Repetition Number of Graphs

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Since \( \text{rep}(G) \geq n/\Delta(G) \) (excluding isolated vertices), \( \text{rep}(G) \leq k \implies \alpha(G) \geq \Delta(G) \geq n/k \) for triangle-free \( G \).
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For bounded \( \text{rep}^# \), no sequence of triangle-free graphs
has \( \alpha(G) \in o(n) \), but \( \exists \) sequence of \( K_4 \)-free graphs with
\( \text{rep}(G) \leq 5 \) and \( \alpha(G) \in o(n) \) (Bollobás [1996]).
Thm. If $G$ has $n$ vertices, avg degree $d$, and min degree $s$, then $\text{rep}(G) \geq \left\lceil \frac{n}{2d-2s+1} \right\rceil$, and this is sharp.
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**Thm.** This is asymptotically sharp for trees ($n/3$), maximal outerplanar graphs ($n/5$), planar triangulations ($n/7$), triangulations with mindegree 4 ($n/5$), triangulations with mindegree 5 ($n/3$).
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**Evidence:** True for trees. For trees with perfect matchings, maximal outerplanar graphs, and triangulations with 2-factors, $\text{rep}(L(G)) = \Theta(m)$. 
General lower bound

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Degree sum: $dn \geq rs + r(s+1) + \ldots r(s+a-1) + b(s+a)$
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General sharpness argument

Given $n, s, r \in \mathbb{N}$ and $d$, seek graph whose degree-sum equals the counting bound: $dn = ns + \frac{n}{2} \left( \frac{n}{r} - 1 \right) + \frac{b(r-b)}{2r}$. 
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**Def.** packed list = \( ar + b \) integers from \( s \) to \( s + a \) having \( r \) copies of each \( s, \ldots, s + a - 1 \) and \( b \) copies of \( s + a \) (also \( 1 \leq b \leq r \)).
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**Thm.** A packed list is graphic if and only if the sum is even and $ar + b > s + a$. 
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Sufficiency uses Erdös–Gallai Conditions in this form:
\[
\sum_{i=1}^{k} (d_i + 1) \leq \sum_{k=1}^{k} d_i^* \text{ for } 1 \leq k \leq \ell(d).
\]
Sketch of sufficiency proof

E–G Conditions: \[ \sum_{i=1}^{k} (d_i + 1) \leq \sum_{k=1}^{\ell} d_i^* \] for \( 1 \leq k \leq \ell(d) \).
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In most cases, $d_k + 1 \leq d_k^*$ for $k = \ell(d)$. 
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Key: As \(k\) decreases, \(d_k\) up by \(\leq 1\), while \(d_k^*\) up by \(r\).
Sketch of sufficiency proof

E–G Conditions: \[ \sum_{i=1}^{k} (d_i + 1) \leq \sum_{k=1}^{l(d)} d_i^* \text{ for } 1 \leq k \leq l(d). \]

Ex. \( r = 3, s = 3, a = 2, b = 1. \)

\[ \begin{array}{c}
d_1 & d_1^* \\
n & n \\
d_1^* \\
\end{array} \quad \begin{array}{c}
d_{1r} & d_{1r}^* \\
n & n \\
d_{1r}^* \\
\end{array} \quad \begin{array}{c}
d_{(d)} & d_{(d)}^* \\
n & n \\
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\end{array} \quad \begin{array}{c}
d_{ar+b} & d_{ar+b}^* \\
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So, \( d_i + 1 \leq d_i^* \) in each term, and sum is okay.
Special Families — $\text{rep}(G) \geq \frac{n}{2d-2s+1}$

Trees: $d = \frac{2n-2}{n} < 2; \quad s = 1; \quad \text{rep}(G) \geq n/3$. 
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Maximal Outerplanar: $d = \frac{4n-6}{n} < 4; \ s = 2; \ \text{rep}(G) \geq \frac{n}{5}$. 
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Triangulations — $d < 6$

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$s = 3 \implies \text{rep}(G) \geq n/7$

$s = 4 \implies \text{rep}(G) \geq n/5$
Triangulations

d < 6; s = 5 ⇒ \text{rep}(G) ≥ n/3
The Augmented Half-Graph

**Ex.** A claw-free graph with repetition number 2.
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\[ x_i y_j \in E(H_p) \iff i + j > p \quad d(x_i) = d(y_i) = i; \]
The Augmented Half-Graph

**Ex.** A claw-free graph with repetition number 2.

Form $H'_p$ by completing $X$ and $Y$ to cliques.

$$d(x_i) = d(y_i) = i;$$

$x_i y_j \in E(H_p) \iff i + j > p$

$H'_p$ is claw-free.
Line Graphs

Edge-degrees: $d_{L(G)}(xy) = d_G(x) + d_G(y) - 2$. 
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**Thm.** If \(G\) has \(m\) edges, then \(\text{rep}(L(G)) \geq \frac{1}{4}m^{1/3}\).

**Pf.** Let \(D = \Delta(G)\) and \(a = m/D\).

\(\exists \leq 2D - 1\) distinct edge-degrees, so \(\text{rep}(L(G)) \geq \frac{m}{2D-1}\).
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Case 1: \( a \geq m^{1/3} \). \( \text{rep}(L(G)) \geq \frac{m}{2D-1} > \frac{a}{2} \geq \frac{1}{2} m^{1/3} \).
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Case 2: \( a \leq m^{1/3} \). Now \( D = m/a \geq m^{2/3} \).

**Aim:** Let \( v \) be a vertex of degree \( D \).

If some degree occurs more than \( \frac{1}{4} D^{1/2} \) times in \( N_G(v) \), then \( \text{rep}(G) \geq \frac{1}{4} D^{1/2} \geq \frac{1}{4} m^{1/3} \), since \( v \) contributes the same to all incident edges.
Since \( D = \frac{m}{\alpha} \geq m^{2/3} \geq \alpha^2 \), we have \( \alpha \leq D^{1/2} \).
Line Graphs - completion of lower bound

Since $D = m/a \geq m^{2/3} \geq a^2$, we have $a \leq D^{1/2}$.

Let $b_1, \ldots, b_D$ be the degrees of vertices in $N_G(\mathcal{V})$:

$$\sum b_i < 2m = 2aD \leq 2D^{3/2}.$$
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Let $r = \frac{1}{4}D^{1/2}$. If each degree occurs $\leq r$ times in $N_G(\nu)$, then the sum is smallest when the list is packed.
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From the original counting lemma (with $s \geq 1$, $n = D$),

$$\sum b_i > D \cdot 1 + \frac{D}{2} \left( \frac{D}{r} - 1 \right) > \frac{D^2}{2r} = 2D^{3/2}.$$
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Since \( D = \frac{m}{a} \geq \frac{m^{2/3}}{} \geq a^2 \), we have \( a \leq D^{1/2} \).

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\sum b_i > D \cdot 1 + \frac{D}{2} \left( \frac{D}{r} - 1 \right) > \frac{D^2}{2r} = 2D^{3/2}
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The contradiction completes the proof.
**Prop.** For infinitely many $m$, there is a graph $G$ with $m$ edges and $\text{rep}(L(G)) \leq \sqrt{4m/3}$. 
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Pf. Make $G$ a disjoint union of stars. Fix $r$. For $1 \leq i \leq r$, include $\lfloor r/i \rfloor$ stars with $i$ edges.
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At most $r$ edges have edge-degr. $i-1$, so $\text{rep}(L(G)) \leq r$. 
Prop. For infinitely many $m$, there is a graph $G$ with $m$ edges and $\text{rep}(L(G)) \leq \sqrt{4m/3}$.

Pf. Make $G$ a disjoint union of stars. Fix $r$. For $1 \leq i \leq r$, include $\lceil r/i \rceil$ stars with $i$ edges.

At most $r$ edges have edge-degr. $i - 1$, so $\text{rep}(L(G)) \leq r$. For $\frac{r}{j} \geq i > \frac{r}{j+1}$, #edges in $i$-edge stars is at least $ji$. 
**Prop.** For infinitely many $m$, there is a graph $G$ with $m$ edges and $\text{rep}(L(G)) \leq \sqrt{4m/3}$.

**Pf.** Make $G$ a disjoint union of stars. Fix $r$. For $1 \leq i \leq r$, include $\lceil r/i \rceil$ stars with $i$ edges.

At most $r$ edges have edge-degr. $i-1$, so $\text{rep}(L(G)) \leq r$. For $\frac{r}{j} \geq i > \frac{r}{j+1}$, #edges in $i$-edge stars is at least $ji$. Summing over $1 \leq i \leq r$ yields $m \geq \frac{3}{4}r^2$, so

$$\text{rep}(L(G)) \leq r \leq \sqrt{4m/3}.$$
Line graphs of sparse graphs

**Conj.** There is a constant $\alpha$ such that if $G$ has $m$ edges, then $\text{rep}(L(G)) \geq \alpha \sqrt{m}$. 
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**Thm.** If $G$ has avg degree $d$, min degree $s$, and $m$ edges, then $\text{rep}(L(G)) \geq \alpha \sqrt{m} - 1$, where $\alpha = s/\sqrt{cd(cd - s)}$ with $c = 2d - 2s + 1$. 
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**Cor.** If $G$ is a tree, $\text{rep}(L(G)) \geq \sqrt{m/30}$. If $G$ is a triangulation, $\text{rep}(L(G)) \geq \sqrt{m/182}$. 
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Sparse graphs with special structure

Using another technical counting lower bound,

**Cor.** If $G$ is a tree with 1-factor, $\text{rep}(L(G)) \geq m/6$.
If $G$ is maximal outerplanar, $\text{rep}(L(G)) \geq m/14$.
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These differ from sharpness by at most a factor of 2.

**Prop.** For $m \equiv 1 \mod 10$, there is a tree $G$ with $m$ edges having a 1-factor and $\text{rep}(L(G)) = (m - 1)/5$. 
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