings by the first lemma, then restore stability by the second. ■

Cor. For $p = \frac{1}{2}$, $m \in \mathbb{N}$, and $c > 4$, almost always the spies win $RS(G, m, r, s)$ when $s \geq c\frac{r}{m}$ and $r \geq cm \ln n$.

Mitsche–Prałat [2012+]: (Using "superspies" occupying a special dominating set in the random graph): $\sigma(G, m, r) \leq (1 + o(1))\frac{r}{m}$ when $r \geq \frac{1}{p} m \ln n$.

Complete k-partite graphs

Let $G_k = K_{n,\dots,n}$ with k parts and $n \geq r$.

Complete k-partite graphs

Let $G_k = K_{n,\ldots,n}$ with k parts and $n \geq r$.

Thm. Spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Complete k-partite graphs

Let $G_k = K_{n,\dots,n}$ with k parts and $n \geq r$.

Thm. Spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

When $k \geq m$, revs win when s is "not much smaller".

Complete k-partite graphs

Let $G_k = K_{n,\dots,n}$ with k parts and $n \geq r$.

Thm. Spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

When $k \geq m$, revs win when s is "not much smaller".

Thm. If $k \geq m$ and $k \mid r$, then at least $\frac{k}{k-1}\frac{r}{m+c} - k$ spies are needed to win on G_k, where $c = 1/(k-1)$.

Complete k-partite graphs

Let $G_k = K_{n,\dots,n}$ with k parts and $n \geq r$.

Thm. Spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

When $k \geq m$, revs win when s is "not much smaller".

Thm. If $k \geq m$ and $k \mid r$, then at least $\frac{k}{k-1}\frac{r}{m+c} - k$ spies are needed to win on G_k, where $c = 1/(k-1)$.

Def. In a game on G_k, an i-swarm sends all revs to part i, filling unguarded partial meetings to size m and then making additional meetings of size m.

Complete k-partite graphs

Let $G_k = K_{n,\dots,n}$ with k parts and $n \ge r$.

Thm. Spies win on G_k if $s \ge \frac{k}{k-1}\frac{r}{m} + k$.

When $k \ge m$, revs win when s is "not much smaller".

Thm. If $k \ge m$ and $k \mid r$, then at least $\frac{k}{k-1}\frac{r}{m+c} - k$ spies are needed to win on G_k, where $c = 1/(k-1)$.

Def. In a game on G_k, an i-swarm sends all revs to part i, filling unguarded partial meetings to size m and then making additional meetings of size m.

Idea: Let $t = r/k$. Revs initially at t verts. in each part.

Complete k-partite graphs

Let $G_k = K_{n,\dots,n}$ with k parts and $n \geq r$.

Thm. Spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

When $k \geq m$, revs win when s is "not much smaller".

Thm. If $k \geq m$ and $k \mid r$, then at least $\frac{k}{k-1}\frac{r}{m+c} - k$ spies are needed to win on G_k, where $c = 1/(k-1)$.

Def. In a game on G_k, an i-swarm sends all revs to part i, filling unguarded partial meetings to size m and then making additional meetings of size m.

Idea: Let $t = r/k$. Revs initially at t verts. in each part.

How many spies are needed to avoid losing by swarm?

Lower Bound (Rev strategy)

Let s_i be the initial #spies in part i (they sit on revs.).

Case 1: $s_i > t$ for some i; revs swarm to part i.
New meetings use m incoming revs, not guardable by spies from part i. At least $\lfloor (k-1)t/m \rfloor$ additional spies must come from other parts, so

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t \left[1 + \frac{k-1}{m}\right] = \frac{k-1+m}{k} \frac{r}{m}.$$

Lower Bound (Rev strategy)

Let s_i be the initial #spies in part i (they sit on revs.).

Case 1: $s_i > t$ for some i; revs swarm to part i.
New meetings use m incoming revs, not guardable by spies from part i. At least $\lfloor (k-1)t/m \rfloor$ additional spies must come from other parts, so

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t\left[1 + \frac{k-1}{m}\right] = \frac{k-1+m}{k}\frac{r}{m}.$$

Case 2: $s_i \leq t$ for all i.
Part i has $t - s_i$ partial meetings; i-swarm can fill them (since $s_i \geq 0$) if $(k-1)t \geq t(m-1)$, implied by $k \geq m$.

Lower Bound (Rev strategy)

Let s_i be the initial #spies in part i (they sit on revs.).

Case 1: $s_i > t$ for some i; revs swarm to part i.
New meetings use m incoming revs, not guardable by spies from part i. At least $\lfloor (k-1)t/m \rfloor$ additional spies must come from other parts, so

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t \left[1 + \frac{k-1}{m} \right] = \frac{k-1+m}{k} \frac{r}{m}.$$

Case 2: $s_i \leq t$ for all i.
Part i has $t - s_i$ partial meetings; i-swarm can fill them (since $s_i \geq 0$) if $(k-1)t \geq t(m-1)$, implied by $k \geq m$.

Hence spies from other parts must guard $\lfloor (r - s_i)/m \rfloor$ new meetings. Summing $s - s_i \geq \frac{r - s_i - m + 1}{m}$ yields

$$(k-1+\tfrac{1}{m})s > k\frac{r-m+1}{m}, \text{ so } s > \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1}\frac{r}{m+c} - k.$$

Lower Bound (Rev strategy)

Let s_i be the initial #spies in part i (they sit on revs.).

Case 1: $s_i > t$ for some i; revs swarm to part i.
New meetings use m incoming revs, not guardable by spies from part i. At least $\lfloor (k-1)t/m \rfloor$ additional spies must come from other parts, so

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t \left[1 + \frac{k-1}{m} \right] = \frac{k-1+m}{k} \frac{r}{m}.$$

Case 2: $s_i \leq t$ for all i.
Part i has $t - s_i$ partial meetings; i-swarm can fill them (since $s_i \geq 0$) if $(k-1)t \geq t(m-1)$, implied by $k \geq m$.

Hence spies from other parts must guard $\lfloor (r - s_i)/m \rfloor$ new meetings. Summing $s - s_i \geq \frac{r - s_i - m + 1}{m}$ yields

$$(k-1+\tfrac{1}{m})s > k\frac{r-m+1}{m}, \text{ so } s > \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1}\frac{r}{m+c} - k.$$

When $k \geq m$, the requirement from Case 2 is weaker (better for spies) than from Case 1. ∎

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. Spy Strategy: Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. Spy Strategy: Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Stability needs $\hat{s} - s_i \geq \frac{\hat{r}}{m}$, so $s_j \geq \frac{\hat{r}}{m(k-1)}$ $\forall j$ suffices.

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Stability needs $\hat{s} - s_i \geq \frac{\hat{r}}{m}$, so $s_j \geq \frac{\hat{r}}{m(k-1)}$ $\forall\, j$ suffices.

By (2), $\frac{\hat{s}}{k} \geq \frac{\hat{r}}{m(k-1)} + 1$ (or $\hat{s} \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$) suffices.

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Stability needs $\hat{s} - s_i \geq \frac{\hat{r}}{m}$, so $s_j \geq \frac{\hat{r}}{m(k-1)}$ $\forall j$ suffices.

By (2), $\frac{\hat{s}}{k} \geq \frac{\hat{r}}{m(k-1)} + 1$ (or $\hat{s} \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$) suffices.

Given: $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Stability needs $\hat{s} - s_i \geq \frac{\hat{r}}{m}$, so $s_j \geq \frac{\hat{r}}{m(k-1)}$ $\forall\, j$ suffices.

By (2), $\frac{\hat{s}}{k} \geq \frac{\hat{r}}{m(k-1)} + 1$ (or $\hat{s} \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$) suffices.

Given: $s \geq \frac{k}{k-1}\frac{r}{m} + k$. Subtract $s - \hat{s} = (r - \hat{r})/m$

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Stability needs $\hat{s} - s_i \geq \frac{\hat{r}}{m}$, so $s_j \geq \frac{\hat{r}}{m(k-1)}$ $\forall\, j$ suffices.

By (2), $\frac{\hat{s}}{k} \geq \frac{\hat{r}}{m(k-1)} + 1$ (or $\hat{s} \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$) suffices.

Given: $s \geq \frac{k}{k-1}\frac{r}{m} + k$. Subtract $s - \hat{s} = (r - \hat{r})/m$

to get $\hat{s} \geq \frac{1}{k-1}\frac{r}{m} + \frac{\hat{r}}{m} + k$

Upper Bound (Spy strategy)

Thm. For $k, m \in \mathbb{N}$, spies win on G_k if $s \geq \frac{k}{k-1}\frac{r}{m} + k$.

Pf. **Spy Strategy:** Stable position after each round.

(1) After revs have moved, cover all new meetings, moving as few spies as possible. (Guaranteed doable after a stable position.)

(2) Restore stability by distributing the $\hat{s}$ spies now free equally among the k parts
(always $|s_i - \hat{s}/k| < 1$, where s_i is #free spies in part i).

Stability needs $\hat{s} - s_i \geq \frac{\hat{r}}{m}$, so $s_j \geq \frac{\hat{r}}{m(k-1)}$ $\forall j$ suffices.

By (2), $\frac{\hat{s}}{k} \geq \frac{\hat{r}}{m(k-1)} + 1$ (or $\hat{s} \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$) suffices.

Given: $s \geq \frac{k}{k-1}\frac{r}{m} + k$. Subtract $s - \hat{s} = (r - \hat{r})/m$

to get $\hat{s} \geq \frac{1}{k-1}\frac{r}{m} + \frac{\hat{r}}{m} + k \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k$. ∎

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Thm. ($m = 2$) Spies win if and only if $s \geq \frac{7r}{10} \approx \frac{7}{5}\frac{r}{m}$.

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Thm. ($m = 2$) Spies win if and only if $s \geq \frac{7r}{10} \approx \frac{7}{5}\frac{r}{m}$.

Thm. ($m = 3$) Spies win if and only if $s \geq \left\lfloor \frac{r}{2} \right\rfloor \approx \frac{3}{2}\frac{r}{m}$.

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Thm. ($m = 2$) Spies win if and only if $s \geq \frac{7r}{10} \approx \frac{7}{5}\frac{r}{m}$.

Thm. ($m = 3$) Spies win if and only if $s \geq \left\lfloor \frac{r}{2} \right\rfloor \approx \frac{3}{2}\frac{r}{m}$.

Thm. ($m \geq 4$, fixed) Spies win only if $s > \frac{3-o(1)}{2}\frac{r}{m}$.

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Thm. ($m = 2$) Spies win if and only if $s \geq \frac{7r}{10} \approx \frac{7}{5}\frac{r}{m}$.

Thm. ($m = 3$) Spies win if and only if $s \geq \left\lfloor \frac{r}{2} \right\rfloor \approx \frac{3}{2}\frac{r}{m}$.

Thm. ($m \geq 4$, fixed) Spies win only if $s > \frac{3-o(1)}{2}\frac{r}{m}$.

Thm. For large fixed m, spies win if $s > \left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m}$.

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Thm. ($m = 2$) Spies win if and only if $s \geq \frac{7r}{10} \approx \frac{7}{5}\frac{r}{m}$.

Thm. ($m = 3$) Spies win if and only if $s \geq \left\lfloor \frac{r}{2} \right\rfloor \approx \frac{3}{2}\frac{r}{m}$.

Thm. ($m \geq 4$, fixed) Spies win only if $s > \frac{3-o(1)}{2}\frac{r}{m}$.

Thm. For large fixed m, spies win if $s > \left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m}$.

• For large fixed m, the threshold t for the number of spies needed to win satisfies $1.5\frac{r}{m} < t < 1.58\frac{r}{m}$.

Complete Bipartite Graphs

$m \geq k = 2$. Proofs more difficult, but same approach.

Lower bd: Strategy for revs to win quickly (small s).

Upper bd: Strategy for spies to maintain invariants that prevent revs winning on next round (large s).

Thm. ($m = 2$) Spies win if and only if $s \geq \frac{7r}{10} \approx \frac{7}{5}\frac{r}{m}$.

Thm. ($m = 3$) Spies win if and only if $s \geq \left\lfloor \frac{r}{2} \right\rfloor \approx \frac{3}{2}\frac{r}{m}$.

Thm. ($m \geq 4$, fixed) Spies win only if $s > \frac{3-o(1)}{2}\frac{r}{m}$.

Thm. For large fixed m, spies win if $s > \left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m}$.

• For large fixed m, the threshold t for the number of spies needed to win satisfies $1.5\frac{r}{m} < t < 1.58\frac{r}{m}$.

Conj. For fixed m, the threshold for the number of spies needed to win is asymptotic to $1.5\frac{r}{m}$.

Open Problems

Ques. Is every interval graph spy-good?

Open Problems

Ques. Is every interval graph spy-good?

Ques. Good upper bounds on $\sigma(Q_d, 2, r)$ for $r > d$?

Open Problems

Ques. Is every interval graph spy-good?

Ques. Good upper bounds on $\sigma(Q_d, 2, r)$ for $r > d$?

Ques. Smallest c so $\sigma(Q_d, m, r) > r - cm$ for $d \geq r$?

Open Problems

Ques. Is every interval graph spy-good?

Ques. Good upper bounds on $\sigma(Q_d, 2, r)$ for $r > d$?

Ques. Smallest c so $\sigma(Q_d, m, r) > r - cm$ for $d \geq r$?

Ques. For sparse random graphs (such as $p = \frac{c \ln n}{n}$) how does $\sigma(G, m, r)$ behave? (Mitsche–Prałat does not apply.)

Open Problems

Ques. Is every interval graph spy-good?

Ques. Good upper bounds on $\sigma(Q_d, 2, r)$ for $r > d$?

Ques. Smallest c so $\sigma(Q_d, m, r) > r - cm$ for $d \geq r$?

Ques. For sparse random graphs (such as $p = \frac{c \ln n}{n}$) how does $\sigma(G, m, r)$ behave? (Mitsche–Prałat does not apply.)

Ques. For complete bipartite, what is $\lim_{r \to \infty} \frac{\sigma(G_2, m, r)}{r/m}$?

Open Problems

Ques. Is every interval graph spy-good?

Ques. Good upper bounds on $\sigma(Q_d, 2, r)$ for $r > d$?

Ques. Smallest c so $\sigma(Q_d, m, r) > r - cm$ for $d \geq r$?

Ques. For sparse random graphs (such as $p = \frac{c \ln n}{n}$) how does $\sigma(G, m, r)$ behave? (Mitsche–Prałat does not apply.)

Ques. For complete bipartite, what is $\lim_{r\to\infty} \frac{\sigma(G_2, m, r)}{r/m}$?

Ques. What is the complexity of determining who wins $RS(G, m, r, s)$?

Open Problems

Ques. Is every interval graph spy-good?

Ques. Good upper bounds on $\sigma(Q_d, 2, r)$ for $r > d$?

Ques. Smallest c so $\sigma(Q_d, m, r) > r - cm$ for $d \geq r$?

Ques. For sparse random graphs (such as $p = \frac{c \ln n}{n}$) how does $\sigma(G, m, r)$ behave? (Mitsche–Prałat does not apply.)

Ques. For complete bipartite, what is $\lim_{r\to\infty} \frac{\sigma(G_2, m, r)}{r/m}$?

Ques. What is the complexity of determining who wins $RS(G, m, r, s)$?

What about testing whether revs can win $RS(G, 2, r, s)$ in two moves when the initial position is specified?

References

- D. Howard and C. Smyth, Revolutionaries and spies, Discrete Math. 312 (2012), 3384–3391.

- D.W. Cranston, C. Smyth, and D.B. West, Revolutionaries and spies on trees and unicyclic graphs, Journal of Combinatorics 3 (2012), 195–205.

- J.V. Butterfield, D.W. Cranston, G.J. Puleo, D.B. West, and R. Zamani, Revolutionaries and spies: Spy-good and spy-bad graphs, Theoretical Computer Science 463 (2012), 35–53.

- D. Mitsche and P. Prałat, Revolutionaries and spies on random graphs, http://arxiv.org/abs/1205.0531, submitted Combinatorics, Probability, and Computing.

[DBW papers (and these slides) available at DBW preprint page (under homepage)]

Rev Strategy for $m = 3$ when $4 \mid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$

Rev Strategy for $m = 3$ when $4 \mid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$

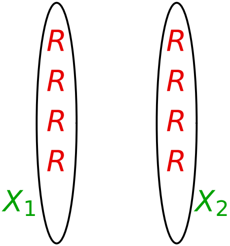
Ex. If $r = 4k$, then revs win against $2k - 1$ spies.

Rev Strategy for $m = 3$ when $4 \mid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$

Ex. If $r = 4k$, then revs win against $2k - 1$ spies.

Start with $2k$ revs each on part X_1 and part X_2.



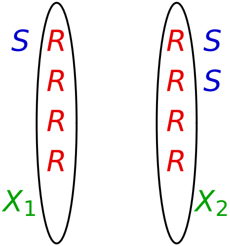
Rev Strategy for $m = 3$ when $4 \mid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$

Ex. If $r = 4k$, then revs win against $2k - 1$ spies.

Start with $2k$ revs each on part X_1 and part X_2.

Start with s_i spies in X_i; may assume $s_1 \le k - 1 < s_2$.



Rev Strategy for $m = 3$ when $4 \mid r$

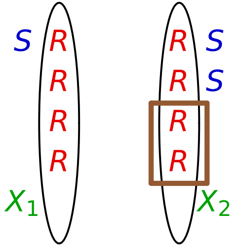
- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$

Ex. If $r = 4k$, then revs win against $2k - 1$ spies.

Start with $2k$ revs each on part X_1 and part X_2.

Start with s_i spies in X_i; may assume $s_1 \le k - 1 < s_2$.

With only $2k-1-s_1$ spies, X_2 has $s_1 + 1$ uncovered revs.



Rev Strategy for $m = 3$ when $4 \mid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$

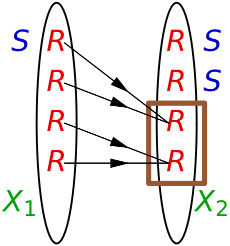
Ex. If $r = 4k$, then revs win against $2k-1$ spies.

Start with $2k$ revs each on part X_1 and part X_2.

Start with s_i spies in X_i; may assume $s_1 \le k-1 < s_2$.

With only $2k-1-s_1$ spies, X_2 has s_1+1 uncovered revs.

Move $2(s_1+1)$ revs from X_1; make s_1+1 meetings in X_2.
Not coverable by the s_1 spies from X_1; spies lose. ∎



Rev Strategy for $m = 3$ when $4 \nmid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$.

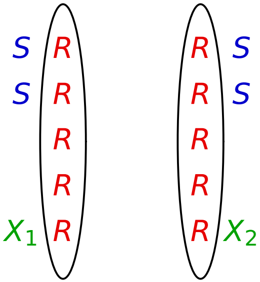
To finish lower bound, we may assume $r = 4k + 2$.

Rev Strategy for $m = 3$ when $4 \nmid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$.

To finish lower bound, we may assume $r = 4k + 2$.

The symmetric strategy fails to defeat $2k$ spies!



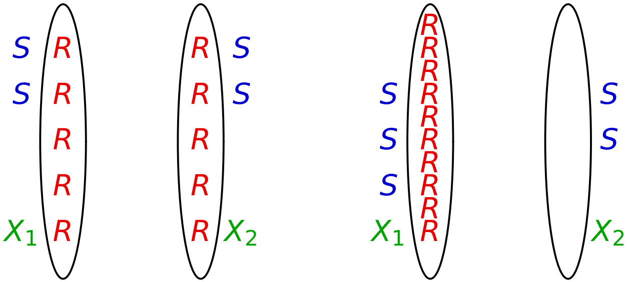
Rev Strategy for $m = 3$ when $4 \nmid r$

- $\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$.

To finish lower bound, we may assume $r = 4k + 2$.

The symmetric strategy fails to defeat $2k$ spies!

By starting all $4k + 2$ revs in X_1 (forcing $\geq \lfloor r/3 \rfloor$ spies to start in X_1), revs can defeat $2k$ spies in two rounds. (How many revs move to X_2 in round 1 depends on how many spies start in X_2.) ■



Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Pf. Let $m' = \lceil m/3 \rceil$. Group revs into cells of size m'.

Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Pf. Let $m' = \lceil m/3 \rceil$. Group revs into cells of size m'.

A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs.

Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Pf. Let $m' = \lceil m/3 \rceil$. Group revs into cells of size m'.

A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs.

The result for size 3 yields $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \lfloor r/m' \rfloor \right\rfloor$. ∎

Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Pf. Let $m' = \lceil m/3 \rceil$. Group revs into cells of size m'.

A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs.

The result for size 3 yields $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \lfloor r/m' \rfloor \right\rfloor$. ∎

Thm. If $m = 2$, then $\sigma(G_2, 2, r) \geq \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$.

Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Pf. Let $m' = \lceil m/3 \rceil$. Group revs into cells of size m'.

A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs.

The result for size 3 yields $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \lfloor r/m' \rfloor \right\rfloor$. ■

Thm. If $m = 2$, then $\sigma(G_2, 2, r) \geq \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$.

Pf. More intricate. Revs win within two rounds against fewer than $\left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$ spies. ■

Lower Bounds for Other m

Cor. $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$.

Pf. Let $m' = \lceil m/3 \rceil$. Group revs into cells of size m'.

A cell moves as one player in a game with meeting size 3 and $\lfloor r/m' \rfloor$ revs.

The result for size 3 yields $\sigma(G_2, m, r) \geq \left\lfloor \frac{1}{2} \lfloor r/m' \rfloor \right\rfloor$. ■

Thm. If $m = 2$, then $\sigma(G_2, 2, r) \geq \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$.

Pf. More intricate. Revs win within two rounds against fewer than $\left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$ spies. ■

The coeff. in $\frac{7}{5}\frac{r}{m}$ for $m = 2$ is less than in $\frac{3}{2}\frac{r}{m}$ for $m = 3$ because the revs now can't efficiently threaten to move $2k$ revs onto k uncovered revs to make k meetings.

Upper Bounds

Upper Bounds

Upper Bounds: $\sigma(G_2, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$,
$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$,
$\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3})\frac{r}{m} + 1$.

Upper Bounds

Upper Bounds: $\sigma(G_2, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$,
$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$,
$\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3})\frac{r}{m} + 1$.

Spies play greedy migration strategy.

Upper Bounds

Upper Bounds: $\sigma(G_2, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$,
$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$,
$\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3})\frac{r}{m} + 1$.

Spies play greedy migration strategy.

In terms of #revs and #covered revs in each part, a desired number of spies in each part is computed.

Upper Bounds

Upper Bounds: $\sigma(G_2, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$,
$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$,
$\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3})\frac{r}{m} + 1$.

Spies play greedy migration strategy.

In terms of #revs and #covered revs in each part, a desired number of spies in each part is computed.

Spies achieve that "greedily", departing vertices with few revs and moving to vertices with many revs.

Upper Bounds

Upper Bounds: $\sigma(G_2, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$,
$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$,
$\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3})\frac{r}{m} + 1$.

Spies play greedy migration strategy.

In terms of #revs and #covered revs in each part, a desired number of spies in each part is computed.

Spies achieve that "greedily", departing vertices with few revs and moving to vertices with many revs.

The maximum #spies move to reach the new distribution: All spies on one part go to the other part, or all new spies in one part come from the other part.

Upper Bounds

Upper Bounds: $\sigma(G_2, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$,
$\sigma(G_2, 3, r) = \lfloor r/2 \rfloor$,
$\sigma(G_2, m, r) \leq (1 + 1/\sqrt{3})\frac{r}{m} + 1$.

Spies play greedy migration strategy.

In terms of #revs and #covered revs in each part, a desired number of spies in each part is computed.

Spies achieve that "greedily", departing vertices with few revs and moving to vertices with many revs.

The maximum #spies move to reach the new distribution: All spies on one part go to the other part, or all new spies in one part come from the other part.

The computed values prevent the revs from winning by swarming a part, and that is shown to be sufficient for a greedy migration strategy to be a winning strategy. ∎