Rainbow Matching
in Edge-Colored Graphs

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Joint work with
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![Latin square](image)

Latin square $\iff$ Edge-colored $K_{n,n}$
**Background**

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![Latin square diagram]

**Latin square** $\iff$ **Edge-colored $K_{n,n}$**

**transversal** $\iff$ **Rainbow perfect matching**
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**Thm.** (X.Li–Z.Xu [2009]) The conjecture holds for properly edge-colored complete graphs.
New Results

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![Diagram of $K_4$ with colors]

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**Thm.** (L–S–W–W) Each condition below yields a rainbow matching of size at least $\lceil \delta(G)/2 \rceil$.

(a) $G$ has more than $\frac{3(\delta(G)-1)}{2}$ vertices.
(b) $G$ is triangle-free.
(c) The coloring is proper, $G \neq K_4$, and $|V(G)| \neq \delta(G)+2$. 

Notation

Fix an edge-colored graph $G$. Let $k = \delta(G)$ and $n = V(G)$. 
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Let $H = G - V(M)$, with $p$ vertices: $p = n - (k - 2c) \geq 2c + 2$. 
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Delete edges within $V(M)$ and $V(H)$ to form $A$.

If $u \in V(M)$, then $\hat{d}_A(u) \geq \hat{\delta}(G) - (2|M| - 1) = 2c + 1$. (1)
**Def.** A color used in $G$ but not in $M$ is free. Let $B$ be the spanning subgraph of $A$ whose edges have free colors. Let $f(S) = \sum_{x \in S} f(x)$ when $f$ is a function on vertices.
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By the choice of $M$, colors in $H$ are not free, so

$$\hat{d}_B(V(H)) \geq p(k/2 + c) \quad (2)$$
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For the upper bound, group $E(B)$ by the endpoints in $M$: Let $M = \{u_j, v_j : 1 \leq j \leq |M| \}$. Let $B_j$ be the subgraph of $B$ induced by $V(H) \cup \{u_j, v_j\}$. 
Slices of $B$

**Lem.** If $\hat{d}_{B_j} \geq 1$ for $w_1, w_2, w_3$ in $V(H)$, then only one can have $\hat{d}_{B_j} = 2$. Furthermore,

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**Pf.** Suppose $\hat{d}_{B_j}(w_1) = \hat{d}_{B_j}(w_2) = 2$ and $w_3 v_j \in E(B_j)$. 

![Diagram of $B$ with vertices $u_j$, $v_j$, $w_1$, $w_2$, $w_3$, and $M$, $H$]
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$\hat{d}_{B_j}(w_2) = 2 \Rightarrow \phi(u_j w_2) \neq \phi(v_j w_2)$. 

\begin{itemize}
  \item $M$
  \item $u_j$
  \item $w_1$
  \item $w_2$
  \item $w_3$
  \item $H$
  \item $\nu_j$
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Now $w_3 v_j$ enlarges $M$ with $u_j w_1$ or $u_j w_2$. 

![Diagram](attachment:image.png)
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Note $p \geq 2c + 2$ and $c \geq 1/2$ imply $p \geq 3$. Since $\hat{d}_{B_j}(w) \leq 2$ for $w \in V(H)$, $\hat{d}_{B_j}(V(H)) \geq p + 2$ requires a bad triple. 

[Diagram showing the relationship between $M$, $w_1$, $w_2$, $w_3$, $u_j$, and $v_j$.]
What is Left to Do?

There are $k/2 - c$ values of $j$, so (2) and (3) imply
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Lem. If \( p \geq 4 \) and \( \hat{d}_{B_j}(V(H)) = p + 1 \) for some \( j \), then \( c \leq 1/2 \), the edge-coloring is not proper, and \( K_3 \subseteq G \).
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**Lem.** If \( p \geq 4 \) and \( \hat{d}_{B_j}(V(H)) = p + 1 \) for some \( j \), then \( c \leq 1/2 \), the edge-coloring is not proper, and \( K_3 \subseteq G \).

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Maximality of $M \implies U = \emptyset$ or $V = \emptyset$, so $\hat{d}_A(v_j) \leq 2$ or $\hat{d}_A(u_j) \leq 2$, but $\hat{d}_A(u) \geq 2c + 1$, so $c \leq 1/2$. ■
**Conclusion**

**Thm.** (LeSaulnier–Stocker–Wenger–West [2009+])

Every edge-colored graph $G$ has a rainbow matching of size $\left\lceil \frac{\delta(G)}{2} \right\rceil$, improving to $\left\lceil \frac{\hat{\delta}(G)}{2} \right\rceil$ under any of:

(a) $G$ has more than $\frac{3(\delta(G)-1)}{2}$ vertices.
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**Pf.** When $p \geq 4$, the Lemma yields $c \leq 1/2$.
When $p \leq 3$, apply $p = n - 2 |M| = n - (k - 2c) \geq 2c + 2$. 
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(a) The stronger bound ($c \leq 0$) holds unless $\hat{d}_{B_j}(V(H)) = p + 1$ for some $j$, and then $c = 1/2$. Now $p(k/2 + c) \leq \hat{d}_B(V(H)) \leq (p + 1)(k/2 - c)$.

simplifies to $2p + 1 \leq k$, or $n \leq 3(k - 1)/2$. 
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(a) The stronger bound ($c \leq 0$) holds unless $\hat{d}_{B_j}(V(H)) = p + 1$ for some $j$, and then $c = 1/2$. Now $p(k/2 + c) \leq \hat{d}_B(V(H)) \leq (p + 1)(k/2 - c)$. Simplifies to $2p + 1 \leq k$, or $n \leq 3(k - 1)/2$.

(b,c) For $n = k + 1$, apply Li–Xu. If $n \geq k + 3$, then $p \geq 4$, and the Lemma $\Rightarrow$ triangles and improper coloring. ■