On r-dynamic Coloring of Graphs

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slides available on DBW preprint page

Joint work with
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k-colorable - having a proper coloring with $\leq k$ colors.
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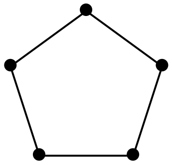
- Always $\chi_r(G) \geq \chi(G)$, and $\chi_1(G) = \chi(G)$.

Ex. $\chi_2(C_5) = 5$. (No two vertices can have same color.)



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$\mathrm{ch}_2(G) \le 4$ for planar graphs with girth at least 7, and
$\mathrm{ch}_2(G) \le 5$ for all planar graphs (Kim–Lee–Park [2011,13]).

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Theme: How do upper bounds on $\chi(G)$ need to be relaxed to obtain bounds on $\chi_r(G)$ as r increases?

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$\mathbb{P}[\ \exists$ bad vertex$] < n^r e^{-\delta(G)\frac{s+1}{r+s}} < n^r n^{-r} = 1$. ∎

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Idea: For $\chi_r(G) \le r\chi(G)$, pair an optimal proper coloring with a random coloring having r colors in each nbhd. (Taherkhani [2014] gave a similar result.)

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Applying the lemma implements the **idea**. ■

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Now $\frac{\chi_r(G)}{r\chi(G)} \approx \frac{1}{t}\sqrt{\frac{4}{3}}\left(\frac{27}{16}\right)^t = r^x$, where $x = \frac{3\lg 3}{2} - 2$. ∎

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What bounds hold for larger diameter or χ_r with $r > 2$?

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The k-sets have degree $k + 1$ and common neighbor v.
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Making v adjacent also to all of $[n]$ yields $\mathrm{diam}(G) = 2$ and $\chi(G) = 3$; still $\chi_r(G) \geq n+1$ when $r > k \geq 2$. ∎

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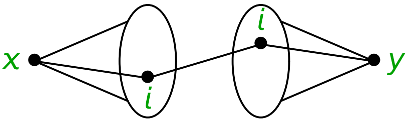

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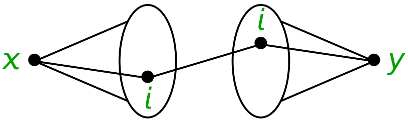

In an intersecting hypergraph, use two colors on a minimal edge and a third color on the other vertices. ■

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c	d	a	b	c	d	a	b	c	d	a	b	c	d
b	a											b	a
d	c											d	c
a	b											a	b
c	d	b	a	c	d	b	a	c	d	b	a	c	d
b	a	c	d	b	a	c	d	b	a	c	d	b	a

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a	b											a	b
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b	*a*	*b*										*b*	*a*
d	*c*	*d*										*d*	*c*
a	*b*											*a*	*b*
c	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*
b	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*

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b	*a*	*b*	*c*	*d*								*b*	*a*
d	*c*	*d*										*d*	*c*
a	*b*											*a*	*b*
c	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*
b	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*

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b	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*		*b*	*a*
d	*c*	*d*										*d*	*c*
a	*b*											*a*	*b*
c	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*
b	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*	*c*	*d*	*b*	*a*