RECTANGLE NUMBER FOR HYPERCUBES AND COMPLETE MULTIPARTITE GRAPHS

Yi-Wu Chang
National Politics University, Taipei, Taiwan

Douglas B. West†
University of Illinois, Urbana, IL 61801-2975, west@math.uiuc.edu

Abstract. The *rectangle number* of a graph G is the minimum t such that G is the intersection graph of sets that are unions of t rectangles in the plane with vertical and horizontal sides. We prove that complete multipartite graphs have rectangle number at most two, and that the k-dimensional hypercube has rectangle number at most $\lceil k/4 \rceil$ (except one more when $k = 4$).

1. INTRODUCTION

The *intersection graph* of a family of sets $\{S_1, \ldots, S_n\}$ is the graph with vertex set $\{v_1, \ldots, v_n\}$ defined by making v_i and v_j adjacent precisely when $X_i \cap S_j \neq \emptyset$. The sets in the family form an *intersection representation* of the corresponding graph. The *interval graphs* are those having intersection representations in which each set is an interval on the real line; this special family has been thoroughly studied in hundreds of papers.

Sets generalizing intervals have been used to permit intersection representations of all graphs. Natural parameters measure how much the sets deviate from being intervals. A *d-box* in $\mathbb{R}^d$ is a cartesian product of d intervals. A *t-interval* in $\mathbb{R}^1$ is a union of t intervals. The *boxicity* $\text{box}(G)$ of a graph G is the minimum d such that G is the intersection graph of a family of d-boxes. The *interval number* $i(G)$ is the minimum t such that G is the intersection graph of a family of t-intervals. Boxicity was introduced by Roberts [3]; interval number by Trotter and Harary [7].

The ideas behind these two parameters can be combined. In any fixed dimension d, we can seek the minimum t such that G is the intersection graph of at most t d-boxes. When $d = 2$, the resulting parameter is the *rectangle number* $r(G)$. In this paper, we compute the rectangle number for all complete multipartite graphs (it is always 2 or 1),

---

†Research supported in part by NSA/MSP Grant MDA904-93-H-3040.
Running head: RECTANGLE NUMBER
AMS codes: 05C35
Keywords: intersection graph, rectangle number, representation parameter, hypercube, complete multipartite graph

and we derive an upper bound on the rectangle number of the k-dimensional hypercube (it is at most $\lceil k/4 \rceil$, except that it equals 2 when $k = 4$).

The result for complete multipartite graphs is striking in relation to their interval number and boxicity. The full spectrum of box intersection parameters starts with the interval number and ends with 1 when the dimension equals the boxicity. Trotter and Harary [7] proved that $i(K_{m,n}) = \lceil \frac{mn+1}{m+n} \rceil$. Hopkins, Trotter, and West [2] proved that $i(G) = \lceil \frac{mn+1}{m+n} \rceil$ when G is a complete multipartite graph in which the two largest partite sets induce $K_{m,n}$, except in rare special cases where it is larger by 1.

Roberts [3] proved that the boxicity of a complete multipartite graph equals the number of partite sets with size at least 2. The boxicity can be interpreted as the number of dimensions that must be allowed for the "multiplicity" of representation to decline from $\lceil \frac{mn+1}{m+n} \rceil$ to 1. Surprisingly, almost all of the collapse in multiplicity occurs on the first step: the rectangle number of every complete multipartite graph is at most 2.

Best-possible bounds on the full spectrum are known for the family of planar graphs. For every planar graph, the interval number is at most 3 [5], the rectangle number is at most 2 [4], and the boxicity is at most 3 [6]. Thus in each dimension, the dimension plus the multiplicity is at most 4, and these bounds are sharp.

For general n-vertex graphs, the interval number is at most $\lceil (n+1)/4 \rceil$ [1], and the boxicity is at most $\lceil n/2 \rceil$ [3]. We do not know the maximum rectangle number for n-vertex graph or any general upper or lower bounds on how the maximum d-box intersection number for n-vertex graphs declines as d increases from 1 to $\lceil n/2 \rceil$. Even for the hypercubes, we have not proved a general lower bound, but we believe that the rectangle number of hypercubes increases linearly with the dimension, and thus that the maximum rectangle number increases at least logarithmically with the number of vertices.

When we speak of the horizontal projection or vertical projection of a rectangle, we mean its projection on the horizontal or vertical axis, respectively. In other words, it is the first or second factor, respectively, in the description of the rectangle as a cartesian product of intervals.

2. COMPLETE MULTIPARTITE GRAPHS

THEOREM 1. If G is a complete multipartite graph, then $r(G) \leq 2$, with equality if and only if G has at least three partite sets of size at least 2.

Proof: If G does not have at least three partite sets of size at least 2, then by Roberts' result [3] the boxicity of G is at most 2, which allows G to be represented using one rectangle per vertex. Indeed, the vertices of one nontrivial partite set can be represented by parallel thin vertical strips, the other vertices by parallel thin horizontal strips, and the vertices in partite sets of size 1 by rectangle that intersect everything.

Conversely, suppose that $r(G) = 1$. Let H be an copy of $C_4 = K_{2,2}$ as an induced subgraph in G, with partite sets X and Y. Since X and Y induce $\overline{K}_2$, we may assume by symmetry that the rectangles for X have disjoint horizontal projections. The horizontal projections for Y must both contain the gap between these and therefore intersect. Thus the rectangles for Y have disjoint vertical projections, and the vertical projections for X intersect. If G contains $K_{2,2,2}$, then the three pairwise edge-disjoint copies of C_4 in this subgraph cannot all satisfy this. Two of the three pairs of independent points have rectangles with disjoint projections in the same direction; thus these cannot represent C_4.

Finally, we construct a rectangle representation with two rectangles per vertex for G. Consider a block adjacency matrix for G with the vertices grouped by partite sets. We obtain a representation from the portion of this matrix above the diagonal. We view the ones in each row as a thin horizontal rectangle for that vertex. We view the ones in each column as a thin vertical rectangle for that vertex. Each partite set is represented by two families of thin parallel rectangles. The rectangles do not extend as far as the "diagonal", so they intersect if and only if the corresponding vertices belong to distinct partite sets. Fig. 1 illustrates the construction. ■

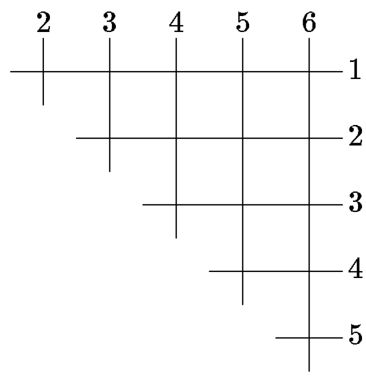

Fig. 1. Rectangle representation for complete multipartite graph.

3. HYPERCUBES

The 4-dimensional hypercube is an annoying complication in our inductive construction of rectangle representations for hypercubes. The induction step is quite easy, but $r(Q_4) = 2$ requires us to provide explicit representations for Q_8 and Q_{12}. A *t-rectangle representation* is an intersection representation assigning each vertex the union of at most t rectangles in the plane (with horizontal and vertical sides). The *multiplicity* of a vertex in a representation is the number of rectangles assigned to it.

In the k-dimensional hypercube Q_k, we view the vertices as subsets of $\{1, \dots, k\}$ adjacent when their subset labels differ by one element. We begin by analyzing Q_4.

LEMMA 2. In every rectangle representation of Q_4, at least two vertices have multiplicity at least two.

Proof: Let f be a rectangle representation of Q_4. We prove that within distance 3 of an arbitrary vertex v in Q_4, there is a vertex with multiplicity at least two. Hence some vertex has multiplicity at least two, and applying the argument again to the complement of that vertex yields a second such vertex.

Without loss of generality, let $v = \emptyset$, and suppose that every vertex other than $\{1, 2, 3, 4\}$ has multiplicity one. Let R_u denote the unique rectangle assigned to u (dropping set brackets). Because Q_4 is triangle-free, no assigned rectangle is contained in another. Also, the rectangle $R_\emptyset$ for v intersects four pairwise disjoint rectangles R_1, R_2, R_3, R_4 for its neighbors. For $i, j \in \{1, 2, 3, 4\}$, we claim that R_i and R_j intersect a common side of $R_\emptyset$. If not, then R_{ij} will also intersect $R_\emptyset$, as shown in Fig. 2.

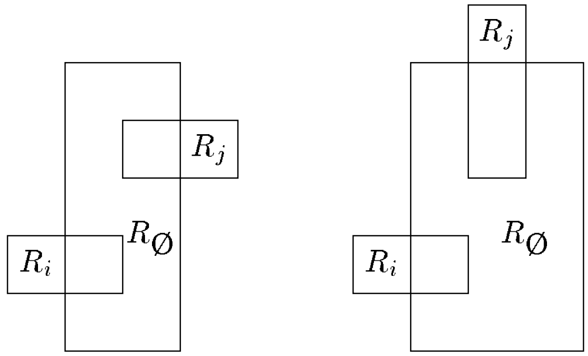

Fig. 2. Restrictions on neighbors in representation of Q_4.

Without loss of generality, we may assume that R_1, R_2, R_3, R_4 intersect the left side of $R_{\emptyset}$ in that order from top to bottom (they may also extend past the right side of $R_{\emptyset}$). Since R_{13} extends vertically from R_1 to R_3 but misses R_2, it must lie entirely to the left or right of $R_{\emptyset}$; by symmetry, we may assume that it lies to the left. Now R_{13} blocks the leftward advance of R_2, and R_3 extends farther left than R_2. Hence R_4 must be to the right of $R_{\emptyset}$ and block the rightward advance of R_3.

Next, R_{23} must extend across the vertical gap between R_2 and R_3 but avoid $R_{\emptyset}$. Hence R_{23} is entirely to the left or to the right of $R_{\emptyset}$; we may assume by symmetry that it lies to the right. Fig. 3 displays the conclusions that will follow from this.

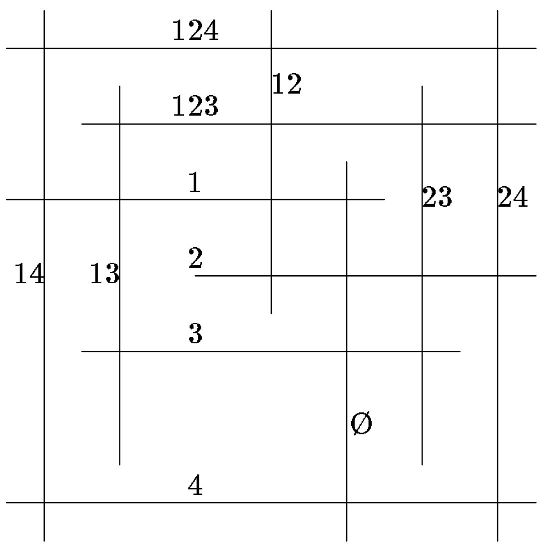

Fig. 3. Arrangement of singleton rectangles in representation of Q_4.

Since R_{23} must meet R_3 and avoid R_{24}, the horizontal extent of R_{23} is now confined between $R_{\emptyset}$ and R_{24}. Since R_{23} must meet R_3 and avoid R_4, the vertical extent of R_{23} is bounded below by R_4. Since R_{123} must meet R_{13} and R_{23} and avoid R, and since R_{23} is bounded below by R_4, we conclude that R_{123} extends from R_{13} to R_{23} above $R_{\emptyset}$. Hence R_{23} blocks the rightward advance of R_1.

Since R_{14} extends vertically from R_1 to R_4, and R_2 extends past the right end of R_1 between them, R_{14} must lie to the left of $R_{\emptyset}$. Indeed, it must also lie to the left of R_3. Now, R_{12} must extend vertically from R_2 to R_1 and must avoid R_{13} and R_{23}. Hence the horizontal extent of R_{12} is bounded between R_{13} and R_{23}. Furthermore, the vertical extent of R_{12} is bounded below by R_3. Since R_{12} and $R_\emptyset$ each extend vertically between R_1 and R_2, their horizontal projections are disjoint; either may be leftmost.

Next, R_{124} must extend horizontally between R_{14} and R_{24}, meeting R_{12} and avoiding R_{13} and R_{23}. Since R_{12} lies above R_3, we conclude that R_{124} lies above R_{123}, and that all of R_{14}, R_{12}, R_{24} extend up far enough to meet it.

We have now located all regions except those whose indices contain both 3 and 4. Both R_{13} and R_{23} lie above R_4. Hence R_{134} and R_{234}, which must avoid R_4, also lie above R_4. Since R_{34} extends vertically between R_3 and R_4 and avoids $R_\emptyset$, it lies entirely to the left or to the right of R. If R_{34} lies to the left [right] of $R_\emptyset$, then R_{234} cannot meet both R_{34} and R_{23} [R_{13}] without crossing $R_\emptyset$, since R_{34} lies below R_1 [R_2].

We have proved that there is no rectangle representation of Q_4 in which every vertex within distance three of a specified vertex has multiplicity 1. ■

LEMMA 3. There is a 2-rectangle representation of Q_4 in which the only vertices with multiplicity 2 are two adjacent vertices x, y, also one rectangle for x emerges vertically in both directions (arbitrarily far) and the other emerges vertically in one direction, and the rectangles for y satisfy the same behavior horizontally.

Proof: By allowing two rectangles for each of 234 and 1234, we can complete the representation begun in Fig. 3 as indicated in Fig. 4. These four rectangles can be extended so that the rectangles for 1234 both extend upward and one extends downward, and the rectangles for 234 both extend rightward and one extends leftward. ■

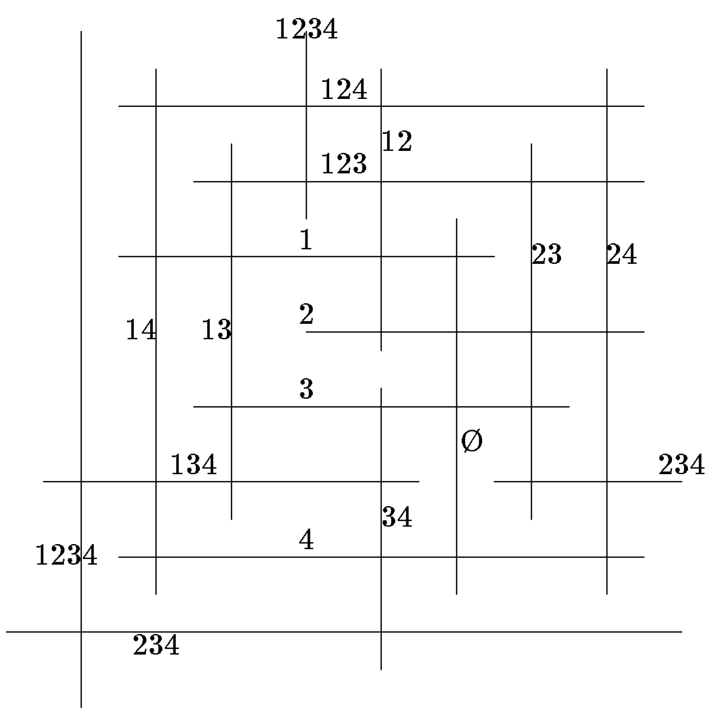

Fig. 4. The canonical 2-rectangle representation of Q_4.

By symmetry, we may choose any adjacent pair of vertices to receive two rectangles each, and we may exchange up/down and right/left at will in this representation. Call this the *canonical representation* of Q_4.

For the main result, we view the vertices of Q_k as binary vectors of length k.

THEOREM 4. The rectangle number of the k-dimensional cube Q_k is at most $\lceil k/4 \rceil$, except that $r(Q_4) = 2$.

Proof: We have proved that $r(Q_4) = 2$. Since Q_4 contains a copy of Q_3 avoiding a pair of adjacent vertices, the canonical representation contains a 1-rectangle representation of Q_3. Since $Q_k \subseteq Q_{k+1}$ for all k, it therefore suffices to prove that $r(Q_{4l} \leq l$ for each $l > 1$.

We use induction on l. For $l > 1$, we view Q_{4l} as $Q_8 \square Q_{4l-8}$, where $\square$ denotes cartesian product of graphs. For each choice of bits in the last $4l - 8$ coordinates of vertices of Q_{4l}, we have a copy of Q_8. If Q_8 has a 2-rectangle representation, then the disjoint union of 2^{4l-8} copies of Q_8 has a 2-rectangle representation. Deleting the edges of these copies of Q_8 leaves 8 disjoint copies of Q_{4l-8}. If $l \geq 4$, then the induction hypothesis allows us to complete the representation using $l - 2$ additional rectangles per vertex.

To complete the proof, we provide constructions for $r(Q_8) = 2$ and $r(Q_{12}) \leq 3$. We use the canonical representation of Q_4 provided by Lemma 3.

2-rectangle representation of Q_8. Express Q_8 as $Q_4 \square Q_4$. View Q_8 as $G \cup H$, where each of G, H is the disjoint union of 16 copies of Q_4. Each vertex appears in one component in each of G and H. Each component of G [resp., H] has a fixed value in the last four [first four] coordinates of its vertex labels. Our representation has eight isomorphic connected *pieces*, each of which represents two components of G and two components of H.

Instead of describing an explicit representation, we describe a class of representations, because we will use eight different 2-rectangle representations of Q_8 to construct a 3-rectangle representation of Q_{12}. We parse each 8-bit label of a vertex of Q_8 as a concatenation $\alpha\beta\gamma\delta$, where α, γ are single bits and β, δ are 3-bit binary vectors. We use vector addition modulo 2. In the 8-bit label of v, we refer to the vector in coordinates 2-4 as $\beta(v)$ and the vector in coordinates 6-8 as $\delta(v)$.

Given a fixed 3-bit vector z, we describe a representation of Q_8; the eight choices for z yield eight different representations, with z as a parameter. Given a 3-bit binary vector x, let $y = x + z$ (modulo 2), and let a, b, c, d respectively denote the four *special* vertices $0x0y, 1x0y, 1x1y, 0x1y$. The subgraph induced by $\{a, b, c, d\}$ is a 4-cycle consisting of one edge from each of two components of G and one edge from each of two components of H. We label these four copies of Q_4 (components in G and H) as AB, BC, CD, DA such that $ab \in AB \subset G$, $bc \in BC \subset H$, $cd \in CD \subset G$, and $da \in DA \subset H$.

We represent $AB \cup BC \cup CD \cup DA$ using two rectangles each for a, b, c, d and one rectangle for the remaining vertices. No vertex outside $\{a, b, c, d\}$ appears in more than one of these four subgraphs. We use four copies of the canonical representation of Q_4, extending the two rectangles for one of $\{a, b, c, d\}$ out vertically in one direction and extending the two rectangles for the appropriate neighbor among $\{a, b, c, d\}$ out horizontally in one direction. The two parallel rectangles for a emerging from the representation of DA serve also as the two parallel rectangles for A emerging from the representation of AB, and similarly for b, c, d. The resulting representation is illustrated in Fig. 5.

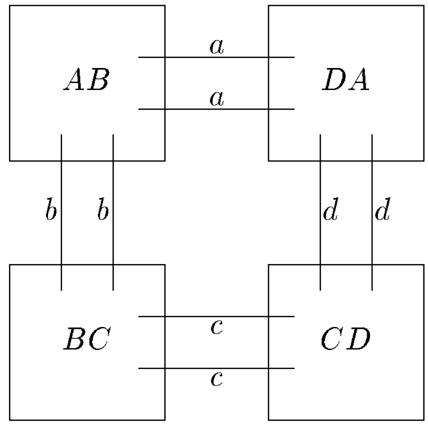

Fig. 5. A *piece* in the representation of Q_8.

For each of the eight choices for x, we have such a piece in our representation f_z. A vertex v is a special vertex (a, b, c, d) in some piece if and only if $\delta(v) = \beta(v) + z$ (modulo 2). A special vertex v appears only in the piece generated by $x = \beta(v)$, and it is assigned two rectangles in representing that component.

A non-special vertex u appears in the piece generated by $x = \beta(u)$ and also in the pieces generated by $x = \delta(u) - z$, assigned one rectangle in each piece. Thus we have represented the 32 pairwise edge-disjoint copies of Q_4 in eight disjoint pieces, obtaining a 2-rectangle representation of Q_8.

3-rectangle representation of Q_{12}. Express Q_{12} as $Q_8 \Box Q_4$. We represent the 16 copies of Q_8 in the manner described above; in total we have 128 pieces. These pieces contain canonical representations of Q_4 in which we have made extra use of the parallel "tails" emerging in common directions. To avoiding adding two intervals for vertices in the remaining 256 disjoint copies of Q_4, we will also make extra use of the third tail emerging in the opposite direction. Each of the 128 pieces has 8 tails emerging (we will use one for each of the four special vertices), and each of the 256 copies of Q_4 needs to use 2 of these tails. We produce a representation in connected *modules*, each containing two of the previously-constructed pieces.

We parse each 12-bit vertex label in Q_{12} as a concatenation $\alpha\beta\gamma\delta\epsilon\zeta$, where α, γ, ϵ are single bits and β, γ, ζ are 3-bit binary vectors. In the 12-bit label of v, we refer to the vector in coordinates 2-4 as $\beta(v)$, in coordinates 6-8 as $\delta(v)$, and in coordinates 10-12 as $\zeta(v)$.

Our copies of Q_8 in Q_{12} have vertex labels fixed in the last four coordinates. Using the ninth coordinate, we view these copies of Q_8 in eight pairs. In the pair for which the last three coordinates are fixed at $\zeta(v) = z$, we use the representation f_z for each of the two copies of Q_8. We combine corresponding pieces of these two representations (and with the pair pick up four of the remaining copies of Q_4) to produce $8 \times 8 = 64$ modules for our full representation.

Let a, b, c, d be the four special vertices in one piece of the representation f_z on the copy of Q_8 having $0z$ in the last four coordinates. Let a', b', c', d' be the corresponding vertices having 1 in the ninth coordinate. Note that w and w' are adjacent, for $w \in \{a, b, c, d\}$. Each of these eight special vertices appears in one un-represented copy of Q_4

in which the first eight coordinates are fixed. Both w and w' appear (and are adjacent) in a single such subgraph, which we call $F(w)$.

We represent $F(w)$ using the canonical representation, letting w, w' be the adjacent pair using two rectangles each. Since we already have used two rectangles for each of w, w' in the pieces of f_z, we can afford only one additional rectangle. This we achieve for each of w, w' by extending a rectangle from a piece of f_z to become a rectangle in the representation of $F(w)$ (one horizontal, one vertical). The explicit geometric arrangement appears in Fig. 6. For fixed z, let f'_z denote the eight modules formed in this way.

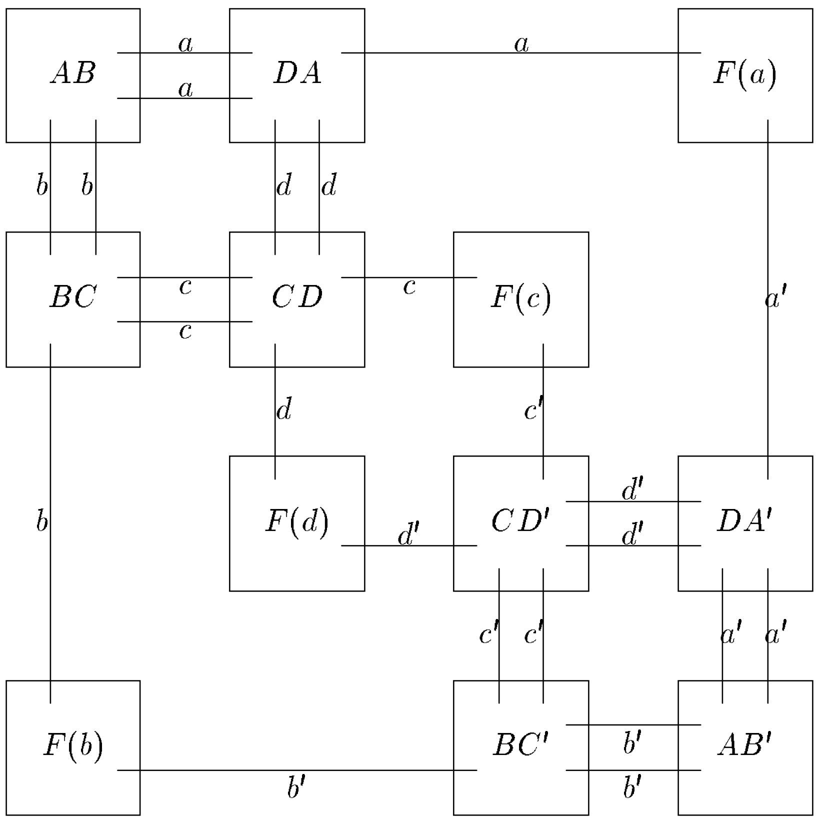

Fig. 6. A *module* in the representation of Q_{12}.

We have one pair of Q_8's for each choice of z. For each special vertex w in each module of f'_z, we have $\delta(w) = \beta(w) + z$ and $\zeta(w) = z$. For each non-special vertex v that belongs to some $F(w)$ in some module of f'_z, we have $\delta(v) = \beta(v) + z$ but $\zeta(v) \neq z$. Thus a vertex v of Q_{12} occurs as a special vertex in some module of some f'_z if and only if $\delta(v) = \beta(v) + \zeta(v)$.

If $\delta(v) = \beta(v) + \zeta(v)$, then v occurs as a special vertex in $f'_{\zeta(v)}$. The edges incident to v via changes in the first eight coordinates occur in an $f_{\zeta(v)}$-piece in an $f'_{\zeta(v)}$-module, and the edges via changes in the last four coordinates occur in the $F(v)$-portion of the $f'_{\zeta(v)}$-module (or $F(w)$ if $v = w'$), which assigns a third rectangle to v. Furthermore, no rectangles for v appear in f'_z with $z \neq \zeta(v)$.

If $\delta(v) \neq \beta(v) + \zeta(v)$, then v never occurs as a special vertex. In the pair of Q_8's corresponding to $z = \delta(v) - \beta(v)$ and represented by f'_z, there are two intervals assigned to v, and all edges incident to v via changes in the first eight coordinates are represented.

Also, v receives one more interval in f'_z, taking care of its incident edges via changes in the last four coordinates. This rectangle appears in $F(w)$ where w agrees with v in the first eight or nine coordinates and is a special vertex.

We have verified that the union of the eight modules of the form f'_z is a 3-rectangle representation of Q_{12}.

REFERENCES

[1] J.R. Griggs, Extremal values of the interval number of a graph, II. *Discrete Math.* 28(1979), 37-47.

[2] L.B. Hopkins, W.T. Trotter, and D.B. West, The interval number of a complete multipartite graph. *Discrete Appl. Math.* 8(1984), 163–187.

[3] F.S. Roberts, On the boxicity and cubicity of a graph. In *Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968)* (Academic Press, 1969), 301–310.

[4] E.R. Scheinerman, *Intersection classes and multiple intersection parameters of graphs.* Ph.D. thesis, Princeton University, 1984.

[5] E.R. Scheinerman and D.B. West, The interval number of a planar graph: three intervals suffice. *J. Combin. Theory Ser. B* 35 (1983), 224–239.

[6] C. Thomassen, Interval representations of planar graphs. *J. Combin. Theory Ser. B* 40(1986), 9–20.

[7] W.T. Trotter and F. Harary, On double and multiple interval graphs. *J. Graph Theory* 2(1978), 137-142.