Sharp bounds for the Chinese Postman Problem in 3-regular graphs and multigraphs

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Abstract

The Chinese Postman Problem in a multigraph is the problem of finding a shortest closed walk traversing all the edges. In a $(2r+1)$-regular multigraph, the problem is equivalent to finding a smallest spanning subgraph in which all vertices have odd degree. In 1994, Kostochka and Tulai established a sharp upper bound for the solution. In this paper, we give simple proofs of their bounds for 3-regular graphs and 3-regular multigraphs and characterize when equality holds in those cases. We conjecture that a more specific construction characterizes equality for $r \geq 2$.

1 Introduction

The Chinese Postman Problem was introduced in the early 1960s by the Chinese mathematician Guan Meigu. Roughly speaking, a postman wishes to traverse every road in a city to deliver the mail, using the least possible total distance. A *postman tour* in a connected multigraph G is a closed walk containing all the edges of G. An *optimal* postman tour in a connected multigraph G is a shortest closed walk traversing all edges in G. Since all edges of G must be used, we are interested only in the additional length needed. Let $p(G) = l - |E(G)|$, where l is the minimum length of a postman tour.

Since a postman tour is an Eulerian supergraph obtained by repeating some edges, $p(G)$ equals the minimum number of edges in a parity subgraph of G, where a *parity subgraph* is a spanning subgraph H of G such that $d_G(v) \equiv d_H(v) \mod 2$ for every vertex v in G. We call

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$p(G)$ the *parity number* of G. Kostochka and Tulai [5] obtained upper bounds on the parity number of $(2r+1)$-regular k-edge-connected multigraphs. We state their result in full, since we are interested in short proofs and characterizations of equality for two special cases.

Definition 1.1. An *odd cut* is an edge cut $[S,\overline{S}]$ such that $|S|$ and $\left|\overline{S}\right|$ are odd. A $(2r+1)$-regular graph or multigraph G is *m-robust* if, for each odd cut $[S,\overline{S}]$ with at most $2r$ edges, both S and $\overline{S}$ have size at least m.

Theorem 1.2 (Kostochka and Tulai [5]). *If G is a $(2r+1)$-regular $2t$-edge-connected m-robust multigraph, where m is odd, then $p(G) \le \frac{n}{2} + \max\{0, \frac{n-2m}{\lfloor (r+t)/(r-t) \rfloor (m+1)}\}$.*

Furthermore, they showed that the bound is sharp when $\left\lfloor \frac{r+t}{r-t} \right\rfloor^2 \ge 2t+1$. We are interested in the case $t=0$, where this inequality always holds.

A $(2r+1)$-regular simple graph is $(2r+3)$-robust, since $1 \le |S| \le 2r+1$ yields $\left|[S,\overline{S}]\right| \ge |S|\,(2r+2-|S|) > 2r$; that is, odd cuts with at most $2r$ edges must have at least $2r+3$ vertices on each side. However, we can only guarantee that a regular multigraph is 3-robust. Thus Theorem 1.2 yields the following special cases. The restrictions stated for n permit dropping the maximization with 0 from the second term.

Corollary 1.3. *If G is a $(2r+1)$-regular graph with n vertices, where $n \ge 4r+6$, then $p(G) \le \frac{n}{2} + \frac{n-(4r+6)}{2r+4}$.*

Corollary 1.4. *If G is a $(2r+1)$-regular multigraph with n vertices, where $n \ge 6$, then $p(G) \le \frac{n}{2} + \frac{n-6}{4}$.*

When t is larger, the family of graphs is more restricted, and the upper bound on $p(G)$ is smaller. For example, it is well known (Bäbler [2]) that every $(2r+1)$-regular $2r$-edge-connected graph has a perfect matching, making the parity number only $n/2$. This generalizes the result of Petersen [8] that 3-regular graphs without cut-edges have perfect matchings.

In Section 3, we give simple proofs of Corollaries 1.3 and 1.4 for $r=1$, using a result of Edmonds [4] about equicovering the edges of regular graphs by perfect matchings. We also determine the families for which equality holds. We describe these families and compute their parity numbers in Section 2. The negative additive terms in the numerators of the upper bounds guarantee that equality cannot hold for disconnected graphs or multigraphs, so we henceforth assume that G is connected.

Our family $\mathcal{H}$ of 3-regular graphs properly contains the 3-regular graphs used by Kostochka and Tulai to show that Corollary 1.3 is sharp for $r=1$. We introduced this family in [6], where we also studied its generalization to $(2r+1)$-regular graphs in relation to maximizing the number of cut-edges and minimizing the size of a maximum matching among connected $(2r+1)$-regular n-vertex graphs. Our family $\mathcal{H}^*$ of 3-regular multigraphs has a very similar structure, so we analyze the two problems simultaneously.

For general r, Kostochka and Tulai [5] provided connected $(2r+1)$-regular graphs that achieve equality in Corollary 1.3. Surprisingly, these graphs have matchings that leave only two vertices unmatched. This is counterintuitive, since the parity number of a regular n-vertex graph of odd degree is minimized (having value $n/2$) if and only if the graph has a perfect matching. In Section 2 we describe for $r \geq 2$ a family $\mathcal{F}_r$ of $(2r+1)$-regular graphs that contains the graphs constructed by Kostochka and Tulai. We conjecture that $\mathcal{F}_r$ is the family for which equality holds in Corollary 1.4.

2 The Constructions

We first construct the classes of 3-regular graphs and multigraphs that we will show are those achieving equality in Corollaries 1.3 and 1.4.

Definition 2.1. A *balloon* in a graph or multigraph G is a maximal 2-edge-connected sub[multi]graph that is incident to exactly one cut-edge of G. Let B denote the graph obtained from the complete graph K_4 by subdividing one edge, and let B^* denote the multigraph obtained from K_3 by duplicating one edge. Let $\mathcal{T}$ be the family of trees (with at least two vertices) such that every non-leaf vertex has degree 3. Let $\mathcal{H}$ [or $\mathcal{H}^*$] be the family of 3-regular [multi]graphs obtained from trees in $\mathcal{T}$ by identifying each leaf of such a tree with the vertex of degree 2 in a copy of B [or B^*, respectively].

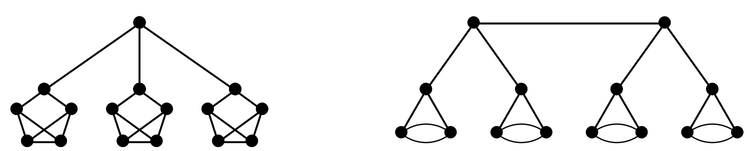

Figure 1: Members of $\mathcal{H}$ and $\mathcal{H}^*$

Remark 2.2. Figure 1 shows graphs in $\mathcal{H}$ and $\mathcal{H}^*$; the smallest such graphs have 10 and 6 vertices, respectively. The copies of B or B^* in a graph in $\mathcal{H}$ or $\mathcal{H}^*$ are balloons corresponding to leaves in a tree in $\mathcal{T}$.

In a 3-regular graph or multigraph, a balloon has one vertex of degree 2, and the others have degree 3; thus B and B^* are the smallest possible balloons, and smallest members of $\mathcal{H}$ and $\mathcal{H}^*$ are the smallest 3-regular graphs or multigraphs that have cut-edges (and balloons).

We begin by computing the parity numbers of graphs in $\mathcal{H}$ and $\mathcal{H}^*$.

Lemma 2.3. *If G is regular of odd degree, then every cut-edge is in every parity subgraph.*

Proof. Let e be a cut-edge in G. To have even degree sum, each component of $G - e$ must have an odd number of vertices. Since a parity subgraph has odd degree at each vertex, the parity subgraph must contain e. □

Proposition 2.4. *Every n-vertex graph in $\mathcal{H}$ has parity number $(2n-5)/3$. Every n-vertex multigraph in $\mathcal{H}^*$ has parity number $(3n-6)/4$.*

Proof. A tree in $\mathcal{T}$ with ℓ leaves has $\ell - 2$ non-leaf vertices and $2\ell - 3$ edges, all cut-edges.

The corresponding graph in $\mathcal{H}$ has ℓ balloons, $6\ell - 2$ vertices, and $2\ell - 3$ cut-edges. After including all the cut-edges (by Lemma 2.3), a parity subgraph must still add edges incident to the four non-cut vertices in each balloon, so smallest parity subgraphs have $4\ell - 3$ edges. With $\ell = (n+2)/6$, the claimed formula follows.

Similarly, the corresponding multigraph in $\mathcal{H}^*$ has ℓ balloons, $4\ell - 2$ vertices, and $2\ell - 3$ cut-edges. In addition to including the cut-edges, a parity subgraph must add another edge in each balloon. Smallest parity subgraphs thus have $3\ell - 3$ edges. With $\ell = (n+2)/4$, the claimed formula follows. □

Thus the graphs in $\mathcal{H}$ and $\mathcal{H}^*$ achieve equality in Corollary 1.3 and Corollary 1.4 when $r = 1$. A natural generalization of $\mathcal{H}$ to $(2r+1)$-regular graphs uses trees whose internal vertices have degree $2r+1$ and attaches balloons with $2r+3$ vertices at the leaves (balloons in $(2r+1)$-regular graphs have at least $2r+3$ vertices). The parity number can be computed as in Proposition 2.4, but the resulting formula is smaller than the bound from Corollary 1.3. Thus we need a different family when $r > 1$.

Let $o(H)$ denote the number of components of odd order in a graph H. In a graph G, the *deficiency* of a vertex subset S is $o(G-S) - |S|$, written $\mathrm{def}(S)$. The *deficiency* of G, written $\mathrm{def}(G)$, is $\max_{S \subseteq V(G)} \mathrm{def}(S)$. By the Berge–Tutte Formula [1], $\mathrm{def}(G)$ equals the minimum number of vertices left uncovered by a matching. A graph G is *factor-critical* if $G - v$ has a perfect matching whenever $v \in V(G)$. We conjecture that the following family is the family of $(2r+1)$-regular graphs (with at least $2r+8$ vertices) achieving equality in Corollary 1.3.

Definition 2.5. For $r \geq 2$, let $\mathcal{F}_r$ be the family of connected $(2r+1)$-regular graphs G satisfying the following conditions:
(i) $|V(G)| = (2r+4)k - 2$ for some positive integer k with $k \geq 3$,
(ii) $\mathrm{def}(G) = 2$,
(iii) There exists a vertex subset S in G with $\mathrm{def}(S) = 2$ such that (1) $o(G - v) = 3$ for $v \in S$, and (2) each component of $G - S$ has $2r + 3$ vertices and is factor-critical.

We show first that this family is nonempty, building examples like those in [5].

Definition 2.6. Let B_r be the complement of the forest consisting of r isolated edges and one component with two edges; note that B_r is the unique graph with $2r+3$ vertices whose vertices all have degree $2r+1$ except for one vertex of degree $2r$. Let Q_r be the complement of the disjoint union of paths with $r+1$ and $r+2$ vertices; note that Q_r also has $2r+3$ vertices, of which $2r-1$ have degree $2r$ and four have degree $2r+1$.

Proposition 2.7. *For $r \geq 2$ and $k \geq 3$, the family $\mathcal{F}_r$ contains a $(2r+1)$-regular graph with $(2r+4)k-2$ vertices.*

Proof. Begin with two disjoint copies of B_r and a path P joining their vertices of degree $2r$. Associate with each internal vertex v of P a copy $Q(v)$ of Q_r. Add edges joining v to the vertices of degree $2r$ in $Q(v)$. Figure 2 shows such a graph for $r=2$, with $|V(P)|=4$. In [5] the construction is not fully explicit; for example, there are other choices for $Q(v)$ with the same vertex degrees. We merely show that it produces graphs in $\mathcal{F}_r$.

By construction, the resulting graph G is $(2r+1)$-regular and has $(2r+4)k-2$ vertices, where $k=|V(P)|$. Each subgraph associated with an internal vertex v has a perfect matching; match v to a vertex having degree $2r$ in $Q(v)$ that lies on the longer path in the complement of $Q(v)$, and match the rest of that path to the other path. Hence G has a matching whose only uncovered vertices are the endpoints of P, yielding $\mathrm{def}(G) \leq 2$.

The needed set S consists of the internal vertices of P. Deleting S leaves $|S|+2$ odd components: balloons at the ends of P plus $\{Q(v)\colon\ v \in S\}$ (hence $\mathrm{def}(S)=2$ and $\mathrm{def}(G)=2$). For $v \in S$, the graph $G-v$ has exactly three components, all with odd order: $Q(v)$ and one component for each copy of B_r. A component containing B_r has $2r+4$ vertices for each internal vertex of the subpath of P joining v and the copy of B_r, so B_r makes the number of vertices odd. Finally, both Q_r and B_r have $2r+3$ vertices and a spanning cycle, so they are factor-critical.

Thus G satisfies the conditions of Definition 2.5. □

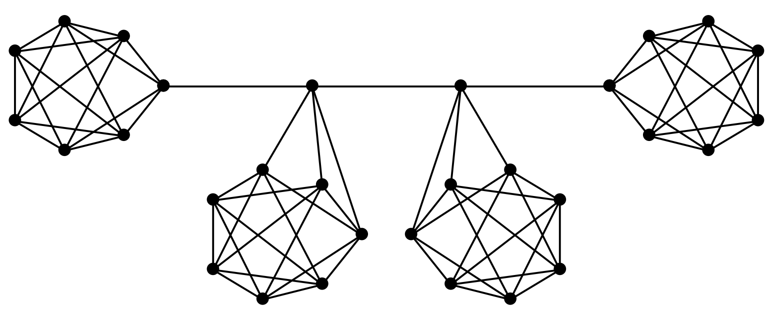

Figure 2: A graph in $\mathcal{F}_2$

We next show that all graphs satisfying the defining properties of $\mathcal{F}_r$ achieve equality in the bound of Corollary 1.3. We begin with a simple observation.

Observation 2.8. *Let G be a graph in which every vertex has odd degree. If $\{X_1,\ldots,X_a\}$ is a partition of $V(G)$, then $\sum_{i=1}^{a} o(G-X_i) \le 2p(G)$.*

Proof. In a parity subgraph H, count the edges joining two members of $\{X_1,\ldots,X_a\}$. At least $o(G-X_i)$ such edges have one end in X_i. Thus $2|E(H)|$ is at least the specified sum. □

Proposition 2.9. *Graphs in $\mathcal{F}_r$ achieve equality in the bound of Corollary 1.3.*

Proof. Let G be an n-vertex graph in $\mathcal{F}_r$, and let S be a vertex subset of G satisfying the conditions in (iii) of Definition 2.5. By these conditions, $n = |S| + (|S|+2)(2r+3)$, so $|S| = \frac{n-(4r+6)}{2r+4}$. We have $o(G-v) \ge 1$ for $v \notin S$, since regular graphs of odd degree have an even number of vertices, and $o(G-v) = 3$ for $v \in S$ by hypothesis. By applying Observation 2.8 to the partition of $V(G)$ into singleton sets, we have

$$2p(G) \;\ge\; \sum_{v\notin S} o(G-v) + \sum_{v\in S} o(G-v) \;\ge\; n + 2|S| \;=\; n + \frac{n-(4r+6)}{r+2},$$

which implies $p(G) \ge \frac{n}{2} + \frac{n-(4r+6)}{2r+4}$, as desired. □

Conjecture 2.10. *For $n > 4r+6$ and $r \ge 2$, equality holds in Corollary 1.3 if and only if $G \in \mathcal{F}_r$.*

There is an analogous definition and conjecture for multigraphs.

Definition 2.11. For $r \ge 2$, let $\mathcal{F}_r^*$ be the family of connected $(2r+1)$-regular multigraphs G satisfying the following conditions:
(i) $|V(G)| = 4k-2$ for some positive integer k with $k \ge 3$,
(ii) $\mathrm{def}(G) = 2$,
(iii) There exists a vertex subset S in G with $\mathrm{def}(S) = 2$ such that (1) $o(G-v) = 3$ for $v \in S$, and (2) each component of $G-S$ has exactly three vertices.

Multigraphs in $\mathcal{F}_r^*$ can be constructed by the same procedure as in Proposition 2.7, using B_r^* instead of B_r and a triangle with appropriate edge-multiplicities instead of Q_r. There is substantial flexibility in the edge-multiplicities; it is possible even to use multiple edges along the path P. A proof like that of Proposition 2.9 shows that the resulting multigraphs satisfy equality in Corollary 1.4. The corresponding conjecture is natural.

Conjecture 2.12. *For $n > 6$ and $r \ge 2$, equality holds in Corollary 1.4 if and only if $G \in \mathcal{F}_r^*$.*

3 The Upper Bound and Characterization of Equality

Among several definitions of "k-graph" in the literature is one by Seymour [9].

Definition 3.1. A *k-graph* is a k-regular multigraph G with an even number of vertices such that for every odd-sized subset X of $V(G)$, the edge cut $[X, \overline{X}]$ has size at least k.

We need a fundamental result about k-graphs due to Edmonds.

Theorem 3.2. (Edmonds [4]) *If G is a k-graph, then there is an integer p and a family $\mathcal{M}$ of perfect matchings such that each edge of G is contained in precisely p members of $\mathcal{M}$. (The members of $\mathcal{M}$ need not be distinct.)*

Lemma 3.3. *If G is a 2-edge-connected 3-regular multigraph with edge weights, then some perfect matching in G has weight at most $W/3$, where W is the sum of the edge weights.*

Proof. Every 2-edge-connected 3-regular multigraph is a 3-graph, since an odd number of edges must leave every odd-sized subset S of $V(G)$. Let $\mathcal{M}$ be a family of perfect matchings as guaranteed by Theorem 3.2. By counting two ways, $|\mathcal{M}|\frac{n}{2} = \frac{3n}{2}p$, which yields $|\mathcal{M}| = 3p$. Let $\mathcal{M} = \{M_1, \ldots, M_{3p}\}$, and let $w(M_i)$ be the total weight of all edges in M_i. Since $\sum w(M_i) = pW$, a lightest matching in the family $\mathcal{M}$ has weight at most $W/3$, by the pigeonhole principle. □

Let $\mathcal{G}_n$ and $\mathcal{G}_n^*$ be the families of connected n-vertex 3-regular graphs and multigraphs, respectively. We need the maximum number of balloons in members of these families. The bound for $\mathcal{G}_n$ is a special case of a lemma in [6]. Let $b(G)$ denote the number of balloons in a graph G.

Lemma 3.4. *If $G \in \mathcal{G}_n$, then $b(G) \le (n+2)/6$, with equality if and only if $G \in \mathcal{H}$.*
If $G \in \mathcal{G}_n^*$, then $b(G) \le (n+2)/4$, with equality if and only if $G \in \mathcal{H}^*$.

Proof. We counted the balloons for members of $\mathcal{H}$ and $\mathcal{H}^*$ in the proof of Proposition 2.4.

For the upper bound, obtain G' from G by shrinking each balloon to a single vertex; G' is connected, and the balloons of G become vertices of degree 1 in G'. The other vertices of G' have degree 3. Let $n' = |V(G')|$ and $m' = |E(G')|$. Since G' is connected, we have $m' \ge n' - 1$, and the degree-sum formula yields $3n' - 2b(G) = 2m' \ge 2n' - 2$. Thus $2b(G) \le n' + 2$. Letting q be the minimum number of vertices in a balloon, we have $n' \le n - (q-1)b(G)$. Combining the inequalities yields $2b(G) \le n + 2 - (q-1)b(G)$, which simplifies to $b(G) \le (n+2)/(q+1)$. Since $q = 5$ when $G \in \mathcal{G}_n$ and $q = 3$ when $G \in \mathcal{G}_n^*$, the bound follows.

Equality in the bound requires equality in each inequality. In particular, G' is a tree whose non-leaf vertices have degree 3, and each balloon has exactly q vertices. That is, $G' \in \mathcal{T}$, and G is in $\mathcal{H}$ or $\mathcal{H}^*$. □

For the proof of the main result, we need the notion of "threads". A *thread* in a [multi]graph G is a maximal path in G whose internal vertices have degree 2 in G.

Theorem 3.5. *If $G \in \mathcal{G}_n$ and $n \geq 10$, then $p(G) \leq \frac{2n-5}{3}$. If $G \in \mathcal{G}_n^*$ and $n \geq 6$, then $p(G) \leq \frac{3n-6}{4}$.*

Proof. Recall B and B^* from Definition 2.1. Let $\hat{B} = B$ when $G \in \mathcal{G}_n$, but $\hat{B} = B^*$ when $G \in \mathcal{G}_n^*$. Again let $q = |V(\hat{B})|$. We use induction on n, with basis $n = 2q$. Let $t(n)$ be the desired bound; note that $t(n) = \frac{n}{2} + \frac{n-2q}{q+1}$ in each case.

If G has no balloons, then G is 2-edge-connected and has a perfect matching, by the result of Petersen [8] (also G has a perfect matching when $n = 2q$ and G has balloons, since G then consists of two copies of $\hat{B}$ and one edge joining them). This yields $p(G) = \frac{n}{2} \leq t(n)$.

Hence we may assume that G has a balloon and that $n > 2q$. Let e be a cut-edge. Let G_1 and G_2 be the components of $G - e$. Since a cut-edge must appear in every parity subgraph (Lemma 2.3), $p(G) = p(G_1) + p(G_2) + 1$.

Let G_1' and G_2' be obtained from G by replacing G_2 or G_1, respectively, with $\hat{B}$. Every parity subgraph of G_i' contains e and a parity subgraph of G_i, and it uses at least $(q-1)/2$ edges in its copy of $\hat{B}$. Thus $p(G_i') = p(G_i) + (q+1)/2$, so $p(G) = p(G_1') + p(G_2') - q$. If neither G_1 nor G_2 is $\hat{B}$, then G_1' and G_2' are smaller than G. Letting $n_i = |V(G_i')|$, we have $n = n_1 + n_2 - 2q$. By applying the induction hypothesis to both G_1' and G_2',

$$p(G) = p(G_1') + p(G_2') - q \leq t(n_1) + t(n_2) - q = t(n). \tag{1}$$

In the remaining case, every cut-edge is incident to a copy of $\hat{B}$. Let each edge have weight 1. Form G' by deleting all the vertices of all the balloons. Each deleted balloon, with q vertices, was incident to $(3q+1)/2$ edges (including the pendant edge). If G' is 2-regular, then G has a perfect matching (by using a perfect matching in each graph consisting of a balloon plus its pendant edge), and

$$p(G) = \frac{n}{2} < t(n). \tag{2}$$

If G has only one cut-edge, then G has two balloons and $n = 2q$. If two cut-edges have a common endpoint, then the third edge incident to it is also a cut-edge, and G is the graph with $3q+1$ vertices in the specified family, since every cut-edge is incident to a copy of $\hat{B}$.

Otherwise, G' has minimum degree 2. Replace each thread of G' through vertices of degree 2 with one edge whose weight is the length of the thread, forming a weighted graph G''. Note that G'' is 3-regular and 2-edge-connected, so G'' is a 3-graph. Applying Lemma 3.3, G'' has a perfect matching M whose weight $w(M)$ is at most 1/3 of the total weight. The total weight is $m - b\frac{3q+1}{2}$, where $m = |E(G)|$ and b is the number of balloons in G.

We obtain a parity subgraph of G by using the threads in G' corresponding to M plus a perfect matching ($\frac{q+1}{2}$ edges) in each subgraph consisting of a balloon plus its pendant edge. Using the bound on b from Lemma 3.4,

$$p(G) \le p(G') + \frac{q+1}{2}b \le \frac{m}{3} + \left[-\frac{3q+1}{6} + \frac{q+1}{2}\right] b \le \frac{n}{2} + \frac{1}{3}\cdot\frac{n+2}{q+1} \le \frac{n}{2} + \frac{n-2q}{q+1} = t(n). \quad (3)$$

A bit of care is needed here, because the last inequality requires $n \ge 3q+1$. Since G has a cut-edge, it has at least two balloons. The balloons were deleted to form G', so G' has at least two vertices of degree 2, suppressed to form G''. Since G'' is a 3-regular multigraph, it has at least 2 vertices. Hence G has at least $2q+4$ vertices, which is enough unless G is simple with 14 vertices. In the one such example (two balloons, whose deletion leaves the graph with degrees $(3,3,2,2)$), G has a perfect matching, and $p(G) = n/2 < t(n)$.

We have proved $p(G) \le t(n)$. □

In this proof, we applied Lemma 3.3 to G'' to bound the parity number of G'. After obtaining the proof, we learned that Bermond, Jackson, and Jaeger [3] proved that the parity number of any 2-edge-connected multigraph is at most 1/3 of the number of edges, which is the same bound we obtained on $p(G')$. The proof of their result also uses Edmonds' result in Theorem 3.2, plus a result by Fleischner, so our argument to bound $p(G')$ can be considered at least as direct.

Since it has a perfect matching, a connected 3-regular G with $2q$ vertices and no cut-edges also achieves equality even though it is not in the specified family. However, an example with more than $2q$ vertices satisfying equality must be in the specified family.

Theorem 3.6. *Let $q = 5$ when $G \in \mathcal{G}_n$ and $q = 3$ when $G \in \mathcal{G}_n^*$. If $n > 2q$, then equality holds in the bound of Corollary 1.3 or 1.4 if and only if $G \in \mathcal{H}$ or $G \in \mathcal{H}^*$, respectively.*

Proof. Let $\hat{\mathcal{H}}$ denote the specified family of interest, $\mathcal{H}$ or $\mathcal{H}^*$. We showed in Proposition 2.4 that equality holds in $\hat{\mathcal{H}}$. Hence it suffices to show that if $p(G) = \frac{n}{2} + \frac{n-2q}{q+1} = t(n)$, then $n = 2q$ or $G \in \hat{\mathcal{H}}$.

For $n > 2q$, we use induction on n as in the proof of Theorem 3.5. To achieve equality in (1), we must have $p(G_i) = t(n_i)$ for $i \in \{1,2\}$. If neither G_1 nor G_2 is $\hat{B}$, then $|V(G_i)| > 2q$. Now the induction hypothesis applies, and $G_i' \in \hat{\mathcal{H}}$. Since shrinking the balloons in G_1' or G_2' yields a tree whose internal vertices have degree 3, the same holds also for G, so also $G \in \hat{\mathcal{H}}$.

In the remaining case, all cut-edges are incident to balloons. Delete the balloons to form G'; we have three subcases. If G' is a cycle, then $p(G) = \frac{n}{2} < t(n)$, as in (2). If G' is a single vertex, then $n = 3q+1$, $b = 3$ and $G \in \hat{\mathcal{H}}$. If G' is a graph with maximum degree 3, then the last part of (3) states $p(G) \le \frac{n}{2} + \frac{1}{3}\cdot\frac{n+2}{q+1} \le \frac{n}{2} + \frac{n-2q}{q+1} = t(n)$, with equality only when $\frac{1}{3}\cdot\frac{n+2}{q+1} = \frac{n-2q}{q+1}$. This simplifies to $n = 3q+1$, which does not occur when G' has maximum degree 3. □

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