Matchings and The Chinese Postman Problem in Odd-Regular Graphs

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Joint work with
Suil O – College of William & Mary

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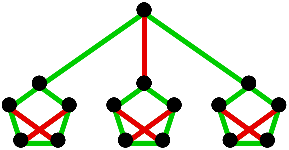

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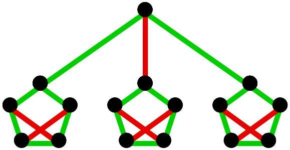

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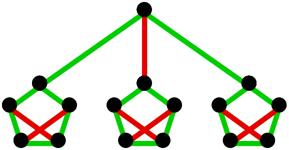

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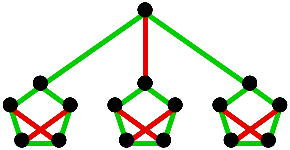

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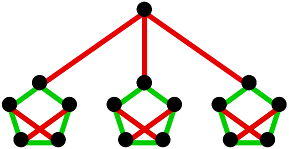

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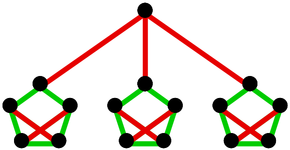

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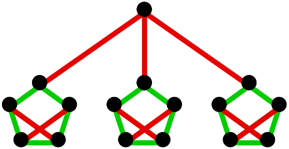

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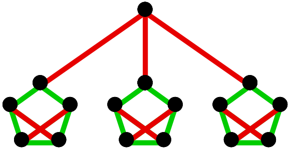

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Thm. (O–West [2011]) Determined $\min \alpha'(G)$ over k-conn. ℓ-regular n-vertex graphs and found those where equality holds. For example, if $G \in \mathcal{F}_{n,r}$ is $(2t+1)$-conn. with $t > 0$, then $\alpha'(G) \geq \frac{n}{2} - \frac{r-t}{2(r+1)^2+t}\frac{n}{2}$.

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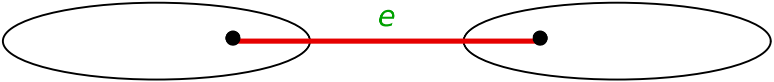

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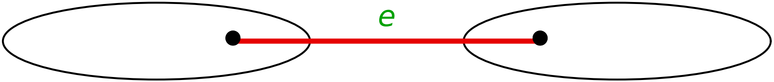

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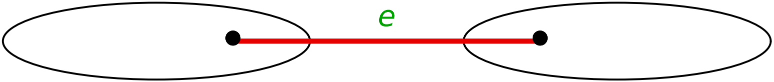

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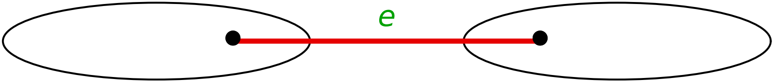

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We seek $(2r+1)$-regular graphs with many cut-edges.

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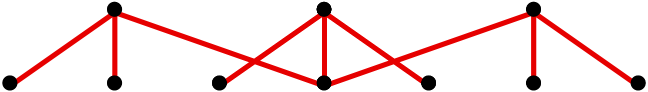

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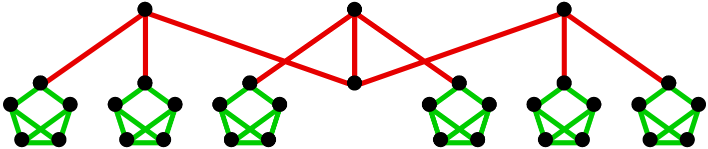

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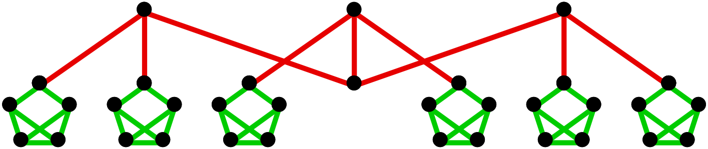

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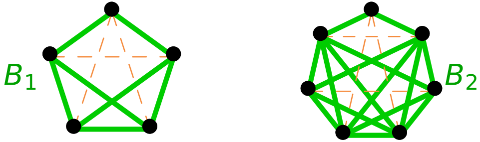

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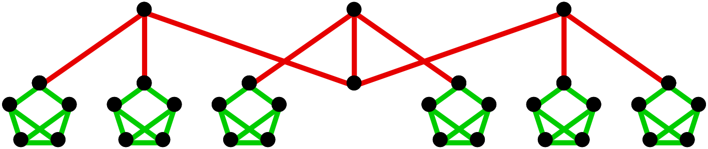

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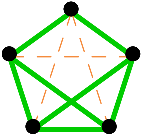
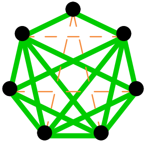
B_1   B_2

B_r is the smallest such graph.

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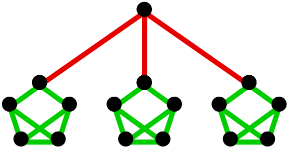

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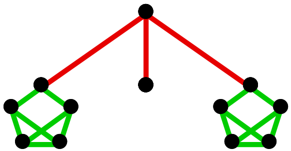

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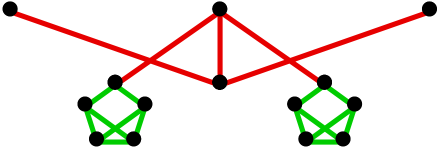

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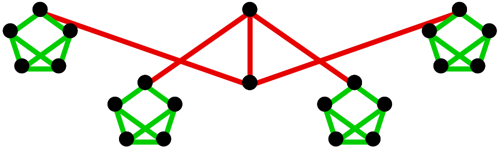

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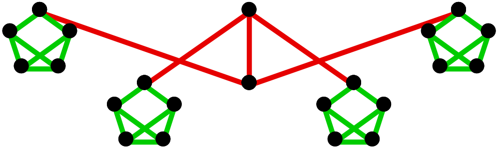

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How do $p(G)$ and the number of cut-edges change?

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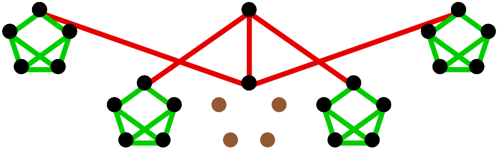

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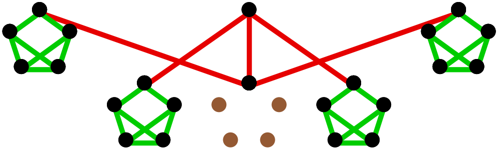

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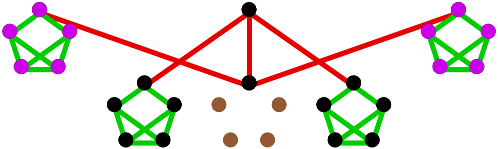

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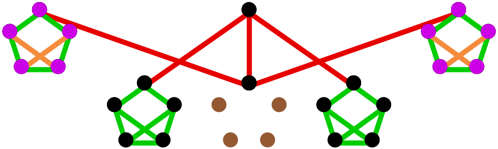

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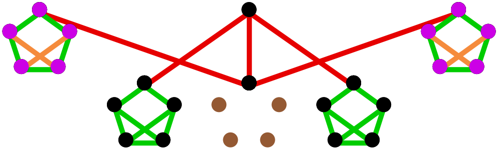

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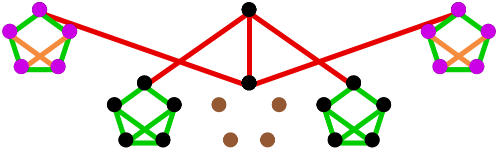

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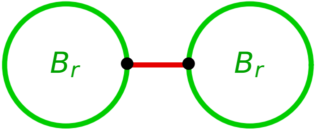
Pf. Basis: $n = 4r+6$. ∎



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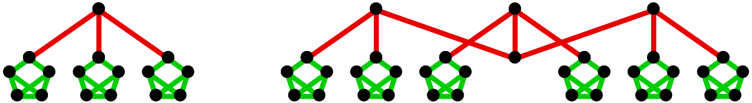

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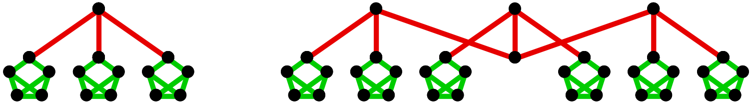

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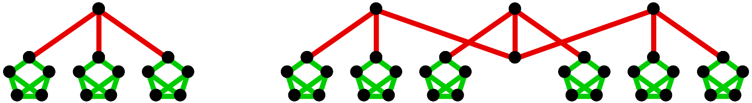

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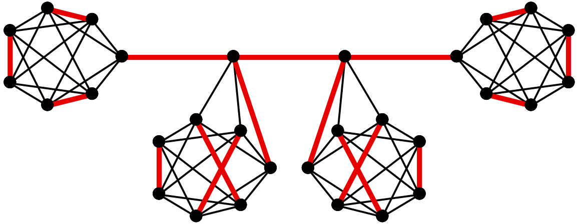
Example achieving equality for $r = 2$ (parity subgraph includes all cut-edges plus a matching on $n-2$ vertices):



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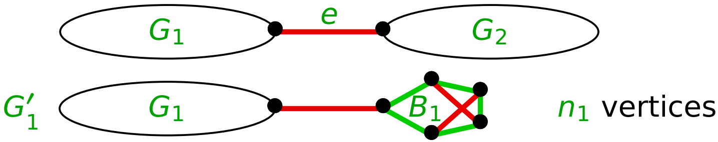

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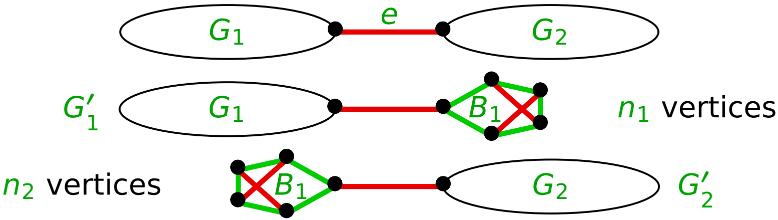

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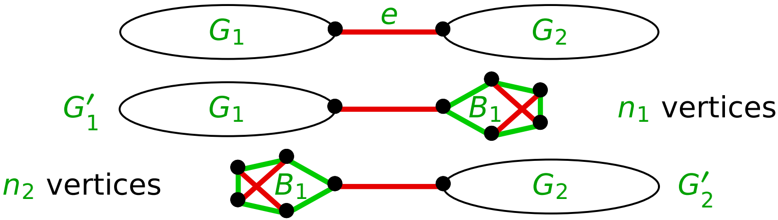

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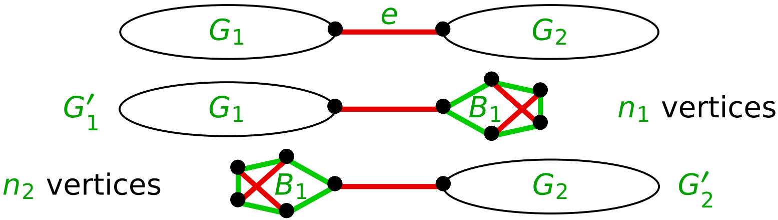

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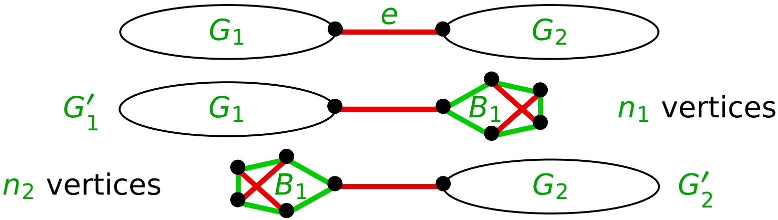

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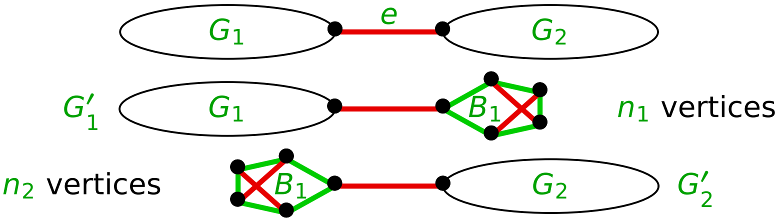

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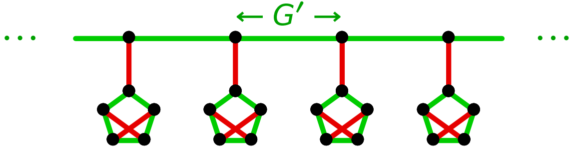

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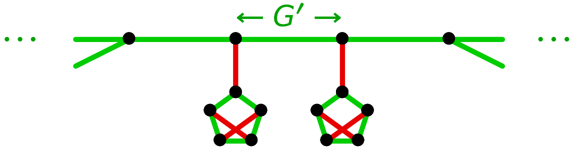

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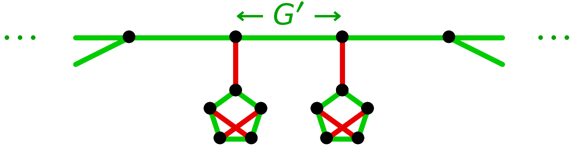

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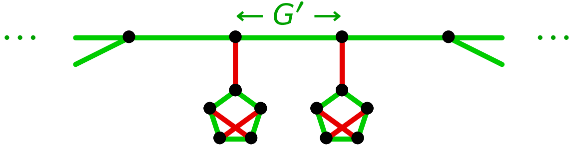

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Idea: Combine a parity subgraph of G'' with the red edges for balloons in G.

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Pf. To cover each edge p times, $|\mathcal{M}| = p(2r+1)$. The total weight over all the matchings is pW; pigeonhole. ■

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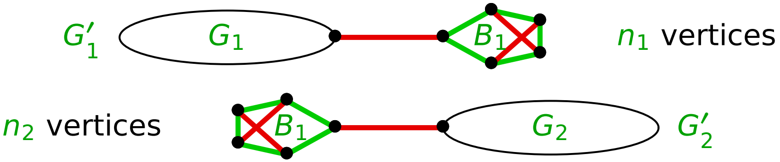

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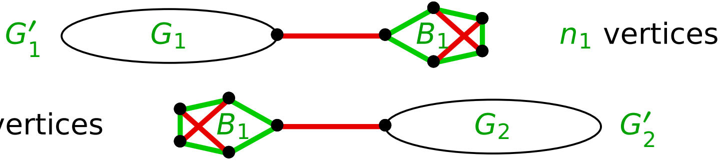

Now G also is constructed by attaching copies of B_1 to the leaves of a tree in $\mathcal{T}_1$, so $G \in \mathcal{H}_1$.

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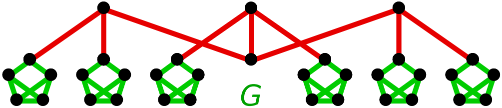

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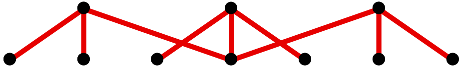

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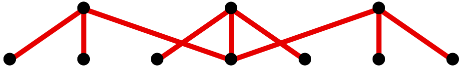

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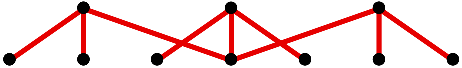

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By vertex degrees, $(2r+1)n' - 2rb(G) = 2m' \ge 2n' - 2$.
$\therefore$ $2rb(G) \le (2r-1)n' + 2$.

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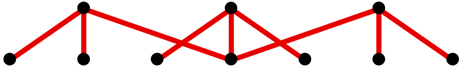

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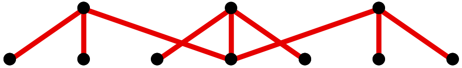

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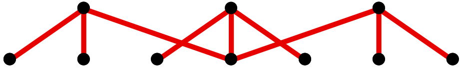

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Pf. More work for cuts with size between 3 and $2r-1$. ■