Rainbow Matching in Edge-Colored Graphs

Timothy D. LeSaulnier[*,†] Christopher Stocker[*] Paul S. Wenger[*] Douglas B. West[*,‡]

January 26, 2010

Abstract

A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors. The *color degree* of a vertex v is the number of different colors on edges incident to v. Wang and Li conjectured that for $k \geq 4$, every edge-colored graph with minimum color degree at least k contains a rainbow matching of size at least $\lceil k/2 \rceil$. We prove the slightly weaker statement that a rainbow matching of size at least $\lfloor k/2 \rfloor$ is guaranteed. We also give sufficient conditions for a rainbow matching of size at least $\lceil k/2 \rceil$ that fail to hold only for finitely many exceptions (for each odd k).

1 Introduction

Given a coloring of the edges of a graph, a *rainbow matching* is a matching whose edges have distinct colors. The study of rainbow matchings began with Ryser, who conjectured that every Latin square of odd order contains a Latin transversal [3]. An equivalent statement is that when n is odd, every proper n-edge-coloring of the complete bipartite graph $K_{n,n}$ contains a rainbow perfect matching.

Wang and Li [4] studied rainbow matchings in arbitrary edge-colored graphs. We use the model of graphs without loops or multi-edges. For a vertex v in an edge-colored graph G, the *color degree* is the number of different colors on edges incident to v; we use the notation $\hat{d}_G(v)$ for this. The *minimum color degree* of G, denoted $\hat{\delta}(G)$, is the minimum of these values. (Subgraphs whose edges have distinct colors have also been called *heterochromatic*, *polychromatic*, or *totally multicolored*, but "rainbow" is the most common term.)

[*]Department of Mathematics, University of Illinois, Urbana, IL 61801.
Email addresses: tlesaul2@illinois.edu, stocker2@illinois.edu, pwenger2@illinois.edu, west@math.uiuc.edu.

[†]Contact author, partially supported by NSF grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students."

[‡]Research supported by NSA grant H98230-09-1-0363.

Wang and Li [4] proved that every edge-colored graph G contains a rainbow matching of size at least $\lceil \frac{5\hat{\delta}(G)-3}{12} \rceil$. They conjectured that a rainbow matching of size at least $\lceil \hat{\delta}(G)/2 \rceil$ can be guaranteed when $\hat{\delta}(G) \geq 4$. A properly 3-edge-colored complete graph with four vertices has no rainbow matching of size 2, but Li and Xu [2] proved the conjecture for all larger properly edge-colored complete graphs. Proper edge-colorings of complete graphs using the fewest colors show that the conjecture is sharp. A survey on rainbow matchings and other rainbow subgraphs appears in [1].

We strengthen the bound of Wang and Li for general edge-colored graphs, proving the conjecture when $\hat{\delta}(G)$ is even. When $\hat{\delta}(G)$ is odd, we obtain various sufficient conditions for a rainbow matching of size $\lceil \hat{\delta}(G)/2 \rceil$. Our results are the following:

Theorem 1.1. *Every edge-colored graph G has a rainbow matching of size at least $\lfloor \hat{\delta}(G)/2 \rfloor$.*

Theorem 1.2. *For an edge-colored graph G, let $k = \hat{\delta}(G)$. Each condition below guarantees that G has a rainbow matching of size at least $\lceil k/2 \rceil$.*

(a) *G contains more than $\frac{3(k-1)}{2}$ vertices.*

(b) *G is triangle-free.*

(c) *G is properly edge-colored, $G \neq K_4$ and $|V(G)| \neq k+2$.*

Condition (a) in Theorem 1.2 implies that, for each odd k, only finitely many edge-colored graphs with minimum color degree k can fail to have no rainbow matching of size $\lceil k/2 \rceil$, where an edge-coloring is viewed as a partition of the edge set. Condition (c) guarantees that failure for a proper edge-coloring can occur only for K_4 or a graph obtained from K_{k+2} by deleting a matching.

2 Notation and Tools

Let G be an edge-colored graph other than K_4, and let $k = \hat{\delta}(G)$. If $|V(G)| = k+1$, then G is a properly edge-colored complete graph and has a rainbow matching of size $\lceil k/2 \rceil$, by the result of Li and Xu [2]. Therefore, we may assume that $|V(G)| \geq k+2$.

Let M be a subgraph of G whose edges form a largest rainbow matching. Let $c = k/2 - |E(M)|$, and let the edges of M be $e_1, \ldots, e_{k/2-c}$, with $e_j = u_j v_j$. We may assume throughout that $c \geq 1/2$, since otherwise G has a rainbow matching of size $\lceil k/2 \rceil$. Let H be the subgraph induced by $V(G) - V(M)$, and let $p = |V(H)|$. Note that $|V(G)| = |V(M)| + |V(H)| = k - 2c + p$. Since $|V(G)| \geq k+2$, we conclude that $p \geq 2c+2$.

Let A be the spanning bipartite subgraph of G whose edge set consists of all edges joining $V(M)$ and $V(H)$ (see Figure 1). We say that a vertex v is *incident* to a color if some edge

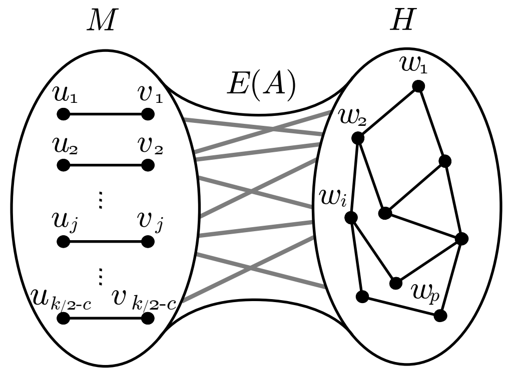

Figure 1: $V(M)$ and $V(H)$ partition $V(G)$.

incident to v has that color. A vertex $u \in V(M)$ is incident to at most $|V(M)|-1$ colors in the subgraph induced by $V(M)$, so u is incident to at least $2c+1$ colors in A. That is,

$$\hat{d}_A(u) \geq 2c+1. \tag{1}$$

We say that a color appearing in G is *free* if it does not appear on an edge of M. Let B denote the spanning subgraph of A whose edges have free colors. We prove our results by summing the color degrees in B of the vertices of H. Consider $w \in V(H)$. There are only $k/2-c$ non-free colors, so w is incident to at least $k/2+c$ free colors. By the maximality of M, no free color appears in H, so the free colors incident to w occur on edges of B. That is, $\hat{d}_B(w) \geq k/2+c$. Summing over $V(H)$ yields

$$\hat{d}_B(V(H)) \geq p(k/2+c), \tag{2}$$

where $f(S) = \sum_{s \in S} f(s)$ when f is defined on elements of S.

For $1 \leq j \leq k/2-c$, let E_j be the subset of $E(B)$ incident to u_jv_j. Let B_j be the graph with vertex set $V(H) \cup \{u_j, v_j\}$ and edge set E_j. Note that $\hat{d}_{B_j}(w) \leq 2$ for $w \in V(H)$.

Lemma 2.1. *If at least three vertices in $V(H)$ have positive color degree in B_j, then only one such vertex can have color degree 2 in B_j. Furthermore,*

$$\hat{d}_{B_j}(V(H)) \leq p+1. \tag{3}$$

Proof. Let w_1, w_2, and w_3 be vertices of H such that $\hat{d}_{B_j}(w_1) = \hat{d}_{B_j}(w_2) = 2$ and $\hat{d}_{B_j}(w_3) \geq 1$. By symmetry, we may assume that $w_3v_j \in E(B_j)$. Maximality of M requires the same color on u_jw_1 and v_jw_2. Since $\hat{d}_{B_j}(w_2) = 2$, the color on u_jw_2 differs from this. Now u_jw_1 or u_jw_2 has a color different from v_jw_3, which yields a larger rainbow matching in G.

Now consider $\hat{d}_{B_j}(V(H))$. Since $p \geq 2c+2$, we have $p \geq 3$. If $\hat{d}_{B_j}(V(H)) \geq p+2$, then $\hat{d}_B(w) \leq 2$ for all $w \in V(H)$ requires three vertices as forbidden above. ∎

If $p = 3$, then color degrees $2, 2, 0$ for $V(H)$ in B_j do not contradict Lemma 2.1. For $p \geq 4$, the next lemma determines the structure of B_j when $\hat{d}_{B_j}(V(H)) = p + 1$. Let $N_G(x)$ denote the neighborhood of a vertex x in a graph G.

Lemma 2.2. *For $p \geq 4$, if $\hat{d}_{B_j}(V(H)) = p + 1$, then u_j or v_j is adjacent in B_j to $p - 1$ vertices of $V(H)$ via edges of the same color.*

Proof. Since $p + 1 \geq 5$, at least three vertices of H have positive color degree in B_j. Now Lemma 2.1 permits only one vertex w such that $\hat{d}_{B_j}(w)=2$, while $\hat{d}_{B_j}(w') = 1$ for each other vertex w' in $V(H)$. Let λ_1 and λ_2 be the colors on $u_j w$ and $v_j w$, respectively. Partition $V(H) - \{w\}$ into two sets by letting $U = N_{B_j}(u_j) - \{w\}$ and $V = N_{B_j}(v_j) - \{w\}$. By the maximality of M, all edges joining u_j to U have color λ_2, and all edges joining v_j to V have color λ_1. If U and V are both nonempty, then replacing $u_j v_j$ with edges to each yields a larger rainbow matching in G. Hence U or V is empty and the other has size $p - 1$. □

Lemma 2.3. *The following imply that $\hat{d}_{B_j}(V(H)) \leq p$ for each j.*

(*a*) *$c \geq 1$.*

(*b*) *G is triangle-free.*

(*c*) *G is properly edge-colored and $p \geq 4$.*

Proof. (a) Since $p \geq 2c + 2$, condition (a) implies $p \geq 4$. If $\hat{d}_{B_j}(V(H)) = p + 1$, then Lemma 2.2 applies, and u_j or v_j is adjacent via the same color to all but one vertex of H. Now $\hat{d}_A(u_j) \leq 2$ or $\hat{d}_A(v_j) \leq 2$, which contradicts (1) when $c \geq 1$.

(b) If G is triangle-free, then no vertex of H is adjacent to both endpoints of an edge in M. Hence, $\hat{d}_{B_j}(w) \leq 1$ for each $w \in V(H)$.

(c) If $p \geq 4$ and $\hat{d}_{B_j}(V(H)) = p + 1$, then Lemma 2.2 applies again and implies that the edge-coloring is not proper. □

3 Proof of the Main Results

Theorem 1.1. *Every edge-colored graph with minimum color degree k has a rainbow matching of size at least $\lfloor k/2 \rfloor$.*

Proof. If the maximum size of a rainbow matching is $k/2 - c$, with $c \geq 1$, then Lemma 2.3(a) yields $\hat{d}_B(V(H)) \leq \sum_{j=1}^{k/2-c} \hat{d}_{B_j}(V(H)) \leq p(k/2 - c)$, which contradicts (2). □

Theorem 1.2. *Let G be an edge-colored graph such that $\hat{\delta}(G) = k$. Each of the following guarantee that G contains a rainbow matching of size at least $\lceil k/2 \rceil$.*

(a) G has more than $\frac{3(k-1)}{2}$ *vertices.*

(b) G is triangle-free.

(c) G is properly edge-colored, $G \neq K_4$*, and* $|V(G)| \neq k+2$*.*

Proof. If G has no rainbow matching of size $\lceil k/2 \rceil$, then for a largest one Theorem 1.1 yields $c = 1/2$. Now (3) implies $\hat{d}_B(V(H)) \leq \sum_{j=1}^{k/2-c} \hat{d}_{B_j}(V(H)) \leq (p+1)(k/2-1/2)$. Combining this with (2) yields $p(k/2+1/2) \leq (p+1)(k/2-1/2)$, which simplifies to $p \leq (k-1)/2$. Hence $|V(G)| \leq 3(k-1)/2$.

If G is a properly edge-colored complete graph other than K_4, then the result of Li and Xu [2] suffices. If G is triangle-free or properly edge-colored with at least $k+3$ vertices, then Lemma 2.3 yields $\hat{d}_B(V(H)) \leq p(k/2-c)$, which again contradicts (2). □

Acknowledgement

The authors would like to thank the participants in the combinatorics group of the Research Experience for Graduate Students program at the University of Illinois during summer 2009. Their interest and input is greatly appreciated.

References

[1] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs—a survey. *Graphs Combin.* 24 (2008), 237–263.

[2] X. Li and Z. Xu, On the existence of a rainbow 1-factor in proper coloring of $K_{rn}^{(r)}$. arXiv:0711.2847 [math.CO] 19 Nov 2007.

[3] H. J. Ryser, Neuere probleme der kombinatorik, in "Vortrage uber Kombinatorik Oberwolfach". Mathematisches Forschungsinstitut Oberwolfach, July 1967, 24-29.

[4] G. Wang and L. Hao, Heterochromatic matchings in edge-colored graphs. *Electron. J. Combin.* 15 (2008), Paper #R138.