# Rainbow Edge-coloring and Rainbow Domination

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slides available on DBW preprint page

Joint work with Timothy D. LeSaulnier

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If the edge-coloring is rainbow, then $\hat{\chi}'(G) = \chi'(G)$.

If the edge-coloring is proper, then $\hat{\gamma}(G) = \gamma(G)$.

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With Pfender: $\hat{\alpha}'(G) \geq \hat{\delta}(G)$ when $n \geq 5.5(\hat{\delta}(G))^2$.

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**Thm.** When $G$ is $t$-tolerant (and no isolated vertices),
$\hat{\gamma}(G) = \frac{t}{t+1}n \iff$ each component is a $t$-flare
(or monochr. $C_3$ ($t=2$) or properly edge-colored $C_4$ ($t=1$)).

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Thus $$\hat{\chi}'(G) \ge \frac{n^2}{n-1} > n+1 = \Delta(G)+1.$$ ■

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**Pf.** $t$-tolerant $\Rightarrow$ $\hat{d}_G(v) \geq d_G(v)/t$. With degree-sum $2m$, the average color degree is $\geq 2m/(nt)$. The lemma yields $H$ with $\hat{\delta}(H) > \frac{2m}{nt(t+1)}$. Now $\hat{\alpha}'(H) \geq \left\lceil \frac{m}{nt(t+1)} \right\rceil$. ■

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Thus $\hat{\chi}'(G) \le j + |E(F_j)|$.

It remains to bound $j$ and $|E(F_j)|$.

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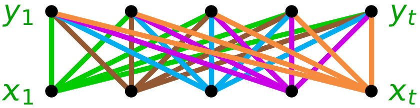
**Note:** Below: a $t$-tolerant edge-colored graph $G$ with avg color degree $(t+1)/2$, but $\hat{\delta}(H) \le 1$ for all $H \subseteq G$.



# Arnautov–Payan bound — Generalization

**Thm.** If $G$ is an $n$-vertex $t$-tolerant edge-colored graph, then $\hat{\gamma}(G) \le \frac{1+\ln k}{k} n$, where $k = \frac{\delta(G)}{t} + 1$.

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Thus $\mathbb{E}(|B|) \le n/k$. We conclude $\mathbb{E}(|A \cup B|) \le \frac{(1+\ln k)}{k} n$. ∎

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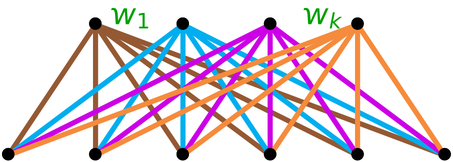


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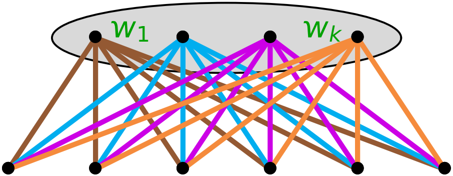


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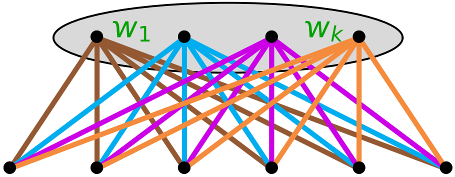


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(No rainbow star covers two vertices of $U$.) ∎

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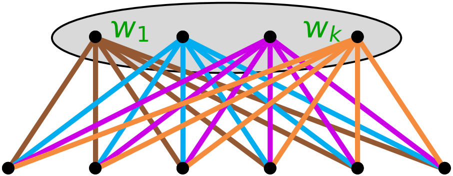


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**Note:** $\hat{\gamma}(G)/n \to 1$, but $t/n \to 1$.

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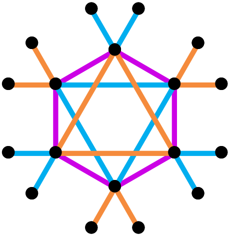
# Characterization of Equality

**Def.** The $t$-corona $H \circ t$ is formed by adding $t$ pendant edges at each vertex of $H$. A $t$-flare is an edge-colored $t$-corona $H \circ t$ that is $t$-tolerant and, for each vertex of $H$, has the same color on all $t$ new pendant edges there.



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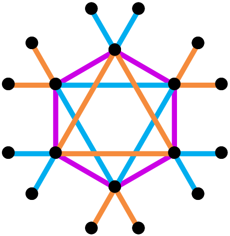

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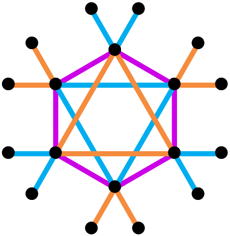

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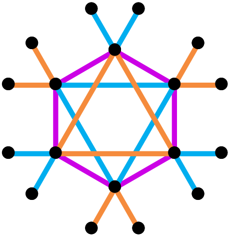

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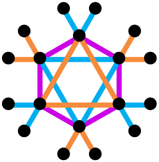
- For $t = 1$ (where $\hat{\gamma}(G) = \gamma(G)$), Payan–Xuong [1982] and Fink–Jacobson–Kinch–Roberts [1985] char'zd $\gamma(G) = n/2$.

# Sketch of Characterizing Equality in $\hat{\gamma}(G) \leq \frac{t}{t+1}n$



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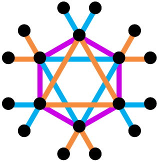

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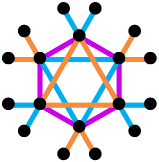

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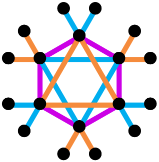

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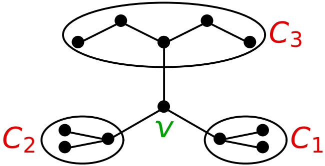

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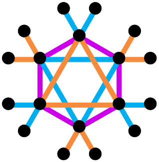

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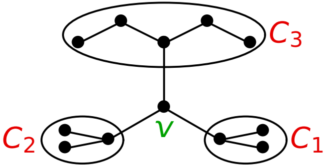


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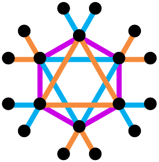

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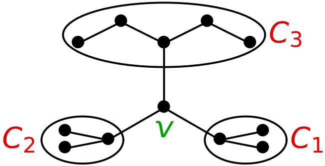
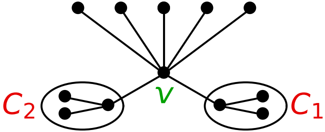



If $(t+1) \nmid |V(C_i)|$, then strict inequality for $C_i$ (and $G$).

Now $(t+1) \nmid n$, and again the inequality is strict for $G$.

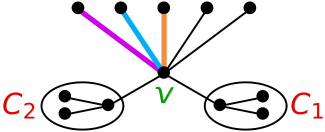
# Idea, continued

∴ $v$ has leaf nbr(s), say $\ell$ of them, with $k$ colors.



# Idea, continued

$\therefore$ $v$ has leaf nbr(s), say $\ell$ of them, with $k$ colors.

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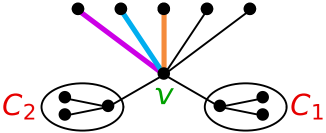


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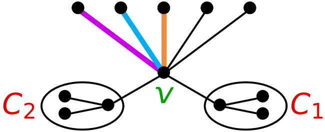


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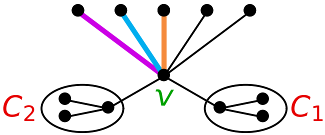


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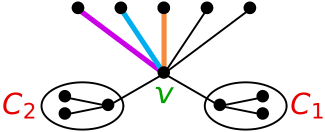


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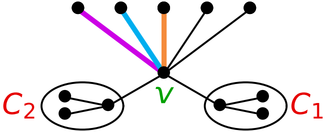


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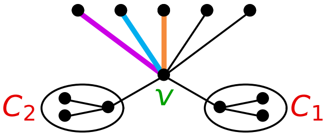


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**Claim:** No other edges at leaves of a spanning tree $T$.
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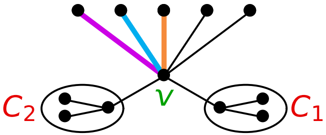


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(The exceptions: monochr. $C_3$ and properly colored $C_4$.) ■

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**Reference:**
T. D. LeSaulnier and D. B. West,
Rainbow edge-coloring and rainbow domination,
*Discrete Math. (2012)*, DOI: 10.1016/j.disc.2012.03.014.