On $r$-dynamic Coloring of Graphs

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slides available on DBW preprint page

Joint work with
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The Problem

**Def.** proper coloring - neighbors have distinct colors. 

$k$-colorable - having a proper coloring with $\leq k$ colors. 

chromatic number $\chi(G)$ - $\min\{k: G \text{ is } k\text{-colorable}\}$. 

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• Always \( \chi_r(G) \geq \chi(G) \), and \( \chi_1(G) = \chi(G) \).

**Ex.** \( \chi_2(C_5) = 5 \). (No two vertices can have same color.)
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**Def.** List analogue: \( \text{ch}_r(G) \) is the least \( k \) such that an \( r \)-dynamic coloring of \( G \) can be chosen from any lists of \( k \) colors at the vertices.
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$\text{ch}_2(G) \leq 4$ for planar graphs with girth at least 7, and $\text{ch}_2(G) \leq 5$ for all planar graphs (Kim–Lee–Park [2011,13]).
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**Theme:** How do upper bounds on $\chi(G)$ need to be relaxed to obtain bounds on $\chi_r(G)$ as $r$ increases?
Our Results

**Thm.** \( \chi_r(G) \leq r\Delta(G) + 1 \) (greedy coloring algorithm)
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**Thm.** $\chi_r(G) \leq \Delta(G) + 2r - 2$ when $\delta(G) > 2r \ln n$. Also, $\chi_r(G) \leq \Delta(G) + r$ when $\delta(G) > r^2 \ln n$. 
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**Thm.** $\chi_2$ is unbounded when $\chi(G) = 2$ and $\text{diam}(G) = 4$. 
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**Thm.** If $|V(G)| = n$, and $\delta(G) > \frac{r+s}{s+1} r \ln n$, then $\chi_r(G) \leq \Delta(G) + r + s$. 
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With positive probability, the coloring is \(r\)-dynamic. This fails at \(v\) only if \(N(v)\) is colored from \(r - 1\) colors.
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$$\mathbb{P}[\text{given } (r - 1)-\text{set bad}] \leq \left(\frac{r-1}{r+s}\right)^{\delta(G)} \leq e^{-\delta(G)\frac{s+1}{r+s}}.$$
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$\#(r - 1)$-sets $= \left(\frac{\Delta(G)+r+s}{r-1}\right) < n^{r-1}$, since $\Delta(G) + r + s < n$. 
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$(r - 1)$-sets = $\binom{\Delta(G)+r+s}{r-1} < n^{r-1}$, since $\Delta(G) + r + s < n$.

Since $G$ has $n$ vertices and $\delta(G) > \frac{r+s}{s+1}r \ln n$,
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#($r - 1$)-sets $= \binom{\Delta(G) + r + s}{r-1} < nr^{-1}$, since $\Delta(G) + r + s < n$.

Since $G$ has $n$ vertices and $\delta(G) > \frac{r+s}{s+1} r \ln n$,

$P[\exists \text{ bad vertex}] < n^{r} e^{-\delta(G)\frac{s+1}{r+s}} < n^{r} n^{-r} = 1.$
Bound in Terms of $\chi(G)$

**Idea:** For $\chi_r(G) \leq r\chi(G)$, pair an optimal proper coloring with a random coloring having $r$ colors in each nbhd. (Taherkhani [2014] gave a similar result.)
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**Lem.** A $k$-uniform $H$ with $\Delta(H) = D$ has an $r$-coloring with all colors on each edge if $re^{-k/r}(k(D-1)+1) \leq e^{-1}$. 
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**Pf.** The nbhd hypergraph is $k$-uniform and $k$-regular. The last inequality on $k$ yields the needed condition. Applying the lemma implements the idea.
Regular with Small Degree

**Def.** Kneser graph $K(n, t)$: vertex set $\binom{[n]}{t}$, with adjacency being disjointness. It is $\binom{n-t}{t}$-regular.

**Thm.** For infinitely many $r$, there is an $r$-regular graph $G$ such that $\chi_r(G) > r^{1.37744}\chi(G)$.

**Pf.** Let $G = K(3t - 1, t)$ and $r = \binom{n-t}{t}$. 
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For $c \in \{\frac{1}{2}, \frac{1}{3}\}$, we use $\binom{m}{cm} \approx \frac{(c^c(1-c)^{1-c})^{-m}}{\sqrt{c(1-c)2\pi m}}$ (Stirling).
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It remains to express $\chi_r(G)/\chi(G)$ in terms of $r$. We have $r = \binom{2t-1}{t} = \frac{1}{2} \binom{2t}{t}$ and $\chi_r(G) = \binom{3t-1}{t} = \frac{2}{3} \binom{3t}{t}$. For $c \in \{\frac{1}{2}, \frac{1}{3}\}$, we use $\binom{m}{cm} \approx \frac{(c^c(1-c)^{1-c})^{-m}}{\sqrt{c(1-c)2\pi m}}$ (Stirling).

Now $\frac{\chi_r(G)}{r\chi(G)} \approx \frac{1}{t} \sqrt{\frac{4}{3}} \left(\frac{27}{16}\right)^t = r^x$, where $x = \frac{3\lg\frac{3}{2}}{2} - 2$. $\blacksquare$
Graphs with Small Diameter

**Thm.** If $\text{diam}(G) = 2$, then $\chi_2(G) \leq \chi(G) + 2$. 
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If $f(N(v)) = \{a\}$, then give a new color $b$ to $v$ and a new color $c$ to some $x \in N(v)$ (we may assume $\delta(G) \geq 2$).
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What bounds hold for larger diameter or $\chi_r$ with $r > 2$?
Constructions

**Thm.** $\chi_2$ is unbounded when $\chi(G) = 2$ and $\text{diam}(G) = 4$. 
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**Pf.** For $\chi_2$ on bipartite with diameter 4, subdivide every edge of $K_n$. The $n$ original vertices need distinct colors.
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**Pf.**
For $\chi_2$ on bipartite with diameter 4, subdivide every edge of $K_n$. The $n$ original vertices need distinct colors.

For $\chi_3$ on bipartite with diameter 3, start with the incidence $[n], (\binom{n}{k})$-bigraph: $j \leftrightarrow A$ if $j \in A$. 
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For \( \chi_3 \) on bipartite with diameter 3, start with the incidence \([n], \binom{n}{k}\)-bigraph: \( j \leftrightarrow A \) if \( j \in A \).

Add \( \nu \) adjacent to \( \binom{n}{k} \), still bipartite.

The \( k \)-sets have degree \( k + 1 \) and common neighbor \( \nu \).

Distance between a \( k \)-set and an element not in it is 3.

Elements of \([n]\) lie in a common \( k \)-set. \( \therefore \text{diam}(G) = 3 \).
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For \(\chi_3\) on bipartite with diameter 3, start with the incidence \([n], \binom{n}{k}\)-bigraph: \(j \leftrightarrow A\) if \(j \in A\).

Add \(v\) adjacent to \(\binom{n}{k}\), still bipartite. The \(k\)-sets have degree \(k + 1\) and common neighbor \(v\). Distance between a \(k\)-set and an element not in it is 3. Elements of \([n]\) lie in a common \(k\)-set. \(\therefore \text{diam}(G) = 3\).

If \(r > k \geq 2\), then the \(k + 1\) neighbors of a \(k\)-set have distinct colors: \(\chi_r(G) \geq n + 1\).
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Elements of \([n]\) lie in a common \( k \)-set. \( \therefore \text{diam}(G) = 3 \).
If \( r > k \geq 2 \), then the \( k + 1 \) neighbors of a \( k \)-set have distinct colors: \( \chi_r(G) \geq n + 1 \).

Making \( v \) adjacent also to all of \([n]\) yields \( \text{diam}(G) = 2 \) and \( \chi(G) = 3 \); still \( \chi_r(G) \geq n + 1 \) when \( r > k \geq 2 \).
Theorem. $\chi_2(G) \leq 3\chi(G)$ when $\text{diam}(G) = 3$, which is sharp.
χ₂ on Graphs with Diameter 3

**Thm.** $\chi_2(G) \leq 3\chi(G)$ when $\text{diam}(G) = 3$, which is sharp.

**Pf. Sharpness:** Form $G$ from $K_{3k}$ by subdividing each edge in $k$ disjoint triangles: $\chi(G) = k$ and $\chi_2(G) = 3k$. 
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**Idea:** Pair a proper coloring $f$ with a 3-coloring that puts two colors in each nbhd that does not already have two colors under $f$. 
\( \chi_2 \) on Graphs with Diameter 3

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Let \( V_i = \{ v : f(v) = i \} \) and \( H_i = \{ \text{vertex nbhds in } V_i \} \). 
Theorem. \( \chi_2(G) \leq 3 \chi(G) \) when \( \text{diam}(G) = 3 \), which is sharp.

Proof. Sharpness: Form \( G \) from \( K_{3k} \) by subdividing each edge in \( k \) disjoint triangles: \( \chi(G) = k \) and \( \chi_2(G) = 3k \).

Idea: Pair a proper coloring \( f \) with a 3-coloring that puts two colors in each nbhd that does not already have two colors under \( f \).

Let \( V_i = \{ v : f(v) = i \} \) and \( H_i = \{ \text{vertex nbhds in } V_i \} \). If \( N(x) \) and \( N(y) \) all have color \( i \), then \( N(x) \cap N(y) \neq \emptyset \).
χ₂ on Graphs with Diameter 3

Thm. χ₂(G) ≤ 3χ(G) when diam(G) = 3, which is sharp.

Pf. Sharpness: Form G from K₃ₖ by subdividing each edge in k disjoint triangles: χ(G) = k and χ₂(G) = 3k.

Idea: Pair a proper coloring f with a 3-coloring that puts two colors in each nbhd that does not already have two colors under f.

Let Vᵢ = {v: f(v) = i} and Hᵢ = {vertex nbhds in Vᵢ}. If N(x) and N(y) all have color i, then N(x) ∩ N(y) ≠ ∅.

In an intersecting hypergraph, use two colors on a minimal edge and a third color on the other vertices.
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\[
\begin{array}{cccccccccccccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} \\
\text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} \\
\text{b} & \text{a} & \text{b} & \text{a} \\
\text{d} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{a} \\
\text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a}
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  a b c d a b c d a b c d a b  
  c d a b c d a b c d a b c d  
  b a b c d                  b a  
  d c d                    d c  
  a b  a b  
  c d b a c d b a c d b a c d  
  b a c d b a c d b a c d b a  
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a b c d a b c d a b c d a b c d a b
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b a b c d a b c d a b c d a b c d
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d c d a b c d a b c d a b c d
a b
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