Rainbow edge-coloring and rainbow domination

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Abstract

Let G be an edge-colored graph with n vertices. A *rainbow subgraph* is a subgraph whose edges have distinct colors. The *rainbow edge-chromatic number* of G, written $\hat{\chi}'(G)$, is the minimum number of rainbow matchings needed to cover $E(G)$. An edge-colored graph is t-*tolerant* if it contains no monochromatic star with $t+1$ edges. If G is t-tolerant, then $\hat{\chi}'(G) < t(t+1)n \ln n$, and examples exist with $\hat{\chi}'(G) \geq \frac{t}{2}(n-1)$. The *rainbow domination number*, written $\hat{\gamma}(G)$, is the minimum number of disjoint rainbow stars needed to cover $V(G)$. For t-tolerant edge-colored n-vertex graphs, we generalize classical bounds on the domination number: (1) $\hat{\gamma}(G) \leq \frac{1+\ln k}{k} n$ (where $k = \frac{\delta(G)}{t} + 1$), and (2) $\hat{\gamma}(G) \leq \frac{t}{t+1} n$ when G has no isolated vertices. We also characterize the edge-colored graphs achieving equality in the latter bound.

1 Introduction

An edge-colored graph is *rainbow* if its edges have distinct colors. Rainbow edge-colored graphs have also been called *heterochromatic*, *polychromatic*, or *totally multicolored*, but "rainbow" is the most common term. A survey of results on rainbow subgraphs, decomposition into rainbow subgraphs, and coverings by rainbow subgraphs appears in [7].

Within an edge-colored graph G, we consider covering the edges by rainbow matchings or covering the vertices by disjoint rainbow stars. The number of rainbow matchings needed to cover $E(G)$ is the *rainbow edge-chromatic number* of G, written $\hat{\chi}'(G)$. The number of disjoint rainbow stars needed to cover $V(G)$ is the *rainbow domination number* of G, written $\hat{\gamma}(G)$. These parameters generalize the edge-chromatic number $\chi'(G)$ and the domination number $\gamma(G)$, respectively; $\chi'(G)$ is the minimum number of matchings needed to cover $E(G)$, and $\gamma(G)$ is the minimum number of stars needed to cover $V(G)$. Note that $\hat{\chi}'(G) = \chi'(G)$ when G itself is rainbow, while $\hat{\gamma}(G) = \gamma(G)$ when the edge-coloring is proper (that is, when all stars are rainbow).

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In studying these parameters on an edge-colored graph G, a useful concept is the *color degree* of a vertex v, written $\hat{d}_G(v)$; it is the number of different colors on edges incident to v. We also write $\hat{\delta}(G)$ and $\hat{\Delta}(G)$ for the minimum and maximum of the color degrees.

Decomposing a graph into few rainbow matchings requires large rainbow matchings. Using this language, we rephrase the conjecture of Ryser [17] that every Latin square of odd order has a transversal. (A *transversal* is a set of positions occupied by distinct labels, one in each row and column). Viewed as a colored matrix, a Latin square of order n corresponds to a proper n-edge-coloring of the complete bipartite graph $K_{n,n}$, and a transversal corresponds to a rainbow perfect matching. Thus Ryser's conjecture states that when n is odd, every proper n-edge-coloring of $K_{n,n}$ contains a rainbow perfect matching.

Proper n-edge-colorings of $K_{n,n}$ can also be viewed as 3-uniform 3-partite hypergraphs with n^2 edges; each edge in $K_{n,n}$ yields a hyperedge consisting of the endpoint in each partite set and the color. A rainbow perfect matching in $K_{n,n}$ corresponds to a perfect matching in the hypergraph. In general, it is NP-hard to determine whether a 3-uniform 3-partite hypergraph has a perfect matching [8].

Let $\hat{\alpha}'(G)$ denote the maximum size of a rainbow matching in G. Wang and Li [18] studied the problem for a general edge-colored graph G, proving that always $\hat{\alpha}'(G) \geq \left\lceil (5\hat{\delta}(G) - 3)/12 \right\rceil$. They also conjectured that $\hat{\alpha}'(G) \geq \left\lceil \hat{\delta}(G)/2 \right\rceil$ when $\hat{\delta}(G) \geq 4$ (the constraint excludes proper 3-edge-colorings of K_4, which have no rainbow matchings of size 2).

Li and Xu [12] proved the conjecture for properly edge-colored complete graphs with more than 4 vertices. LeSaulnier, Stocker, Wenger, and West [11] showed that always $\hat{\alpha}'(G) \geq \left\lfloor \hat{\delta}(G)/2 \right\rfloor$. Exploiting this result and its lemmas, Kostochka and Yancey [9] proved the conjecture of Wang and Li completely:

Theorem 1.1 ([9]). $\hat{\alpha}'(G) \geq \left\lceil \hat{\delta}(G)/2 \right\rceil$ *for every edge-colored graph* G.

Theorem 1.1 is a key ingredient in proving our upper bound on $\hat{\chi}'(G)$. Define a graph to be *t-tolerant* if it contains no monochromatic star with $t+1$ edges. We consider t-tolerant edge-colored graphs with n vertices. In Section 2 we construct examples with $\hat{\chi}'(G) \geq \frac{t}{2}(n-1)$ and prove that always $\hat{\chi}'(G) < t(t+1)n \ln n$. There is potential for improving the upper bound: Kostochka, Pfender, and Yancey [10] showed further that if $|V(G)| > 5.5\hat{\delta}(G)^2$, then G has a rainbow matching of size at least $\hat{\delta}(G)$.

For rainbow domination, our aim in Section 3 is to generalize classical bounds on $\gamma(G)$ for n-vertex graphs. The trivial bound $\gamma(G) \leq n - \Delta(G)$ (Berge [5]) generalizes as trivially to $\hat{\gamma}(G) \leq n - \hat{\Delta}(G)$ and remains sharp. For graphs without isolated vertices, Ore [14] showed $\gamma(G) \leq n/2$. We prove more generally that $\hat{\gamma}(G) \leq \frac{t}{t+1}n$ when G is a t-tolerant edge-colored graph without isolated vertices; this reduces to Ore's result when $t = 1$. Our most difficult result characterizes the edge-colored graphs satisfying equality in this bound.

We also generalize the upper bound on $\gamma(G)$ proved independently by Arnautov [4], Payan [15], and Lovász [13]. Later, Alon and Spencer gave a probabilistic proof [2].

Theorem 1.2 ([4, 15, 13, 2]). *If G is an n-vertex graph, then $\gamma(G) \le \frac{(1+\ln(\delta(G)+1))}{\delta(G)+1}n$.*

Many authors refer to Alon [1] for the asymptotic tightness of Theorem 1.2, but that reference does not provide a complete proof. Instead, Alon and Wormald [3] show that if c is fixed and less than 1, and k is sufficiently large, then the expected number of dominating sets of size $(1+o(1))\frac{c\ln k}{k}n$ in a random k-regular n-vertex graph tends to 0 as $n \to \infty$. Since $\frac{1+\ln(k+1)}{k+1} \sim \frac{\ln k}{k}$, with high probability $\gamma(G) = (1+o(1))\frac{n(1+\ln(k+1))}{k+1}$ for such graphs.

The proof of our generalization mirrors the probabilistic proof in [2]. We show that if G is a t-tolerant edge-colored n-vertex graph, then $\hat{\gamma}(G) \le \frac{(1+\ln k)}{k}n$, where $k = \frac{\delta(G)}{t} + 1$.

2 Rainbow Edge-Chromatic Number

In this section we study the maximum of $\hat{\chi}'(G)$ among t-tolerant edge-colored graphs with n vertices. We begin by constructing examples with $\hat{\chi}'(G) \ge \frac{t}{2}(n-1)$. Later we show that always $\hat{\chi}'(G) < t(t+1)n\ln n$.

Proposition 2.1. *There exist infinitely many t-tolerant edge-colored graphs G such that $\hat{\chi}'(G) \ge \frac{t}{2}(|V(G)|-1) = \frac{t}{2}\Delta(G)$.*

Proof. For $t, p \in \mathbb{N}$, start with a proper tp-edge-coloring of K_{tp}. Obtain a t-tolerant edge-colored graph G by combining t-tuples of color classes into single colors. In G there are only p colors, so $\hat{\alpha}'(G) \le p$. Hence $\hat{\chi}'(G) \ge \frac{1}{p}|E(G)| \ge \frac{t}{2}(tp-1) = \frac{t}{2}(|V(G)|-1) = \frac{t}{2}\Delta(G)$. □

Although Vizing's Theorem states that $\chi'(G) \le \Delta(G)+1$ always, making t large in Proposition 2.1 shows that $\hat{\chi}'(G)$ can be much larger than $\Delta(G)+1$. We may also have $\hat{\chi}'(G) > \Delta(G)+1$ when G is properly edge-colored, such as when G is a properly edge-colored copy of K_4. For an infinite family of examples, we turn to Latin squares and transversals.

Proposition 2.2. *When $n \equiv 2 \mod 4$, there is an edge-colored graph G such that $\hat{\chi}'(G) > \Delta(G)+1$ and G is a proper n-edge-coloring of $K_{n,n}$.*

Proof. As noted earlier, proper n-edge-colorings of $K_{n,n}$ correspond to Latin squares of order n. Each rainbow matching corresponds to a partial transversal of the Latin square, so $\hat{\chi}'(G)$ is the minimum number of partial tranversals covering the square. Latin squares of even order need not have transversals. To construct such squares when $n \equiv 2 \mod 4$, let $k = n/2$, and let A and B be latin squares of order k, using disjoint sets of k labels in the two squares. Let $C = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$.

Although C is a Latin square of order n, it has no transversal. A transversal must use each of the $2k$ labels. Since k is odd, some quadrant must contribute at least $\lceil k/2 \rceil$

positions. Now each of the other three quadrants is limited to $\lfloor k/2 \rfloor$ contributions, so a partial transversal has size at most $n-1$.

Thus at least $n^2/(n-1)$ partial transversals are needed to cover C. Since $(n-1)(n+1) < n^2$, we have $\hat{\chi}'(G) \geq n^2/(n-1) > n+1 = \Delta(G)+1$. □

In the rest of this section we obtain an upper bound on $\hat{\chi}'(G)$ when G is t-tolerant. The *average color degree* of G is the average of the color degrees of the vertices: $\sum_{v \in V(G)} \hat{d}_G(v)/|V(G)|$. The usual argument for graphs with large average degree having subgraphs with large minimum degree applies also for color degree with a generalized ratio.

Lemma 2.3. *Fix $t \in \mathbb{N}$ and $c \in \mathbb{R}$ with $c > 0$. Every t-tolerant edge-colored graph G with average color degree at least c has an edge-colored t-tolerant subgraph H with $\hat{\delta}(H) > \frac{c}{t+1}$.*

Proof. The claim holds with $H = G$ unless $\hat{d}_G(v) \leq \frac{c}{t+1}$ for some vertex v. Deleting v decreases the color degree of each neighbor by at most 1, but v may have up to $t\hat{d}_G(v)$ neighbors. Thus deleting v reduces the sum of the color degrees by at most $(t+1)\hat{d}_G(v)$. Since $\sum_{u \in V(G-v)} \hat{d}_{G-v}(u) \geq \sum_{u \in V(G)} \hat{d}_G(u) - (t+1)\hat{d}_G(v) \geq c|V(G)| - c$, deleting v does not reduce the average color degree. Furthermore, every subgraph of a t-tolerant graph is also t-tolerant. Iteratively deleting vertices with color degree at most $c/(t+1)$ must end; since the average color degree never decreases, it ends with a subgraph H having no vertex with color degree at most $c/(t+1)$. □

Corollary 2.4. *Every n-vertex t-tolerant edge-colored graph with m edges contains a rainbow matching of size at least $\left\lceil \frac{m}{nt(t+1)} \right\rceil$.*

Proof. Since G is t-tolerant, always $\hat{d}_G(v) \geq d_G(v)/t$. Since the degrees sum to $2m$, the color degrees sum to at least $2m/t$. With average color degree at least $2m/(nt)$, Lemma 2.3 guarantees a subgraph H with $\hat{\delta}(H) > \frac{2m}{nt(t+1)}$. By Theorem 1.1, $\hat{\alpha}'(H) \geq \left\lceil \frac{m}{nt(t+1)} \right\rceil$. □

Theorem 2.5. *If G is an n-vertex t-tolerant edge-colored graph, then $\hat{\chi}'(G) < t(t+1)n \ln n$.*

Proof. Extend G to an edge-colored copy of K_n by assigning new colors to the missing edges; this does not reduce $\hat{\chi}'$, and the resulting edge-colored graph is t-tolerant. Hence we may assume that G is a edge-coloring of K_n (and that $n \geq 2$).

We greedily delete largest rainbow matchings. With $F_0 = G$, for $i > 0$ obtain F_i from F_{i-1} by deleting a largest rainbow matching, M_{i-1}. Let $a_i = |E(F_i)|/\binom{n}{2}$, so $a_0 = 1$. By Corollary 2.4, $|M_{i-1}| \geq \frac{|E(F_{i-1})|}{nt(t+1)} = a_{i-1}\frac{n-1}{2t(t+1)}$. Let j be the least index such that $a_j \frac{n-1}{2t(t+1)} \leq 1$. Note that $E(F_j)$ decomposes into $|E(F_j)|$ rainbow matchings of size 1. Using also $\{M_i \colon 0 \leq i < j\}$, we obtain $\hat{\chi}'(G) \leq j + |E(F_j)|$. It remains only to bound j and $|E(F_j)|$.

For $i \geq 1$, the definition of a_i yields

$$a_i \binom{n}{2} \;=\; |E(F_i)| \;\leq\; a_{i-1}\binom{n}{2} - a_{i-1}\frac{n-1}{2t(t+1)} \;\leq\; a_{i-1}\binom{n}{2}\left(1 - \frac{1}{nt(t+1)}\right).$$

Thus $a_i \le a_{i-1}\left(1 - \frac{1}{nt(t+1)}\right)$. Since $a_0 = 1$, we obtain $a_i \le \left(1 - \frac{1}{nt(t+1)}\right)^i < e^{\frac{-i}{nt(t+1)}}$.

With $\frac{2t(t+1)}{n-1} < a_{j-1} \le e^{\frac{-j+1}{nt(t+1)}}$, we have $j < nt(t+1)\ln\frac{n-1}{2t(t+1)} + 1$. Now F_0 decomposes into $j + a_j\binom{n}{2}$ rainbow matchings, where $a_j \le \frac{2t(t+1)}{n-1}$, so

$$j + a_j\binom{n}{2} \;<\; nt(t+1)\ln\frac{n-1}{2t(t+1)} + 1 + \frac{2t(t+1)}{n-1}\frac{n(n-1)}{2} \;<\; t(t+1)n\ln(n-1),$$

where the last inequality follows from $n \ge 2$ and $t \ge 1$. Thus $\hat{\chi}'(G) < t(t+1)n\ln n$. ☐

The argument of Lemma 2.3 seems slack when $t > 1$, since if deleting v reduces the sum of color degrees by $(t+1)\hat{d}_G(v)$, then deleting a neighbor u of v does not reduce the sum by as much as $(t+1)\hat{d}_G(u)$. Nevertheless, average color degree c cannot guarantee a subgraph with minimum color degree more than 1, no matter how large c is.

Proposition 2.6. *There is a t-tolerant edge-colored graph with average color degree $(t+1)/2$ having no subgraph with minimum color degree more than* 1.

Proof. Consider the complete bipartite graph $K_{t,t}$ with partite sets $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$. Use color i on all edges incident to x_i; thus $\hat{d}(x_i) = 1$ and $\hat{d}(y_j) = t$. The average color degree is $(t+1)/2$, and the edge-coloring is t-tolerant. Every subgraph using a vertex of $x_1, \ldots, x_t$ has minimum color degree at most 1, and all other subgraphs have no edges. ☐

This example shows that eliminating a factor of t in the upper bound in Theorem 2.5 will require finding rainbow matchings using something other than minimum color degree.

3 Rainbow Domination

In this section, we generalize classical upper bounds on domination number to the setting of rainbow domination. We begin by generalizing the Arnautov–Payan bound (Theorem 1.2). Our proof mirrors the probabilistic proof of Theorem 1.2 by Alon and Spencer [2]. The asymptotic sharpness of Theorem 1.2 applies here when G is properly edge-colored, since then $\hat{\gamma}(G) = \gamma(G)$. We make no attempt to show sharpness in other cases.

Theorem 3.1. *If G is an n-vertex t-tolerant edge-colored graph, then $\hat{\gamma}(G) \le \frac{1+\ln k}{k}n$, where $k = \frac{\delta(G)}{t} + 1$.*

Proof. For each vertex v, form a largest rainbow star S_v centered at v by including, for each color incident to v, an incident edge of that color chosen uniformly at random. For each edge vw, $\mathbb{P}(vw \in E(S_v)) \ge 1/t$, since G is t-tolerant.

Set $p = \frac{\ln k}{k}$. Form a set $A \subset V(G)$ by including each vertex v with probability p, independently; the expected size of A is pn. Let $B = V(G) - \bigcup_{v\in A} V(S_v)$. By including each vertex of B as a rainbow star by itself, we have $\hat{\gamma}(G) \le |A| + |B|$.

The event $w \in B$ requires $w \notin A$ and, for each neighbor v of w, either $v \notin A$ or $w \notin S_v$. Therefore, $\mathbb{P}(w \in B) \le (1-p)[(1-p)+p(1-1/t)]^{\delta(G)}$. We compute

$$\mathbb{P}(w \in B) \le (1-p)(1-\frac{p}{t})^{\delta(G)} \le e^{-p}e^{-\delta(G)p/t} = e^{-pk} = \frac{1}{k}.$$

Hence $\mathbb{E}(|B|) \le n/k$ and $\mathbb{E}(|A \cup B|) \le \frac{(1+\ln k)}{k}n$, so $\hat{\gamma}(G) \le \frac{(1+\ln k)}{k}n$. □

One would prefer to generalize the Arnautov-Payan bound by replacing $\delta(G)/t + 1$ in Theorem 3.1 with $\hat{\delta}(G)+1$, but this is not valid. For example, the monochromatic star $K_{1,t}$ is t-tolerant, and its minimum color degree is 1. Its rainbow domination number is $n-1$, so $\delta(G)/t$ cannot be strengthened to $\hat{\delta}(G)$ in the bound.

Now we consider the generalization of the older elementary bounds. The inequality of Berge [5] and its generalization are quite straightforward.

Proposition 3.2. *If G is an edge-colored graph with n vertices, then $\hat{\gamma}(G) \le n - \hat{\Delta}(G)$, and this is sharp among edge-colored graphs (with connectivity $\hat{\Delta}(G)$) no matter how many times colors are allowed to appear at vertices.*

Proof. For the upper bound, use a rainbow star centered at a vertex of maximum color degree, and cover the remaining vertices with trivial stars.

The bound is sharp for stars and also for more highly connected graphs. Begin with an independent set U of size $n-k$. Add vertices $w_1, \ldots, w_k$, making w_i adjacent to all of U by edges with color i. Finally, add edges to make $w_1, \ldots, w_k$ pairwise adjacent, using distinct new colors. Each w_i has $n-k$ incident edges of the same color. All vertices have color degree k. The rainbow domination number equals the upper bound $n-k$, since no rainbow star covers two vertices of U. □

It is not easy to characterize equality in Proposition 3.2. A necessary condition is that the vertices left by deleting those in any largest rainbow star form an independent set, but this is not sufficent. Even among properly edge-colored graphs, consider a graph consisting of two complete subgraphs sharing a common vertex v, plus a few pendant edges at one other vertex in each of the two complete subgraphs. The largest rainbow star is at v, and deleting it leaves an independent set, but the (rainbow) domination number is 2.

The construction in Proposition 3.2 shows that $\hat{\gamma}(G)$ cannot be bounded by cn for $c < 1$; raising the "tolerance" of vertices for multiple edges with the same color allows the ratio of $\hat{\gamma}(G)$ to the number of vertices to tend to 1. This contrasts with Ore's bound, $\gamma(G) \le n/2$ when $\delta(G) \ge 1$. Restricting to t-tolerant edge-colored graphs permits a generalization of Ore's bound. We will also characterize when equality holds in the generalized bound.

Ore's bound follows immediately from the next lemma, which is useful in the generalization. A graph is *nontrivial* if it has at least one edge.

Proposition 3.3. *If G is a graph with no isolated vertices, then $V(G)$ can be covered by a family $\mathcal{F}$ of disjoint nontrivial stars in G.*

Proof. We may assume that G is a forest, since deleting an edge of a cycle cannot isolate a vertex. Select a non-leaf vertex v adjacent to a leaf. Put into $\mathcal{F}$ the star H consisting of v and its leaf neighbors; H is nontrivial. Also, $G - V(H)$ has no isolated vertex, since the remaining neighbors of v were not leaves. Iterating the process on $G - V(H)$ to complete the covering. □

Theorem 3.4. *If G is an n-vertex t-tolerant edge-colored graph with no isolated vertices, then $\hat{\gamma}(G) \le \frac{t}{t+1}n$.*

Proof. Let $\mathcal{F}$ be a family of disjoint nontrivial stars covering $V(G)$, as guaranteed by Proposition 3.3. For $F \in \mathcal{F}$ with central vertex v_F, a largest *rainbow* star contained in F has $\hat{d}_F(v_F)$ leaves. Let $\mathcal{F}'$ be a family of rainbow stars constructed by choosing a largest rainbow substar inside each member of $\mathcal{F}$. Let $s = \sum_{F\in\mathcal{F}} \hat{d}_F(v_F)$. If $|\mathcal{F}| = k$, then $\mathcal{F}'$ covers $k + s$ vertices using k rainbow stars. Extend $\mathcal{F}'$ to cover $V(G)$ by adding the uncovered vertices as 1-vertex stars. Since s vertices come "for free" in $\mathcal{F}'$, we have $\hat{\gamma}(G) \le n - s$.

Since the stars in $\mathcal{F}$ are nontrivial, $s \ge k$. For $F \in \mathcal{F}$, each color counted by $\hat{d}_F(v_F)$ appears on at most t edges of F, so $|V(F)| \le t \cdot \hat{d}_F(v_F) + 1$. Since $\bigcup_{F\in\mathcal{F}} F$ is a spanning subgraph, summing over the members of $\mathcal{F}$ yields $n \le ts + k \le (t + 1)s$. Now $s \ge n/(t + 1)$ and $\hat{\gamma}(G) \le n - s \le \frac{t}{t+1}n$. □

Our aim now is to characterize when equality holds in the bound of Theorem 3.4. When $t = 1$, we have proper colorings and $\hat{\gamma}(G) = \gamma(G)$, so in that case the characterization must reduce to the characterization of $\gamma(G) = n/2$ among graphs without isolated vertices. That characterization was proved independently by Payan and Xuong [16] and by Fink, Jacobson, Kinch, and Roberts [6]. It uses the "corona" operation. For general t, we generalize this operation. There will also be one exception for each of $t = 1$ and $t = 2$.

Example 3.5. The *t-corona* of a graph H is the graph obtained by adding t pendant edges at each vertex of H (note that H may have isolated vertices; its t-corona does not). Let A denote the independent set of $t|V(H)|$ added vertices. An edge-colored t-corona of H is a *t-flare* if it is t-tolerant and for each vertex $v \in V(H)$, the edges joining v to A have the same color. Figure 1 shows a 2-flare and a 3-flare. Note that in a t-flare the color on the edges joining a vertex v of H to A may not appear on any edge of H incident to v, but it may appear elsewhere in H.

The key observation is that in a t-flare G, no rainbow star covers two vertices of A. Since $|A| = \frac{t}{t+1}V(G)$, a t-flare is a sharpness example for Theorem 3.4.

Theorem 3.6. *Let G be an n-vertex t-tolerant edge-colored graph with no isolated vertices. If $\hat{\gamma}(G) = \frac{t}{t+1}n$ and D is a component of G, then D is*

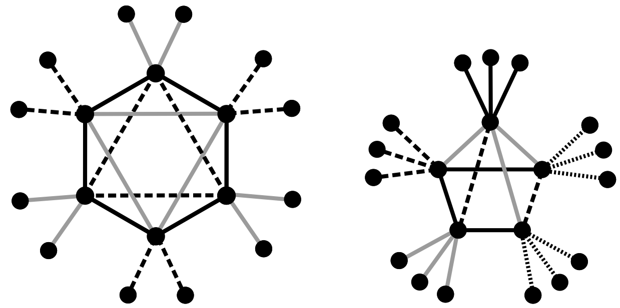

Figure 1: A 2-flare and a 3-flare.

(a) *a t-flare, or*
(b) *a monochromatically edge-colored C_3 if $t = 2$, or*
(c) *a properly edge-colored C_4 if $t = 1$.*

Proof. It suffices to prove the claim for connected graphs. Let T be a spanning tree of G, and let v be a vertex that is not a leaf of T. If v has no leaf neighbor in T, as in Figure 2, then let $C_1, \ldots, C_r$ be the components of $T - v$; none are isolated vertices.

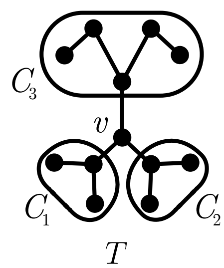

Figure 2: The vertex v has no leaf neighbor in T.

If $|V(C_i)| \not\equiv 0 \pmod{t+1}$, then $\hat{\gamma}(C_i) < \frac{t}{t+1}|V(C_i)|$ by Theorem 3.4, and also $\hat{\gamma}(G - V(C_i)) \le \frac{t}{t+1}(n - |V(C_i)|)$. Together, these yield the contradiction $\hat{\gamma}(G) < \frac{t}{t+1}n$, so $|V(C_i)| \equiv 0 \pmod{t+1}$ for all i. Now $n \equiv 1 \pmod{t+1}$, which with Theorem 3.4 requires $\hat{\gamma}(G) < \frac{t}{t+1}n$. We conclude that every non-leaf vertex of T has a leaf neighbor in T.

Let v have ℓ leaf neighbors in T, so $T - v$ has ℓ isolated vertices and nontrivial components $C_1, \ldots, C_r$, as in Figure 3.

Let k be the number of distinct colors on edges joining v to isolated vertices of $T - v$. We have $k \ge 1$ and $k \ge \ell/t$, since G is t-tolerant. Within T there is a rainbow star F with $k + 1$ vertices centered at v whose leaves are isolated vertices in $T - v$. Taking the $\ell - k$ isolated vertices in $T - v$ not covered by F as rainbow 1-vertex stars, we obtain $\hat{\gamma}(G) \le \sum_{i=1}^{r} \hat{\gamma}(V(C_i)) + \ell - k + 1$. By Theorem 3.4, $\sum_{i=1}^{r} \hat{\gamma}(V(C_i)) \le \frac{t}{t+1}(n - \ell - 1)$, so $\hat{\gamma}(G) \le \frac{t}{t+1}(n - \ell - 1) + \ell - k + 1$. We conclude that $\frac{t}{t+1}(n - \ell - 1) + \ell - k + 1 \ge \frac{t}{t+1}n$, which yields $k \le \frac{\ell+1}{t+1}$. Recalling that $k \ge \ell/t$, we have $\ell \le t$. On the other hand, $k \ge 1$ yields $\ell \ge t$. Thus $\ell = t$ and $k = 1$.

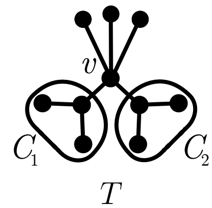

Figure 3: The vertex v has ℓ leaf neighbors in T.

Since T and v are arbitrary, we have shown that in every spanning tree T, every non-leaf vertex v has exactly t leaf neighbors, and the edges joining these leaves to v all have the same color. Thus every spanning tree is a t-flare. If for some fixed spanning tree T there are no edges in G joining leaves of T, then G is a t-flare as well. It remains to show that for some spanning tree T, there are no edges in G joining leaves of T, unless $G = C_3$ or $G = C_4$.

Let T be a fixed spanning tree of G, and let w_1w_2 be an edge in G joining leaves of T. If w_1 and w_2 have a common neighbor in T, then let it be u. Since u is a non-leaf vertex of T with two leaf neighbors in T, and T is a t-flare, we have $t \geq 2$, and the edges uw_1 and uw_2 have the same color.

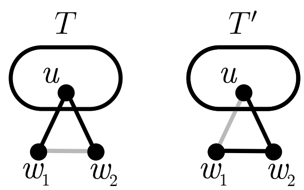

Figure 4: Spanning trees T and T'; the gray edges are omitted.

Consider the spanning tree T' obtained from T by replacing uw_1 with w_1w_2, as shown in Figure 4. In T', vertex w_2 is not a leaf, so it must have t leaf neighbors. Since its neighbors in T' are w_1 and u, we have $t = 2$, and u is a leaf in T'. Also, the edges w_1w_2 and w_2u have the same color. Thus G is a monochromatically edge-colored 3-cycle.

Suppose now that w_1 and w_2 have no common neighbor in T. Let u_1 and u_2 be their neighbors, respectively. Consider the spanning tree T' obtained from T by replacing u_1w_1 with w_1w_2, as shown in Figure 5.

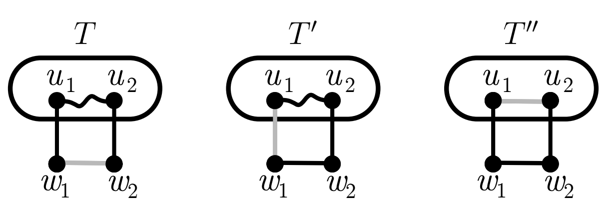

Figure 5: Three spanning trees T, T', and T''; the gray edges are omitted.

In T', vertex w_2 is not a leaf and has w_1 as its only leaf neighbor, so $t = 1$. Adding the edge w_1w_2 to T produces a cycle C. Deleting any edge e of C not in $\{u_1w_1, w_1w_2, u_2w_2\}$ produces a spanning tree T'' in which w_1 and w_2 are not leaves. Each has exactly one leaf neighbor in T''. Hence u_1 and u_2 are leaves of T''. Since u_1 and u_2 are not leaves in T, we conclude that G has four vertices and that the deleted edge e is u_1u_2, so $C_4 \subseteq G$. Neither u_1w_2 nor u_2w_1 can lie in $E(G)$, since otherwise G contains a spanning tree isomorphic to $K_{1,3}$ which is not a t-corona when $t = 1$. Now $G = C_4$ and, since $t = 1$, G is properly edge-colored. □

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