JOURNAL OF COMBINATORIAL THEORY, Series A 41, 105–116 (1986)

"Poly-unsaturated" Posets: The Greene–Kleitman Theorem Is Best Possible

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Communicated by the Managing Editors

Received October 4, 1983

Let $C = \{C_i\}$ be a partition of a partially ordered set into chains; C is *k-saturated* if $\sum_{C_i \in C} \min\{k, |C_i|\}$ equals the size of the largest union of k antichains. Greene and Kleitman proved that for any poset P and any integer k, P has a chain partition that is both k and $k+1$-saturated. We prove that this is best possible in a strong sense, by exhibiting a poset of each height $n \geqslant 3$ in which no partition is k-saturated for any two nonconsecutive values of $k \leqslant n$. Furthermore, by showing that these posets are 2-dimensional, we obtain a poset of each width $n \geqslant 4$ that has the same property for antichain partitions. © 1986 Academic Press, Inc.

1. INTRODUCTION AND CONSTRUCTION

A celebrated theorem of Greene and Kleitman [7] proved equality for a particular pair of dual integer packing and covering problems in any partially ordered set (henceforth "poset"). A *k-family* in an arbitrary poset is a collection of elements containing no chain of $k+1$ elements (henceforth "$k+1$-chain"). A collection of elements is a k-family if and only if is the union of k antichains. For any chain partition C of the poset, $m_k(C) = \sum_{C_i \in C} \min\{k, |C_i|\}$ yields a trivial upper bound on the size of any k-family. The maximum size of a k-family is denoted d_k and called the kth "Dilworth number"; Dilworth [2] proved that $d_1 = \min_C m_1(C)$. In general, a partition C is called *k-saturated* if $d_k = m_k(C)$. Greene and Kleitman proved that for every k and every poset P, P has a k-saturated partition. Later proofs include [1, 3, 4, 8, 9]; see [11] for a discussion of these. Greene and Kleitman proved that not only is there a k-saturated partition, in fact there is a partition that is both k- and $k+1$-saturated. Hoffman and Schwartz [8] call this the t-phenomenon.

In general, one cannot guarantee more than the Greene–Kleitman result. The standard example is the 6-point poset of height 3 appearing in Fig. 1 under the name P_3. The partition into two 3-chains is 1- and 2-saturated but not 3-saturated; the partition into a 4-chain and two isolated points is

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2- and 3-saturated but not 1-saturated. In this note we show that the Greene–Kleitman result is best possible in a strong sense. A poset has *height n* (notation $h(P)$) if its longest chain has $n+1$ elements. Beginning with P_3, we inductively obtain a poset P_n of height n that has no partition that is k-saturated for nonconsecutive values of k $(1 \leqslant k \leqslant n)$. The proof of this is completed in Section 3. To start the lemmas, we need some information about the maximum k-families, also called *Sperner* k-families, in P_n. Although the chain partitioning result is of considerably more interest, we consider the antichain partitions first in Section 2 because a minor part of this is used on the way to the chain partitioning result. The result about antichain partitions says that P_n has an antichain partition that is as well behaved as its chain partitions are badly behaved. In particular, it has antichain partition that is k-saturated for all k, as defined below.

Greene [5, 6] proved that the t-phenomenon holds also for the conjugate problem of saturated antichain partitions. Just as a k-family is the union of k antichains, so a family of poset elements containing no $k+1$-antichain is the union of k chains, by Dilworth's theorem. Such families are called k-cofamilies. Let $\hat{d}_k$ be the size of the largest such union. For any antichain partition $A = \{A_i\}$, the same function $m_k(A) = \sum_{A_i \in A} \min\{k, |A_i|\}$ provides an upper bound on $\hat{d}_k$. If $\hat{d}_k = m_k(A)$, the antichain partition A is k-*saturated.* By Greene's result, every poset has simultaneously k, $k+1$-saturated antichain partitions (also proved by Frank [4]). Again, we show this is best possible, by using posets obtained from $\{P_n\}$. We show that P_n is 2-dimensional by inductively constructing two linear extensions whose intersection is P_n. Reversing one of the extensions yields a conjugate poset in which every chain of P_n becomes an antichain, and vice versa. Hence a chain partition of P_n is k-saturated if and only if the partition of its conjugate into the corresponding antichains is also k-saturated.

This investigation was motivated by a search for a counterexample to a conjecture of West and Saks [10], which would generalize the Greene–Kleitman theorem to a setting involving direct products of posets. West and Tovey [13] pointed out that certain posets with completely saturated partitions (k-saturated for all k) satisfy even the strongest versions of that conjecture; the posets P_n represent the greatest possible departure from the existence of completely saturated partitions. The conjecture—or perhaps we should say the question—is whether the largest semiantichain and smallest unichain covering in a direct product of posets have the same size, where a *unichain* in a direct product is a chain in which one coordinate remains fixed, and a *semiantichain* is a collection of elements containing no pair on a unichain. Despite the apparent departure from "nice" behavior by $\{P_n\}$, equality has been verified for all $P_m \times P_n$ [12], based on properties of $\{P_n\}$ not discussed here.

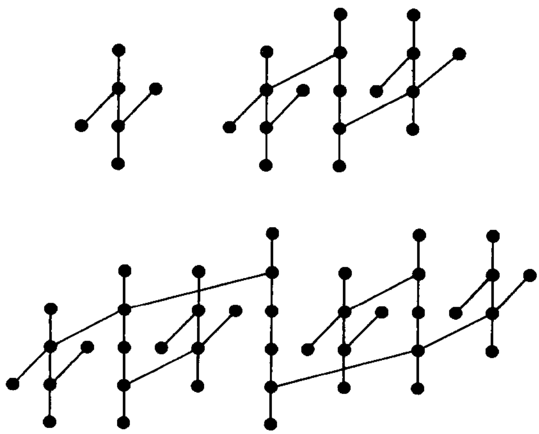

FIG. 1. P_3, P_4, P_5.

To construct P_n for $n > 3$, begin by taking two copies of P_{n-1} and a "central chain" of size $n+1$. Regard the two copies of P_{n-1} as the "left" or "lower" copy P_n^- and the "right" or "higher" copy P_n^+. This makes it natural to call the "central chain" P_n^0. Denote the top two and bottom two elements of P_n^0 by a_n, b_n, c_n, d_n, in descending order. Denote the similarly defined elements in P_n^- and P_n^+ by a_n^-, b_n^-, c_n^-, d_n^- and a_n^+, b_n^+, c_n^+, d_n^+. To complete the construction of P_n, add two more relations $b_n > b_n^-$ and $c_n < c_n^+$, and those implied by transitivity.

For P_3 the central chain is the unique chain of size 4. The construction and terminology are illustrated in Fig. 1. For $n \geqslant 4$ the decomposition of P_n into P_{n-1}, P_n^0, P_n^+ is unique, since although there are $n+1$-chains other than P_n^0, there is no other whose deletion leaves two copies of P_{n-1}.

Several other bits of notation will be useful. For the purpose of calculating m_k, we can index the chains of any partition in decreasing order of size, $|C_1| \geqslant |C_2| \geqslant \cdots$, and associate C with this partition of the integer $|P_n|$. We also summarize the k-family sizes and partition bounds by writing $\bar{d}(P) = (d_1(P), d_2(P), \ldots)$ and $\bar{m}(C) = (m_1(C), m_2(C), \ldots)$. Since every partition is k-saturated for all $k > h(P)$, we stop these vectors at $h(P)$. For P_4, $\bar{d} = (5,10,13,15)$. P_4 has three partitions of interest, whose size sequen-

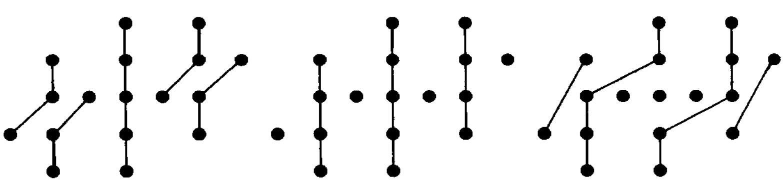

FIG. 2. Saturated chain partitions of P_4.

ces are (53333), (5441111), and (5522111), as illustrated in Fig. 2. The m-vectors for these are, respectively, (**5**,**10**,15,16), (7,**10**,**13**,16), and (7,11,**13**,**15**), where the saturated values are boldface.

2. The structure of Maximum k-Families in P_n

We begin by computing $d_k^n = d_k(P_n)$. The first lemma describes the construction of all maximum k-families in P_n.

Lemma 1. *For all k, there exist maximum k-families in P_n that contain either one of $\{a_n, d_n\}$. (In fact, maximum k-families exist having any subset of $\{a_n, d_n\}$, except that "both" requires $k \geq 2$, and "neither" forbids $n = k = 3$.) Furthermore, if $n \geq 4$,*

(a) *$d_n^n = |P| - 2$. For $n \geq 5$ the two deleted elements consist of one from $\{a_n, b_n, b_n^-\}$ and one from $\{c_n, d_n, d_n^+\}$. For $n = 4$ they are one of $\{a_n, b_n, b_n^-, c_n^-, d_n^-\}$ and one of $\{c_n, d_n, c_n^+, b_n^+, a_n^+\}$. In either case, they must include at least one of $\{a_n, b_n, c_n, d_n\}$.*

(b) *For $1 \leq k \leq n-1$, $F \subset P_n$ is a maximum k-family if and only if it can be expressed as $F^- \cup F^+ \cup S$, where F^- and F^+ are maximum k-families in P_n^- and P_n^+ and S consists of k elements from P_n^0 satisfying*

$$|\{a_n, b_n\} \cap S| \leq |\{a_n^-\} \cap F^-| \qquad \text{and} \qquad |\{c_n, d_n\} \cap S| \leq |\{d_n^+\} \cap F^+|.$$

Proof. The assertion about $\{a_n, d_n\}$ can be verified for $n = 3$ by listing all maximum k-families.

Now assume $n \geq 4$. To obtain a maximum n-family, we must delete the smallest set that will destroy all $n+1$-chains. If $n \geq 5$, there are $n+1$-chains that intersect only in $\{a_n, b_n, b_n^-\}$, and $n+1$-chains disjoint from them that intersect only in $\{c_n, d_n, d_n^+\}$, so we must delete one from each set. Doing so breaks all $n+1$-chains, as long as we delete at least one from $\{a_n, b_n, c_n, d_n\}$. Thus there are 8 maximum n-families; and the claim about $\{a_n, d_n\}$ holds. If $n = 4$, the same analysis applies with $\{b_n^-\}$ replaced by $\{b_n^-, c_n^-, d_n^-\}$ and $\{c_n^+\}$ replaced by $\{c_n^+, b_n^+, a_n^+\}$.

For $k < n$, any k-family formed as described in (b) must be a maximum k-family, since it achieves the bound given by any chain partition consisting of P_n^0 and k-saturated partitions of P_n^- and P_n^+. If there is any such k-family, then all maximum k-families use maximum k-families from both P_n^- and P_n^+. To show that there is such a k-family, we use induction. We know by induction that there are maximum k-families $F^- \subset P_n^-$ and $F^+ \subset P_n^+$ with $a_n^- \in F^-$ and $d_n^+ \in F^+$. Let $F = F^- \cup F^+ \cup S$, where S is any subset of P_n^0 containing at most one of $\{a_n, b_n\}$ and at most one of

$\{c_n, d_n\}$. If F contains a $k+1$-chain, the chain must consist of a k-chain from F^- topped by one of $\{a_n, b_n\}$ (or similarly with F^+ and $\{c_n, d_n\}$), but then F^- itself contains a $k+1$-chain, since a_n^- has the same comparabilities in P_n^- as $\{a_n, b_n\}$. Note that the induction hypothesis is preserved, since with $k<n$ either of $\{a_n, d_n\}$ could be included (any subset if $k \geqslant 2$).

If a_n^- does not belong to a maximum k-family F^- chosen from P_n^-, then neither of $\{a_n, b_n\}$ can be added in constructing a maximum k-family in P_n (similarly for d_n^+ and $\{c_n, d_n\}$). Again this follows from having identical comparabilities in P_n^-. If adding one of $\{a_n, b_n\}$ does not create a $k+1$-chain, then adding a_n^- to F^- does not create a $k+1$-chain in P_n^-, so F^- was not a maximum k-family. Note that when $k=n-1$, this implies that $\{a_n^-, d_n^+\}$ belongs to every maximum $n-1$-family, and when $k=n-2$ at least one of them must belong. ∎

Lemma 1 allows us to compute d_k^n, by providing the recurrence $d_k^n = 2d_k^{n-1}+k$ for $k<n$. Using $d_n^n = |P_n|-2$ for $n \geqslant 4$, $|P_n| = 2|P_{n-1}|+n+1$ for $n \geqslant 4$, $|P_3|=6$, and $d^3=(2, 4, 5)$, the recurrence for d_k^n yields

$$|P_n| = 3 \cdot 2^{n-1} - n - 3$$

$$d_k^n = 3 \cdot 2^{n-1} - k - 5 \cdot 2^{n-k} \qquad \text{for} \quad 4 \leqslant k \leqslant n$$

$$d_k^n = 3 \cdot 2^{n-1} - k - (k+1) \cdot 2^{n-k} \qquad \text{for} \quad 2 \leqslant k \leqslant 4$$

$$d_k^n = k(3 \cdot 2^{n-3} - 1) \qquad \text{for} \quad 1 \leqslant k \leqslant 2.$$

In particular, $d_{n-1}^n = |P_n| - 6$ for $n \geqslant 5$, and so on.

In some sense, a maximum k-family is well behaved if it can be obtained by adding an antichain to a maximum $k-1$-family. Knowing the structure of these families in P_n allows us to prove that this always holds here.

LEMMA 2. *Let $\Delta_k^n = d_k^n - d_k^{n-1}$. Then P_n has a partition into antichains of sizes $\Delta_1^n, \ldots, \Delta_{n+1}^n$. For $n \geqslant 4$, there is such a partition in which the smallest antichain consists of $\{b_n, c_n^+\}$, the next smallest contains $\{b_n^-, c_n\}$, and the two largest consist of all the maximal elements and all the minimal elements.*

Proof. For $n=3$, 4, 5, these partitions of $|P_n|$ are 2211, 55322, and 11 11 7 5 4 2. For $n=3$, the antichain partition is found by inspection; let $A_1(P_3)$ be the maximal elements, $A_2(P_3)$ the minimal elements, $A_3 = \{c_n\}$ and $A_4 = \{b_n\}$. For $n \geqslant 4$, the values for d_k^n above imply $\Delta_{n+1}^n = 2$, $\Delta_n^n = 2\,\Delta_n^{n-1}$, and $\Delta_k^n = 2\,\Delta_k^{n-1}+1$ when $k<n$. We construct the desired partition $A_1(P_n), \ldots, A_{n+1}(P_n)$ of P_n using the partitions already constructed for P_n^- and P_n^+. Begin by letting $B_i = A_i(P_n^-) \cup A_i(P_n^+)$ for $i \leqslant n$. Let $A_1 = B_1 \cup \{a_n\}$, $A_2 = B_2 \cup \{d_n\}$, and augment B_i by an element of $P_n^0 - \{a_n, b_n, c_n, d_n\}$ to obtain $A_i(P_n)$ for $3<i<n$. Let $A_n =$

$(B_n - \{c_n^+\}) \cup \{c_n\}$, and let $A_{n+1} = \{b_n c_n^+\}$. These sets are disjoint and have the desired sizes; we must show they are antichains. A_1 (A_2) consists solely of maximal [minimal] elements, hence is an antichain. $A_3,..., A_{n-1}$ and A_{n+1} are clearly antichains. For A_n, note that B_n contains c_n^+ (and b_n^-) by induction, hence it cannot contain any other element of P_n^+ comparable to c_n. Hence, A_n is an antichain. The point is that replacing c_n^+ by c_n to form A_n leaves the 2-element remainder $\{b_n, c_n^+\}$ as an antichain. ▮

THEOREM 1. *Every maximum k-family in* P_n *has a partition into antichains of sizes* $\Delta_1^n,..., \Delta_k^n$.

Proof. Again we use induction, with the result for P_3 verified by inspection. First suppose $k = n$, and recall the characterization in Lemma 1. For the maximum n-families that omit c_n^+, we use the first n of the antichains obtained in Lemma 2, replacing a_n by b_n in A_1 if necessary. For those omitting b_n^-, we use a dual set obtained by an obvious order-reversing map that interchanges P_n^- and P_n^+ and turns P_n^0 upside down, then replacing d_n with c_n in A_1 if necessary. For those omitting two of $\{a_n, b_n, c_n, d_n\}$, we use B_n as the last antichain instead of A_n, and again replace a_n or d_n by b_n or c_n in A_1 or A_2 if necessary.

Now suppose $k < n$, and again recall Lemma 1. The maximum k-family F is the union of maximum k-families from P_n^- and P_n^+, plus k elements from P_n^0. Unite the ith-largest antichains in the partitions of $P_n^{\pm}$ guaranteed by induction, and add one more element from P_n^0 to each antichain as follows. If one of $\{a_n, b_n\}$ appears in F, a_n^- must also appear; add the element from $\{a_n, b_n\}$ to the antichain containing a_n^-. Similarly, if one of $\{c_n, d_n\}$ appears, add it to the antichain containing d_n^+. Finally, add one element of F from $P_n^0 - \{a_n, b_n, c_n, d_n\}$ to each antichain that has not yet received one, arbitrarily. ▮

This theorem is best possible in the following sense. It is not true that an arbitrary k-family F in P_n must have a partition into antichains whose sizes obey the conditions $\sum |A_i| = |F|$ and $\sum_{i=1}^{j} |A_i| \leqslant \sum_{i=1}^{j} d_i^n$. In particular, the following 3-family of size 11 in P_4 has no partition into antichains of sizes 5, 5, 1: with $n = 4$, take 1 element each from $\{a_n, b_n\}$, $\{c_n, d_n\}$, and $P_n^0 - \{a_n, b_n, c_n, d_n\}$, and 4 elements each from $P_n^- - \{a_n^-\}$ and $P_n^+ - \{d_n^-\}$. This example generalizes easily for larger n.

3. THE STRUCTURE OF SATURATED PARTITIONS IN P_n

Next we construct a set of consecutively saturated chain partitions, indexed by $C^{n,k}$ for the simultaneously k- and $k+1$-saturated partition of P_n. An important part of the proof of the polyunsaturation theorem is that these we construct are almost the only saturated partitions of P_n. Begin by letting $C^{3,1}$ and $C^{3,2}$ be the unique partitions of P_3 with size sequences (33) and (411). For $k<n-1$, construct $C^{n,k}$ by taking the chain P_n^0 and copies of $C^{n-1,k}$ from each of P_n^- and P_n^+. The partitions $C^{n,n-1}$ must be defined separately. For $n=4$, let $C^{4,3}$ be the unique partition of P_4 with size sequence (5522111), illustrated in Fig. 2. In particular, note that 4-saturation in P_4 requires two 5-chains and that any partition of P_4 with two 5-chains has no other chain with more than 2 elements.

From the formulas for d_k^n and the fact that $h(P_n)=n$, it is clear that when $n>4$ a chain partition will be both $n-1$- and n-saturated if and only if it has exactly two $n+1$-chains and exactly two n-chains. We construct $C^{n,n-1}$ inductively so that one of the $n+1$-chains contains $\{a_n, b_n\}$ and the other contains $\{c_n, d_n\}$. $C^{4,3}$ satisfies this. Assuming that $C^{n-1,n-2}$ has been so constructed, construct $C^{n,n-1}$ by taking copies of $C^{n-1,n-2}$ from P_n^- and P_n^+ and modifying them as follows: on the n-chains where they appear, replace a_n^- by $\{a_n, b_n\}$ and d_n^+ by $\{c_n, d_n\}$, thus turning two n-chains into $n+1$-chains. The replaced element a_n^- (d_n^+) can be added to the end of any available chain; it is convenient to add it to the end of the chain of $n-4$ central elements of the central chain of P_n^- (P_n^+). The other chains from the copies of P_{n-1} remain as they were, including the other n-chain from each, and the $n-3$ uncovered elements from P_n^0 are placed in a single chain. Now $C^{n,n-1}$ has exactly two $n+1$-chains and exactly two n-chains (hence is $n-1$- and n-saturated), and the hypothesis about the location of the longest chains has been preserved.

The polyunsaturation theorem follows easily once we establish the structure of k-saturated partitions in P_n. Several lemmas lead up to this.

LEMMA 3. (a) *P_n contains at most two disjoint $n+1$-chains, achievable only by using one that contains $\{a_n, b_n, b_n^-\}$ and another that contains $\{c_n, d_n, d_n^+\}$. P_3 does not contain two disjoint 4-chains.*

(b) *P_n contains at most five disjoint chains having at least n elements each, achievable only by P_n^0 (or an n-subchain of it) with two chains each from P_n^- and P_n^+, or by a slight variation of this in which one element of P_n^0 replaces an extreme element of one of the four other chains. If $n \leqslant 4$, then at most $n-1$ such disjoint chains can be found.*

Proof. The possibilities for disjoint $n+1$-chains have been discussed already, so consider (b). If P_n^0 appears in its entirety as one of the chains, then the remaining poset $P_n^- \cup P_n^+$ consists of disjoint copies of P_{n-1}, to which we apply (a). Next, if P_n^0 contributes an n-chain, then the situation is the same unless the remaining available element is one of $\{a_n, b_n, c_n, d_n\}$. In that case, the remaining available poset is as before, except that a_n^- or d_n^+ is "duplicated," so that a substitution is possible in one of the n-chains.

Finally, suppose none of the chains are contained in P_n^0. Then we can delete $P_n^0 - \{a_n, b_n, c_n, d_n\}$ from consideration, since those elements appear only on chains contained in P_n^0. Again we have two disjoint posets in which to look for n-chains, and we can view the elements a_n^- and d_n^+ as triplicated. However, this time obtaining five n-chains requires that we take at least three from one copy. This is impossible, since all n-chains in P_{n-1} must use b_{n-1} or c_{n-1}. By inspection, P_4 has at most three disjoint 4-chains, achievable only using the central chain or a 4-chain from it, and P_3 has only two disjoint 3-chains. ∎

LEMMA 4. *Suppose C is a k-saturated partition of P_n with $k \leqslant n-2$. Then every element of P_n^0 appears in C on a chain with more than k elements.*

Proof. Suppose this fails for some $x \in P_n^0$. Then delete x from C and P_n^0 to obtain a chain partition C' of a poset P'. We have $m_k(C') = m_k(C) - 1$ and $d_k(P') = d_k(P_n)$, since by Lemma 1 there is some maximum k-family in P_n that omits x. This means C could not have been k-saturated. ∎

LEMMA 5. *If $n \geqslant 4$ and C is a k-saturated partition of P_n containing the chain P_n^0, then $k < n$ and the rest of C consists of k-saturated partitions of P_n^- and P_n^+.*

Proof. $P_n - P_n^0$ has no $n+1$-chains, so $d_n^n = |P| - 2$ implies $k < n$ if P_n^0 appears in C. Let C^- and C^+ be the restrictions of C to P_n^- and P_n^+. Evaluating $m_k(C)$ yields

$$m_k(C^-) + m_k(C^+) + k = m_k(C) = d_k^n = 2d_k^{n-1} + k.$$

The fact that $d_k \leqslant m_k$ for any partition of any poset completes the proof. ∎

Now we can prove the needed facts about the structure of k-saturated partitions in P_n. Item (d) below helps the induction go through.

LEMMA 6. *Suppose C is a k-saturated partition of P_n with $n \geqslant 4$:*

(a) *If $k = n$, then C contains two $n+1$-chains.*

(b) *If $k < n-1$, then C consists of P_n^0 and k-saturated partitions of the subposets P_n^+ and P_n^-.*

(c) *If $k = n-1$, then one of* (a) *and* (b) *holds.*

(d) *Every chain of C that has at least k elements in P_n^- has a top element not less than b_n. Similarly, every one with at least k elements in P_n^+ has a bottom element not greater than c_n.*

Proof. For $k = n$, the presence of two $n+1$-chains follows from $h(P) = n$

and $d_k^n = |P| - 2$. Any chain of P_n that has n elements in P_n^- has a maximal element at its top (similarly for P_n^+), so (d) holds in this case. We prove the rest of the cases by induction. To do this we need to verify (d) for $n = 3$.

The only 1-saturated partition of P_3 is $C^{3,1}$, and $C^{3,2}$ is its only 3-saturated partition. Hence P_3 has no 1,3-saturated partition. Furthermore, $C^{3,1}$ and $C^{3,2}$ are the only 2-saturated partitions of P_3, so inspection yields the other statement needed for the basis step. In particular, in any k-saturated partition of P_3, any chain with at least k members has its top member not less than b_3, and its bottom member not greater than c_3.

Now suppose $n \geqslant 4$ and $k < n - 1$. If P_n^0 appears in C, then Lemma 5 implies (b), and (d) follows by induction, since anything in P_n^- not less than b_n^- is also not less than b_n (similarly for P_n^+).

So, assume P_n^0 does not appear in C. By Lemma 4, the chain of C containing any element of P_n^0 has more than k members. Consider a chain C_0 containing an element of $P_n^0 - \{a_n, b_n, c_n, d_n\}$. C_0 contains only members of P_n^0. Modify C by pulling all other members of P_n^0 from their chains in C and adding them to C_0; let the resulting partition be C'. Since $|C_0| > k$ originally, $m_k(C') \leqslant m_k(C)$, so C' is also k-saturated.

C' contains P_n^0, so we can apply Lemma 5; the remainder of C' consists of k-saturated partitions C^- and C^+ of P_n^- and P_n^+. However, the chains in these partitions are subchains of chains in C. A closer look at $m_k(C')$ and $m_k(C)$ shows that if in fact $m_k(C') = m_k(C)$ (i.e., no decrease due to the change), then every chain in C other than C_0 that contains any elements of P_n^0 must also contain at least k elements of P_n^- (or of P_n^+).

The restriction of such a chain to P_n^- or P_n^+ is a chain in C^- or C^+. Since these are k-saturated partitions of copies of P_{n-1}, we can apply part (d) of the induction hypothesis. This says that any such chain in C^- has a top element not less than b_n^-. Such elements are unrelated to every element of P_n^0, so no element of P_n^0 can belong to this chain! (Similarly for C^+.) The contradiction completes this case.

Now suppose $k = n - 1$. As before, if C contains P_n^0 we are done. If C contains two $n+1$-chains, we must verify (d). Consider any chain in C having at least $n-1$-elements in P_n^-. Since P_n^0 has no chains with more than k elements, the highest of these elements must be maximal or covered by a maximal element. Except for b_n^+, every such element is not less than b_n. However, the chain containing b_n^- must belong to the $n+1$-chain in C with a_n at its top, so for even this chain the top element is not less than b_n.

Finally, assume C does not contain P_n^0 and does not contain two $n+1$-chains. If $n \geqslant 5$, then $d_{n-1}^n = |P| - 6$. Achieving $m_k(C) = |P| - 6$ then requires C to have six n-chains or to have four n-chains and one $n+1$-chain. By Lemma 3, six n-chains is impossible, and the only way to obtain four n-chains and one $n+1$-chain is to use P_n^0! For $n = 4$, with $d_{n-1}^n = |P| - 4$, this situation is similar. Here C must have four n-chains or two n-

chains and one $n+1$-chain. Again, this is possible only by using P_n^0. This finishes the last case. ∎

Lemma 6 yields the polyunsaturation theorem itself as a corollary.

THEOREM 2. *P_n has no chain partition that is k-saturated for any nonconsecutive values of k.*

Proof. Proof by induction. Recall that the concept of k-saturated has meaning only for $k \leqslant n$. The basis $n=3$ has been verified. Now, suppose $n \geqslant 4$ and C is k,l-saturated, with $k < l-1$. Since $k < n-1$, a k-saturated partition must use P_n^0, by Lemma 6(b). Since this partition is also l-saturated, Lemma 6(a) implies $l < n$. Now Lemma 6(b) states that C^- and C^+ are k, l-saturated partitions of P_n^- and P_n^+, which by the induction hypothesis is impossible. ∎

This investigaton suggests several related questions. First, what posets have this "polyunsaturated" property? As a start, perhaps they can be shown to be trees. Is there a relationship between poorly behaved chain partitions and well-behaved k-families; i.e., does the polyunsaturated property imply that every maximum k-family has an antichain partition with size sequence $\Delta_1, \ldots, \Delta_k$?

4. ANTICHAIN PARTITIONS IN THE CONJUGATES TO P_n

Another natural question has a simple answer: what is a correspondingly bad property for antichain partitions? As discussed in the introduction, there is a natural definition of k-saturation for antichain partitions, and Greene [5, 6] showed that every poset has a simultaneously k, $k+1$-saturated antichain partition for every k. This is best possible in the same strong sense that the Greene–Kleitman theorem is best possible. Instead of inductively defining posets of arbitrarily width and proving by induction as above that they have no "nonconsecutively saturated" antichain partitions, we can easily obtain these posets from $\{P_n\}$, as summarized in the introduction.

A *linear extension* of a poset is a total order on its elements that contains all the relations of the poset. The *dimension* of a poset is the smallest number of linear extensions whose intersection is the poset, i.e., $x < y$ in P if and only if $x < y$ in each of the extensions.

THEOREM 3. *The dimension of P_n is 2. It is realized by two linear extensions L_1, L_2 such that every element appearing below b_n in L_2 appears below it in P_n, and every element appearing above c_n in L_2 appears above it in P_n.*

Proof. We inductively construct the desired extensions $L_1(P_n)$ and $L_2(P_n)$. For P_3 denote the four elements of the central chain by a, b, c, d as usual, and let e be the other minimal element and f the other maximal element. Let $L_1(P_3) = a, b, e, f, c, d$, and $L_2(P_3) = f, a, b, c, d, e$; by inspection, these intersect to form P_3, and $L_2(P_3)$ has the required behavior for $\{b, c\}$.

Now suppose $n \geqslant 4$. In the construction, we concatenate various extensions (chains) to form others. The resulting extension reads in the natural order, with x preceding y meaning $x > y$. Let $L_1(P_n) = L_1(P_n^+)$, P_n^0, $L_1(P_n^-)$. To define $L_2(P_n)$, begin with $L_2(P_n^-)$, $P_n^0 - \{a_n, b_n, c_n, d_n\}$, $L_2(P_n^+)$. Insert a_n, b_n into $L_2(P_n^-)$ immediately preceding b_n^-, and insert c_n, d_n into $L_2(P_n^+)$ immediately following c_n^+.

By induction, these intersect to form the subposets P_n^-, P_n^0, P_n^+, as desired. It is also clear that the elements of $P_n^0 - \{c_n, d_n\}$ are unrelated to those of P_n^+, and symmetrical for $P_n^0 - \{a_n, b_n\}$. Finally, $\{a_n, b_n\}$ precede all of P_n^- in $L_1(P_n)$, so they will be unrelated to those they follow in $L_2(P_n)$ and greater than those they precede. Their relations are the same as b_n^-, except that they are unrelated to a_n^-, and the inductive hypothesis about L_2 guarantees that everything is in the right place. The fact that a_n^- must precede b_n^- in both orderings of P_n^- guarantees that the placement of $\{a_n, b_n\}$ preserves the induction hypothesis. ∎

Let Q_n be the poset formed by inverting $L_2(P_n)$ and intersecting it with $L_1(P_n)$. Elements are related in Q_n if and only if they are unrelated in P_n, so a set of elements is a chain, antichain, k-family, or union of k chains in one of $\{Q_n, P_n\}$ if and only if it is a antichain, chain, union of k chains, or k-family in the other. Q_3 and Q_4 appear in Fig. 3.

Translating Theorems 1 and 2 for Q_n yields most of the following theorem.

THEOREM 4. *Q_n is a poset of width $n+1$ with no antichain partition that is k-saturated for nonconsecutive values of $k \leqslant n$. Also, every maximum-sized union of k chains has a partition into k chains of sizes $\Delta_1^n, \ldots, \Delta_k^n$. Furthermore, Q_n has a completely saturated chain partition, and P_n has a completely saturated antichain partition.*

Proof. We need only verify the last sentence. It suffices to show that the

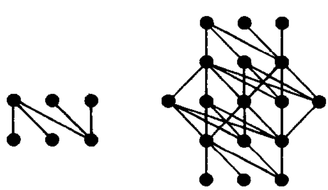

FIG. 3. Q_3, Q_4.

antichain partition A^n constructed for P_n in the proof of Lemma 2 is completely saturated; i.e., $m_k(A^n) = \hat{d}_k^n$, where $\hat{d}_k^n = \hat{d}_k(P_n)$. We could prove this explicitly by induction, but it also follows immediately from Theorem 1 and the theorems of Greene [5, 6].

Greene proved that the antichains in a maximum k-family, together with the rest of the poset as small antichains (such as singletons), form h-saturated antichain partitions for appropriate values of h. Specifically, if $T_h(S)$ is a partition of S into antichains of size at most h, and A is a collection of k antichains, then a necessary and sufficient condition for A together with any $T_h(P - \cup A)$ to form a k-saturated antichain partition is that $\Delta_k \geqslant h \geqslant \Delta_{k+1}$ and A is a maximum k-family. Similar statements hold for maximum h-cofamilies and saturated chain partitions.

Suppose P has a partition into $h(P)+1$ antichains of sizes Δ_i, such as the partition A^n of P_n constructed in Lemma 2. By applying Greene's theorem, it is easy to see that such a partition must be completely saturated. Similarly, the corresponding chain partition of Q_n must be a completely saturated chain partition. ∎

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