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Note

Covering a Poset by Interval Orders

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The number of interval orders needed to cover the elements of an n-element poset is at most $\lceil\sqrt{n}\rceil$. More precisely, any poset with at most k^2 elements can be covered by k interval orders, with k interval orders needed if and only if the poset is the disjoint union of k chains of size k. © 1994 Academic Press, Inc.

In this note, we consider the problem of covering the elements of a poset (here a finite partially ordered set) with the minimum number of subposets that are interval orders, by analogy with the problem of covering a poset by the minimum number of chains. Interval orders are those posets representable by assigning an interval $I(x) \subseteq \mathbb{R}$ to each element $x \in P$ such that $x < y$ in P if and only if the right endpoint of $I(x)$ is to the left of the left endpoint of $I(y)$. Fishburn [2] characterized these posets as those with no 4-element subposet isomorphic to the disjoint union of two 2-element chains (denoted $\mathbf{2}+\mathbf{2}$). Note that any union of a chain and an antichain is thus an interval order.

The chain-covering problem is well-solved algorithmically, since Dilworth's min-max theorem [1] equating minimum chain covering and maximum antichain size is easy to implement by network flow methods [3]. No such min-max theorem holds for covering by interval orders. For chains, the forbidden subposet is a pair of incomparable elements, and any union of k pairwise incomparable elements requires k chains to cover it. However, the union of arbitrarily many disjoint 2-element chains can be covered by two interval orders, namely the antichain consisting of the minimal elements and the antichain consisting of the maximal elements.

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Maurice Pouzet asked at the 1991 meeting on Combinatorics of Ordered Sets at Oberwolfach for the complexity of minimum covering by interval orders, but this has apparently not been determined.

In this note, we solve the corresponding extremal problem in terms of the number of elements of the poset. Any poset with n elements can be covered by $\lceil\sqrt{n}\rceil$ interval orders. With the appropriate inductive hypothesis, it is not difficult to prove this and determine the extermal posets.

THEOREM. *If P is a poset with at most k^2 elements, then P can be covered by k interval orders, with equality required if and only if P is the poset Q_k formed by the disjoint union of k chains of size k. Furthermore, if $P \neq Q_k$, then these $k-1$ interval orders can all be assumed to be the disjoint union of a chain and an antichain, except possibly for one copy of the poset N whose comparability graph is a 4-vertex path, needed only if $|P| = k^2$.*

Proof. First we verify that Q_k requires k interval orders, by induction on k. The basis is trivial. For $k > 1$, no interval order contained in Q_k includes two elements from each of two different chains. This implies that deleting the elements of any interval order contained in Q_k leaves a copy of Q_{k-1}, which requires $k-1$ more interval orders.

For the upper bound, call a poset that is the union of a chain and an antichain an *L-poset*, with arbitrary relations allowed between the chain and the antichain. Note that every L-poset is an interval order. For the basis, consider $k=2$. Every poset other than $Q_2 = \mathbf{2}+\mathbf{2}$ that has at most four elements is an L-poset except for the poset N, which is nevertheless also an interval order.

Now consider an arbitrary P with at most k^2 elements, for $k > 2$. The largest L-subposet L of P is the union of a largest chain and a largest antichain (sizes $h(P)$ and $w(P)$, respectively, called the *height* and *width* of P), so $|L| = h(P) + w(P) - 1$. Since P can be partitioned into $w(P)$ chains (by Dilworth's Theorem [1]), we have $|P| \leqslant w(P)\,h(P)$. This implies $w(P) + h(P) \geqslant 2\sqrt{|P|}$, with equality if and only if $w(P) = h(P) = \sqrt{|P|}$. This yields $|P - L| < (k-1)^2$ if $|P| < k^2$ or $w(P) + h(P) > 2k$, in which case $P - L$ can be covered by $k-2$ additional L-posets, by the induction hypothesis.

This leaves the case where $|P| = k^2$ and $h(P) = w(P) = k$ but $P \neq Q_k$. By Dilworth's Theorem, we can partition the elements of P into k chains of size k. For each $x \in P$, let $h(x)$ denote the height of x on its chain in this decomposition; note that the elements of fixed height form an antichain and that there are k elements of each height. Since $P \neq Q_k$, there is some additional relation $x < y$ between elements of different chains. Choose one of the chains in the decomposition that does not contain x or y, and choose a value j not equal to $h(x)$ or $h(y)$. Then the union of the chosen chain and the elements of height j is an L-poset with $2k-1$ elements and $P-L$ is a poset on $(k-1)^2$ elements other than Q_{k-1}. By the induction hypothesis, $P-L$ can be covered by $k-3$ additional L-posets and one copy of N. ∎

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