Repetition Number of Graphs

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For bounded rep#, no sequence of triangle-free graphs has $\alpha(G) \in o(n)$, but $\exists$ sequence of K_4-free graphs with $\mathrm{rep}(G) \leq 5$ and $\alpha(G) \in o(n)$ (Bollobás [1996]).

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Evidence: True for trees. For trees with perfect matchings, maximal outerplanar graphs, and triangulations with 2-factors, $\mathrm{rep}(L(G)) = \Theta(m)$.

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$$= ra\left(\frac{2s+a-1}{2}\right) + b(s+a) = ns + \frac{n}{2}\left(\frac{n}{r} - 1\right) + \frac{b(r-b)}{2r}.$$

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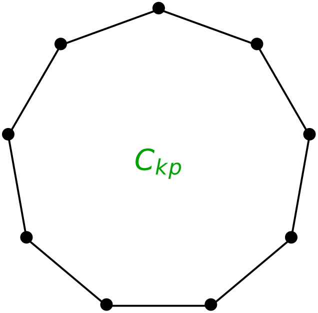

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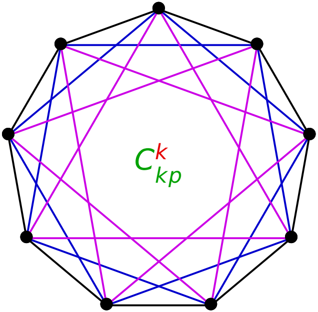

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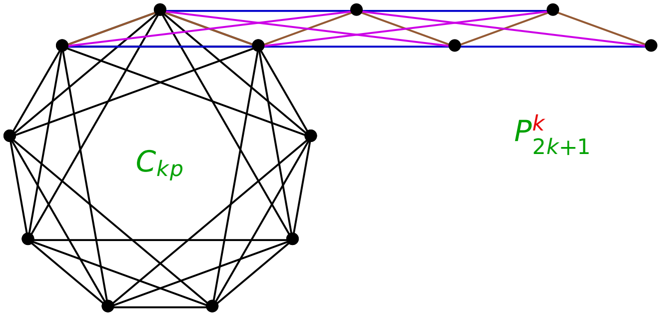

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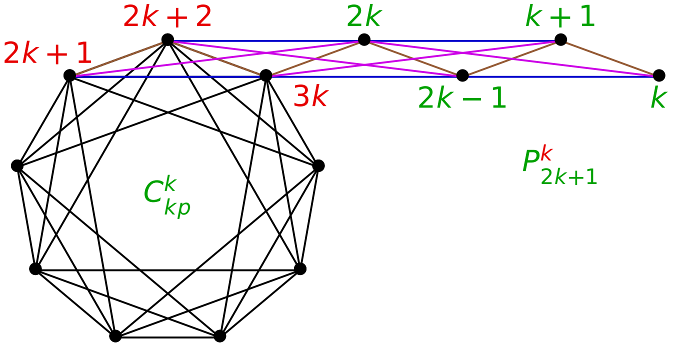

General sharpness argument

Given $n, s, r \in \mathbf{N}$ and d, seek graph whose degree-sum equals the counting bound: $dn = ns + \frac{n}{2}(\frac{n}{r} - 1) + \frac{b(r-b)}{2r}$.

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Sufficiency uses Erdös–Gallai Conditions in this form:
$\sum_{i=1}^{k}(d_i + 1) \le \sum_{k=1}^{k} d_i^*$ for $1 \le k \le \ell(d)$.

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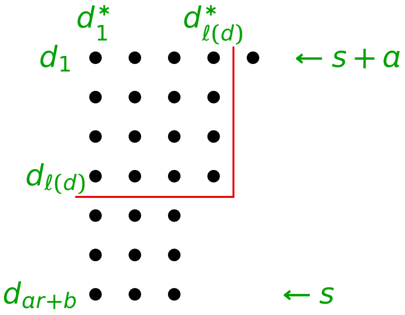

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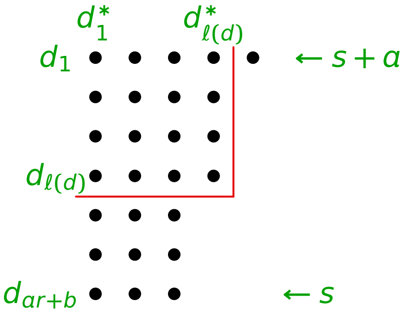

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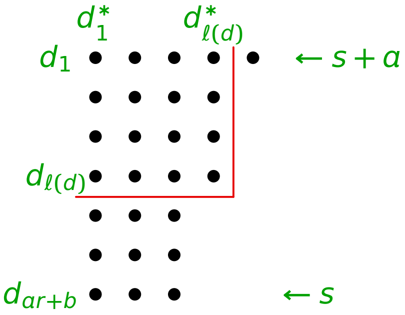

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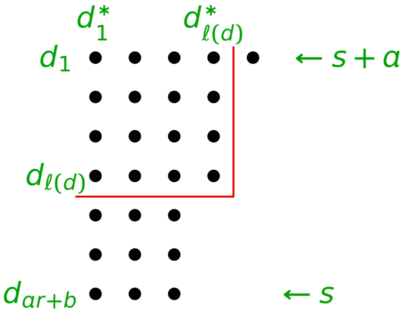

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So, $d_i + 1 \leq d_i^*$ in each term, and sum is okay.

Special Families — $\text{rep}(G) \geq \frac{n}{2d-2s+1}$

Trees: $d = \frac{2n-2}{n} < 2$; $s = 1$; $\text{rep}(G) \geq n/3$.

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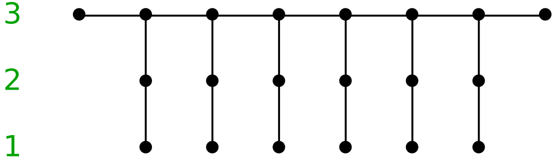

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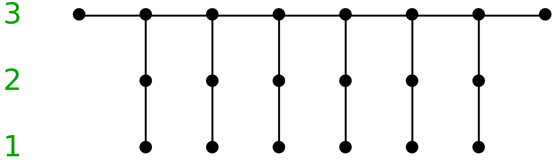

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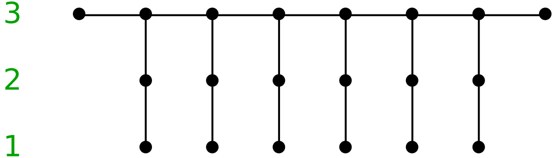

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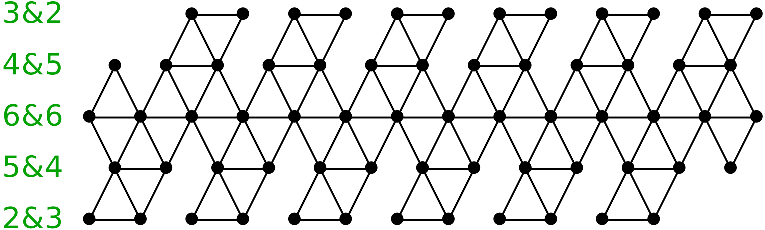

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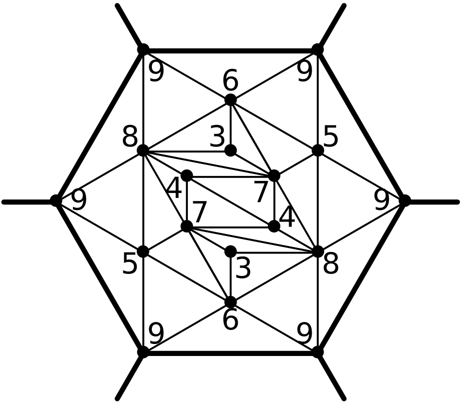

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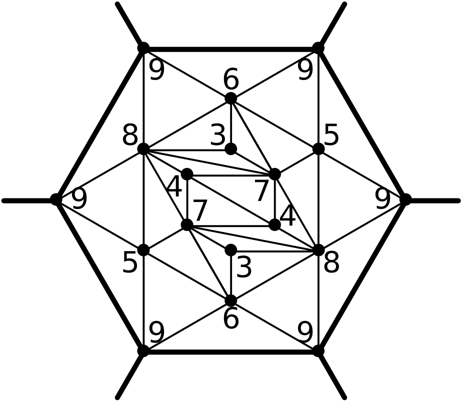
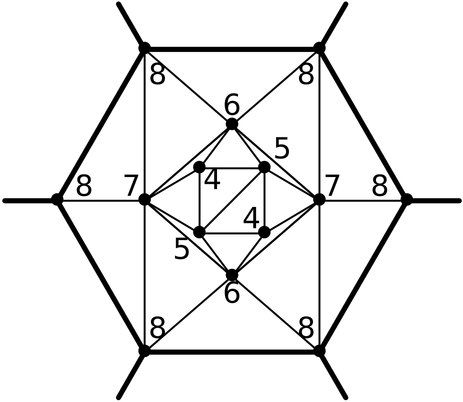


$s = 4 \Rightarrow \operatorname{rep}(G) \geq n/5$



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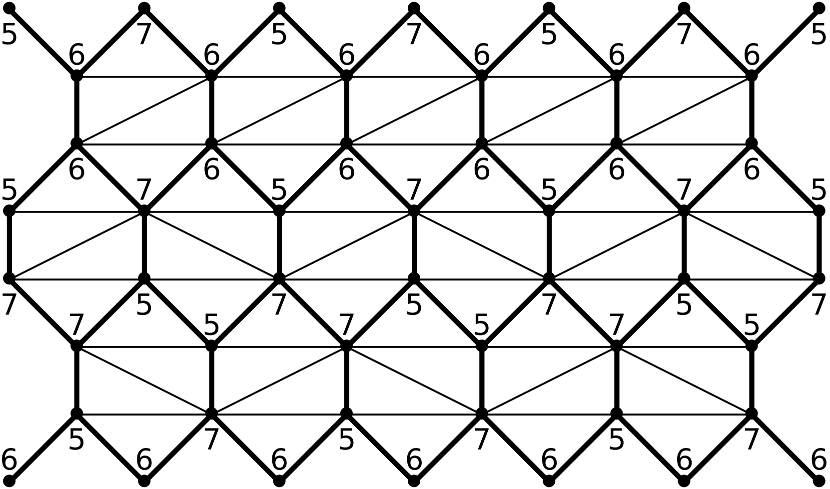
$d < 6$; $s = 5 \;\Rightarrow\; \mathrm{rep}(G) \geq n/3$



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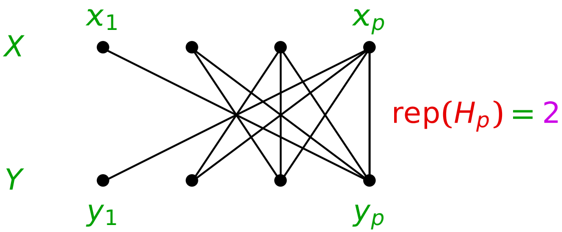

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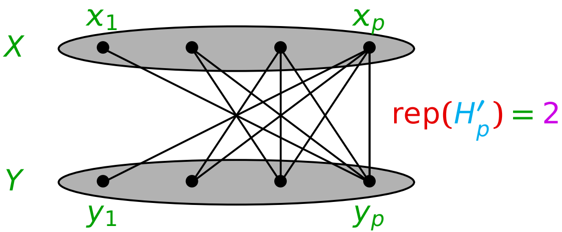

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Form H'_p by completing X and Y to cliques.

$d(x_i) = d(y_i) = i + p - 1$. $\qquad H'_p$ is claw-free.

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Aim: Let v be a vertex of degree D.
If some degree occurs more than $\frac{1}{4}D^{1/2}$ times in $N_G(v)$,
then $\text{rep}(G) \geq \frac{1}{4}D^{1/2} \geq \frac{1}{4}m^{1/3}$,
since v contributes the same to all incident edges.

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$$\sum b_i < 2m = 2aD \leq 2D^{3/2}.$$

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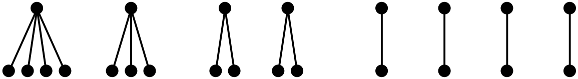

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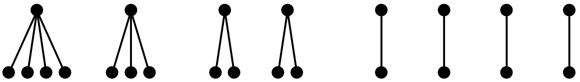

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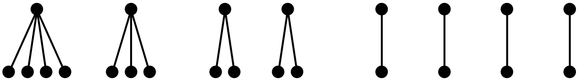

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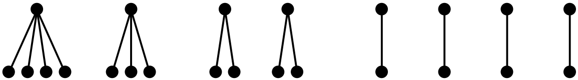

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Summing over $1 \leq i \leq r$ yields $m \geq \frac{3}{4}r^2$, so

$$\mathrm{rep}(L(G)) \leq r \leq \sqrt{4m/3}.$$

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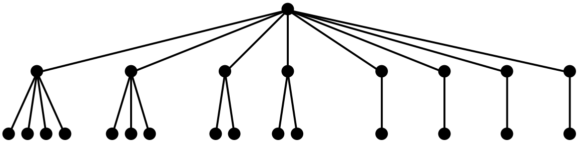

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Using another technical counting lower bound,

Cor. If G is a tree with 1-factor, $\mathrm{rep}(L(G)) \geq m/6$.
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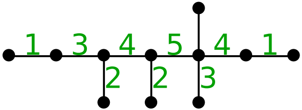

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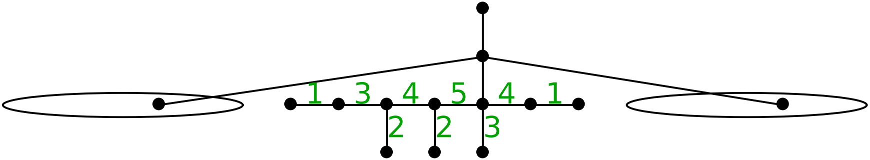

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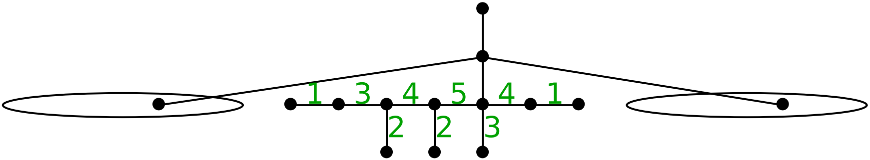

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