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Vertex Degrees in Planar Graphs

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ABSTRACT. For a planar graph on n vertices we determine the maximum values for the following: 1) The sum of the m largest vertex degrees. 2) For $k \geq 12$, the number of vertices of degree at least k and the sum of the degrees of vertices with degree at least k. 3) For $6 \leq k \leq 11$, upper and lower bounds for the latter two values, which match for certain congruence classes of n.

1. Introduction

We consider the sum of large vertex degrees in a planar graph. One approach to this is to specify a threshold k and maximize the sum of the vertex degrees that are at least k; let $f(n,k)$ denote the maximum value of this for an n-vertex planar graph. Since $K_2 \vee P_{n-2}$ is planar, we have $f(n,k) \geq 2(n-1)$ for any fixed k as long as $n \geq k+1$. Paul Erdős and Andy Vince asked whether $f(n,k) \leq 2n$ for sufficiently large fixed k; the answer is no. For $k \geq 12$ we prove:

$$f(n,k) = \begin{cases} 2n-2 & k+1 \leq n < \frac{3}{2}k - 1 \\ 2n - 16 + 6\lfloor \frac{2n-16}{k-6} \rfloor & \frac{3}{2}k - 1 \leq n \end{cases}$$

Craig Tovey independently found examples for fixed k where $f(n,k) \geq (2 + \frac{8}{k})n$; within an additive constant the optimum is $(2 + \frac{12}{k-6})n$. Fan Chung earlier observed that the sum of $o(n)$ vertex degrees in a planar graph is bounded by $2n + o(n)$. The reason for this is that the total degree in the subgraph induced by vertices of high degree is bounded by $o(n)$, and the bipartite subgraph consisting

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of edges from high degree to low degree vertices has at most $2n-4$ edges. So, exceeding $2n$ by a linear amount requires a linear number of vertices of high degree.

To obtain upper bounds we first determine the maximum sum of the m largest vertex degrees. To obtain some of the lower bounds we construct triangulations in which all vertices have degree 3, or k. The constructions therefore are related to a question posed by Jerry Griggs; what is the fewest number of vertices of degree less than k in an n-vertex triangulation? We present answers for all n when $k \geq 12$ and special congruence classes of n when $6 \leq k \leq 11$. For $k < 6$ the minimum is 0, but the maximum number of vertices of degree k is $n-2$ if $k = 4$ or if $k = 5$ and n is even.

2. The m Largest Vertex Degrees

Given a planar graph G on n vertices, let B be a set of m vertices of G with largest degree and let $D = \sum_{v \in B} d(v)$. We obtain an upper bound on D by studying the structure of the subgraph induced by B. We use $N(v)$ to denote the set of neighbors of a vertex v and G_T to denote the subgraph of G induced by a set of vertices $T \subseteq V(G)$.

LEMMA 2.1. *If $m \geq 3$ and G is a plane graph maximizing D, then G_B is a triangulation.*

PROOF. Let $S = V(G) - B$ be the set of vertices of small degree in G. We may assume that G has no edges within S, since deleting such edges does not reduce D. Hence every face of G contains a vertex of B. If G is not connected, then we can increase D by joining vertices of B on a face bounding two components of G. Hence we may assume G is connected.

Suppose $v \in S$, and consider two rotationally-consecutive edges vx, vy at v, if $d(v) \geq 2$. The independence of S implies $x, y \in B$, and the maximality of D implies $xy \in E(G)$. Hence $G_{N(v)}$ is connected. If G_B is not connected, the connectivity of G guarantees a path of length two joining components of G_B through a vertex $v \in S$. This contradicts our conclusion that $G_{N(v)}$ is connected; we conclude that G_B is connected.

If G_B is not a triangulation, then we can find three vertices x, y, z consecutive along a face of G_B, with $xz \notin E(G)$; we may assume yz is clockwise from yx at y through this face. Let the neighbors of y in clockwise order from x to z be $x, a_1, \ldots, a_p, z$. By the independence of S and maximality of D, the neighbors of x in $\{a_i\}$ are an initial segment of $\{a_i\}$; let a_r be the last neighbor of x in the order, if any. We delete the edge $a_r y$ (if $r > 0$) and replace the edges $ya_{r+1}, \ldots, ya_p$, if any, by $xa_{r+1}, \ldots, xa_p$. We can now add xz for a net increase in D. (Note: if the removal of edges from x reduces its degree so it is no longer among the m largest, we still have contradicted the maximality of D.) We conclude that G_B has no face of length exceeding 3. □

LEMMA 2.2. *Let C be a closed walk in a simple plane graph G. Let S be a set of s vertices in a region bounded by C, and let R be a specified set of $r \geq 2$ vertices on the portion of C bounding it. Then there are at most $r + 2(s-1)$ edges between R and S, and this is achievable if $s \geq 1$.*

PROOF. By induction on r. For $r = 2$, the fact that G is simple yields the desired bound $2s$. Suppose $r \geq 3$. If every vertex of S has at most two neighbors in R, then the number of edges is at most $2s < r+2(s-1)$. Hence we may assume there is a vertex $v \in S$ with $k \geq 3$ neighbors in R. Without loss of generality we may assume that $N(v) \subseteq R$. Thus the edges from v to $N(v)$ complete k closed walks with segments of C bounded by vertices of R; call these $C_1, ..., C_k$. Let s_i be the number of vertices of S in the portion of the original region bounded by C_i. Since each C_i is missing at least one element of R, we can apply induction with $R_i = C_i \cap R$ to obtain a total bound of

$$k + \sum_{i=1}^{k}(|R_i| + 2(s_i - 1)) = \sum_{i=1}^{k}|R_i| + 2\sum_{i=1}^{k} s_i - k.$$

Since $\sum_{i=1}^{k} |R_i| = |R| + k$ and $\sum_{i=1}^{k} s_i = s - 1$, this simplifies to the desired bound. If $s \geq 1$, the bound can be achieved by connecting one inside vertex to all elements of R and the remaining inside vertices to two consecutive neighbors of the first inside vertex. □

THEOREM 2.1. *The maximum of the sum of the m largest vertex degrees in an n-vertex planar graph is*

$$D(n,m) = \begin{cases} n-1 & \text{for } m = 1 \\ 2n-2 & \text{for } m = 2 \\ 2n-16+6m & \text{for } 3 \leq m \leq \frac{1}{3}(n+4) \\ 3n-12+3m & \text{for } m > \frac{1}{3}(n+4) \end{cases}$$

PROOF. If $m = 1$, then $n - 1$ is clearly an upper bound achieved by a star. For $m = 2$ the bound is still clear and can be achieved by $K_2 \vee P_{n-2}$. For $m \geq 3$, let G be a planar graph maximizing D; we know that G_B is a triangulation with $3m - 6$ edges and $2m - 4$ faces. Note that each vertex of a triangulation has degree at least three if $m \geq 4$. The question then becomes how can the remaining $n - m$ vertices be added to produce the maximum value for D. Since we add edges from these vertices only to B, they will have degree at most 3, and the vertices claimed to have the m largest degrees in fact will have the m largest degrees. We know by Lemma 2.2 that the maximum contribution due to s vertices inside any triangle is $3 + 2(s-1)$. Therefore, D is maximized by greedily distributing one vertex per face until each face has a vertex inside, for a contribution of 3 for each such vertex, and additional vertices contribute only 2. If $n \leq 3m - 4$, then the total is $2(3m-6) + 3(n-m) = 3n - 12 + 3m$; if $n \geq 3m-4$, then total is $2(3m-6)+3(2m-4)+2(n-3m+4) = 2n-16+6m$. □

3. The Vertex Degrees Above a Threshold

3.1. Upper Bounds. In this section we consider $f(n,k)$, the maximum possible degree sum of the vertices with degree above a threshold k. Our strategy to bound $f(n,k)$ is to first bound the maximum number of vertices of degree at least k in an n-vertex planar graph; we denote this by $m(n,k)$. Then, since the bound $D(n,m)$ obtained in the previous section is monotone in m, we obtain the bound $f(n,k) \le D(n, m(n,k))$. Returning to the planar graph $K_2 \vee P_{n-2}$, we see that $m(n,k) \ge 2$ as long as $n \ge k+1$. If $m(n,k) \ge 3$, then the sum of the three largest degrees is at least $3k$, which by the previous theorem is at most $D(n,3) = 2n+2$. This implies that the special case $m(n,k) = 2$ and $f(n,k) = 2n-2$ occurs only when $k+1 \le n \le \frac{3}{2}k - 2$. Henceforth we focus on the case $n \ge \frac{3}{2}k - 1$.

THEOREM 3.1. *If $n \ge \frac{3}{2}k - 1$ and $k \ge 6$, then*

$$m(n,k) \le \begin{cases} \frac{2n-16}{k-6} & \text{if } k \ge 12 \text{ or } n \le \frac{4(k+6)}{12-k} \\ \frac{3n-12}{k-3} & \text{if } 6 \le k \le 11 \text{ and } n > \frac{4(k+6)}{12-k} \end{cases}$$

PROOF. Let $m = m(n,k)$. We first note that when $n \ge \frac{3}{2}k - 1$ it is easy to construct a planar graph with three vertices of degree k. Hence we may assume $m \ge 3$. Since these m vertices are those of largest degree, $km \le D(n,m)$. Using the bound obtained in the last section, we have

$$km \le \begin{cases} 2n - 16 + 6m & 3 \le m \le \frac{1}{3}(n+4) \\ 3n - 12 + 3m & m > \frac{1}{3}(n+4) \end{cases}$$

Hence whenever $3 \le m \le \frac{1}{3}(n+4)$ we have the bound $m \le \frac{2n-16}{k-6}$, and whenever $m > \frac{1}{3}(n+4)$ we have the bound $m \le \frac{3n-12}{k-3}$.

If $k \ge 12$ and $m > \frac{1}{3}(n+4)$, then the second bound yields

$$12m \le km \le 3(3m-4) - 12 + 3m = 12m - 24.$$

Hence if $k \ge 12$, then $m \le \frac{1}{3}(n+4)$ and the first bound always holds.

Henceforth suppose $6 \le k \le 11$. If $n \le \frac{4(k+6)}{12-k}$ and $m > \frac{1}{3}(n+4)$, then the second bound says $m \le \frac{3n-12}{k-3}$. Together these imply $\frac{1}{3}(n+4) < \frac{3n-12}{k-3}$, which is equivalent to $n > \frac{4(k+6)}{12-k}$ and contradicts the hypothesis. Hence we must have the first bound when $n \le \frac{4(k+6)}{12-k}$. Note also that this implies $k > \frac{12(n-2)}{n+4}$, which is at least 6 when $n \ge \frac{3}{2}k - 1$, so the bound is meaningful.

If $n > \frac{4(k+6)}{12-k}$ and $m > \frac{1}{3}(n+4)$, then the second bound $m \le \frac{3n-12}{k-3}$ applies as claimed, so suppose $m \le \frac{1}{3}(n+4)$. As noted above, $n > \frac{4(k+6)}{12-k}$ is equivalent to $\frac{1}{3}(n+4) < \frac{3n-12}{k-3}$, so in this case we obtain $m \le \frac{3n-12}{k-3}$ again. □

COROLLARY 3.1. *If $n \ge \frac{3}{2}k - 1$ and $k \ge 6$, then*

$$f(n,k) \le \begin{cases} 2n - 16 + 6\lfloor \frac{2n-16}{k-6} \rfloor & \text{if } k \ge 12 \text{ or } \frac{3}{2}k - 1 \le n \le \frac{4(k+6)}{12-k} \\ 3n - 12 + 3\lfloor \frac{3n-12}{k-3} \rfloor & \text{if } 6 \le k \le 11 \text{ and } n > \frac{4(k+6)}{12-k} \end{cases}$$

PROOF. As was our goal, we now use the fact that $D(n,m)$ is monotone increasing in m to obtain the bound $f(n,k) \le D(n,m(n,k))$. If $k \ge 12$ or $n \le \frac{4(k+6)}{12-k}$, then by the last theorem $m(n,k) \le \frac{2n-16}{k-6}$, and hence $f(n,k) \le D(n, \frac{2n-16}{k-6})$. The hypothesis of this case also guarantees $3 \le \frac{2n-16}{k-6} \le \frac{1}{3}(n+4)$, so we can evaluate $D(n, \frac{2n-16}{k-6}) = 2n - 16 + 6\lfloor\frac{2n-16}{k-6}\rfloor$. This leaves the case $6 \le k \le 11$ and $n > \frac{4(k+6)}{12-k}$, where the last theorem gives $m \le \frac{3n-12}{k-3}$. If $\frac{3n-12}{k-3} > \frac{1}{3}(n+4)$, we obtain the bound claimed for T. Because $D(n,m)$ is monotonic in m, this bound on T also holds if $\frac{3n-12}{k-3} \le \frac{1}{3}(n+4)$. □

3.2. Lower Bounds. The constructions in this section all begin with the definition of a set B of m "big" vertices (intended to have degree above the threshold), and the description of a triangulation G_B. After this we add the remaining $n-m$ vertices to faces or edges of G_B. Adding a vertex to a face means placing it inside the face and joining it to each vertex on the face. Adding a vertex to an edge uv means placing it in a face bounded by uv and joining it to both u and v. In all cases except $k=11$, vertices are not added to edges unless a vertex has been already been added to every face. The proof of Theorem 3 guarantees that the sum of the degrees of the m big vertices will be $D(n,m)$. For $k \ge 12$ we are able to match the upper bound on $f(n,k)$ for all n. For $k < 12$ our upper bounds split into cases depending on n. We construct matching lower bounds for large n in certain congruence classes.

We note in passing that the combination of our bounds and our constructions answers some cases of a question posed by Jerry Griggs. He asked for the minimum number of vertices of degree less than k in a planar n-vertex triangulation, which is equivalent to determining the maximum number of vertices of degree at least k. For $k \ge 12$, this maximum number is always $\lceil\frac{2n-16}{k-6}\rceil$. For $6 \le k \le 11$, we determine the minimum number of vertices of degree less than k for appropriate congruence classes of n. In particular, for $k=6$ and n even, this minimum number is 4, which requires four vertices of degree 3. Griggs reports that when n is odd, there is no triangulation with four vertices of degree 3 and the rest of degree (at least) 6, as proved by Grünbaum and Motzkin [2], so here the answer is 5 (we omit this construction). Griggs also reports that Yan-Chyuan Lin has determined the answer within 1 for the remaining cases between $k=7$ and $k=11$. The remainder of this section contains constructions that prove our results by showing $f(n,k) = D(n,m(n,k))$ in various cases, except that for $k=11$ we must also improve the upper bound slightly. For $k+1 \le n \le \frac{3}{2}k-2$ we have already shown $m(n,k)=2$, and $f(n,k) = 2n-2 = D(n,2)$, so again we focus on the case $n \ge \frac{3}{2}k-1$.

THEOREM 3.2. *If $k \ge 12$, then $f(n,k) = D(n,m(n,k))$.*

PROOF. Let $2n-16 = m(k-6)+r$ with $0 \le r < k-6$, so that $m = m(n,k)$. Let $B = \{u_1, ..., u_m\}$. The graph G_B consists of the edges $\{u_i u_j\}$ such that $|j-i| \le 3$. There are $m-1+m-2+m-3 = 3m-6$ edges. Figure 1 illustrates

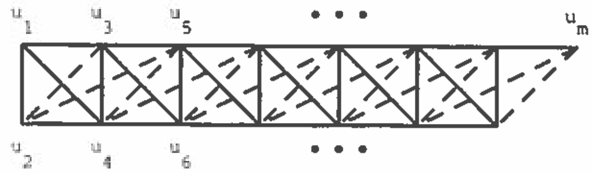

FIGURE 1.

the graph for odd m. We obtain a planar representation by drawing the dashed path around the outside.

Adding a vertex to each of the $2m-4$ faces of G_B doubles the degree of each vertex in B, giving them all degree 12 except for $d(u_1) = d(u_m) = 6$, $d(u_2) = d(u_{m-1}) = 8$, and $d(u_3) = d(u_{m-2}) = 10$. To bring all vertices of B to degree 12, add 2 vertices to each of u_1u_3 and u_mu_{m-2}, and add 4 vertices to each of u_1u_2 and u_mu_{m-1}. To reach degree k, we add $k-12$ vertices to each edge $u_{2i-1}u_{2i}$ for $1 \leq i \leq \lfloor m/2 \rfloor - 1$. If m is even, we also add $k-12$ vertices to $u_{m-1}u_m$. If m is odd, we add $\lfloor (k-12)/2 \rfloor$ to $u_{m-2}u_{m-1}$, and we add $\lceil (k-12)/2 \rceil$ to each of $u_{m-2}u_m$ and $u_{m-1}u_m$. Altogether, the number of vertices used is $m + (2m-4) + 12 + (k-12)(\lfloor m/2 \rfloor) + \epsilon$, where $\epsilon = 0$ if m is even and $\epsilon = \lceil (k-12)/2 \rceil$ if m is odd. Considering all cases for the parity of m and k, the formula equals $8 + \lceil m(k-6)/2 \rceil$. Since $m(k-6) = 2n - 16 - r$, we have used $n - \lfloor r/2 \rfloor$ vertices; add the remaining $\lfloor r/2 \rfloor$ vertices to the edge u_1u_2. In all cases for the parity of m and k, the construction achieves the bound $f(n,k) \leq 2n - 16 + 6m$ from the previous section. □

THEOREM 3.3. *If $6 \leq k \leq 8$ and $n = r_kj_k + 4$, where $j_k = 2, 8, 10$ for $k = 6, 7, 8$, and $r_k \geq 2, 1, 1$ for $k = 6, 7, 8$, then $f(n,k) = D(n, m(n,k))$.*

PROOF. For these cases our base graph G_B is a triangulation on an even number of vertices with 4 vertices of degree 4, 4 vertices of degree 5, and the remainder of degree 6. Let $m = 2p$; the graphs in Figure 2 (without the dots) illustrate G_B when $m = 24$. Let $B = \{a_1, \dots, a_{\lceil p/2 \rceil}\} \cup \{b_1, \dots, b_{\lfloor p/2 \rfloor}\} \cup \{c_1, \dots, c_{\lceil p/2 \rceil}\} \cup \{d_1, \dots, d_{\lfloor p/2 \rfloor}\} \cup$. The graph G_B consists of the sets $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$ inducing paths (with indices in order), the four-cycles $(a_ib_ic_id_i)$ and $(a_{i+1}b_ic_{i+1}d_i)$ for for $1 \leq i \leq \lfloor p/2 \rfloor$, and the two edges a_1c_1 and either $b_{p/2}d_{p/2}$ (if p is even) or $a_{\lceil p/2 \rceil}c_{\lceil p/2 \rceil}$ (if p is odd). All vertices have degree 6 except those on the cycle $(a_1b_1c_1d_1)$ of degrees 4 and 5 and those on the cycle $a_{\lceil p/2 \rceil}b_{\lfloor p/2 \rfloor}c_{\lceil p/2 \rceil}d_{\lfloor p/2 \rfloor}$ of degrees 4 and 5.

In each case, we begin with this triangulation G_B and add vertices into selected faces to achieve the bounds of the preceding section.

For $k = 6$, suppose $n = 2r + 4$. Using the graph G_B described above for $m = 2r$, add the 4 vertices into the faces indicated in Figure 2. Note that this graph is 6-regular, except for four vertices of degree 3.

For $k = 7$, suppose $n = 8r + 4$. Begin with the graph constructed for $k = 6$

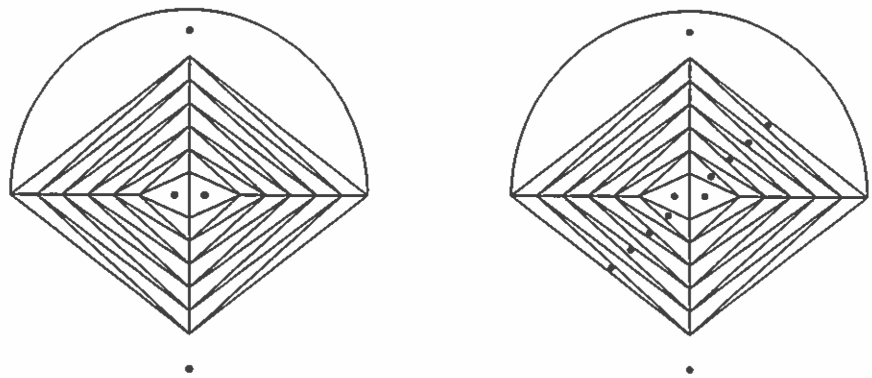

FIGURE 2.

when $m = 6r$. We increase the degree of the $6r$ vertices in B by one by adding one to each of a set of $2r$ vertex-disjoint triangles. The triangles are the partitions into consecutive triples of the sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ and the sequence $c_1, d_1, c_2, d_2, c_3, d_3, \ldots$. Since m is divisible by 6, this covers the triangles exactly.

For $k = 8$, suppose $n = 10r + 4$. This is very similar to $k = 7$. To the graph constructed above for $k = 7$ and $n = 8r + 4$, add the same pattern of r additional vertices into each of the two empty wedges. That is, we place vertices in the triangles consisting of the partitions into consecutive triples of the sequence $a_1, d_1, a_2, d_2, a_3, d_3, \ldots$ and the sequence $c_1, b_1, c_2, b_2, c_3, b_3, \ldots$. □

THEOREM 3.4. *If $9 \leq k \leq 10$ and $n = rj_k + s_k$, where $j_k = 18, 21$ and $s_k = 10, 11$ for $k = 9, 10$, and $r \geq 1$, then $f(n,k) = D(n, m(n,k))$.*

PROOF. For $k = 9, 10, 11$ we need a new base graph. Now G_B will be a triangulation on $m = 3p$ vertices with 12 vertices of degree 5 and the remaining vertices of degree 6. Such a graph on 30 vertices appears in Figure 3 (ignoring the dots). The construction of this graph is much like the construction of the earlier triangulation, but with six spokes instead of 4. Let the vertices be $\{a_{i,j} : 0 \leq i \leq 2, 1 \leq j \leq \lceil p/2 \rceil\}$ and $\{b_{i,j} : 0 \leq i \leq 2, 1 \leq j \leq \lfloor p/2 \rfloor\}$. The edges are the "spoke-paths" $(a_{i,1} \ldots a_{i,\lceil p/2 \rceil})$ and $(b_{i,1} \ldots b_{i,\lfloor p/2 \rfloor})$, the 6-cycles $(a_{0,j} b_{0,j} a_{1,j} b_{1,j} a_{2,j} b_{2,j})$ and $(a_{0,j+1} b_{0,j} a_{1,j+1} b_{1,j} a_{2,j+1} b_{2,j})$ for $1 \leq j \leq \lfloor p/2 \rfloor$, and the two triangles $a_{0,1} a_{1,1} a_{2,1}$ and either $b_{0,p/2} b_{1,p/2} b_{2,p/2}$ (if p is even) or $a_{0,\lceil p/2 \rceil} a_{1,\lceil p/2 \rceil} a_{\lceil p/2 \rceil}$ (if p is odd). The vertices all have degree 6, except the first and last of each spoke, which have degree 5. The four faces involving $\{a_{i,1}\}$ and $\{b_{i,1}\}$ we call the "inner" faces; similarly there are four "outer" faces involving $\{a_{i,\lceil p/2 \rceil}\}$ and $\{b_{i,\lfloor p/2 \rfloor}\}$. The remaining faces are the triangles formed by any three consecutive vertices in the six sequences $a_{i,1}, b_{i',1}, a_{i,2}, b_{i',2}, \ldots$, where $i' \equiv i \pm 1$ *mod* 3. The triangles in each such sequence form a wedge of $p - 2$ triangles.

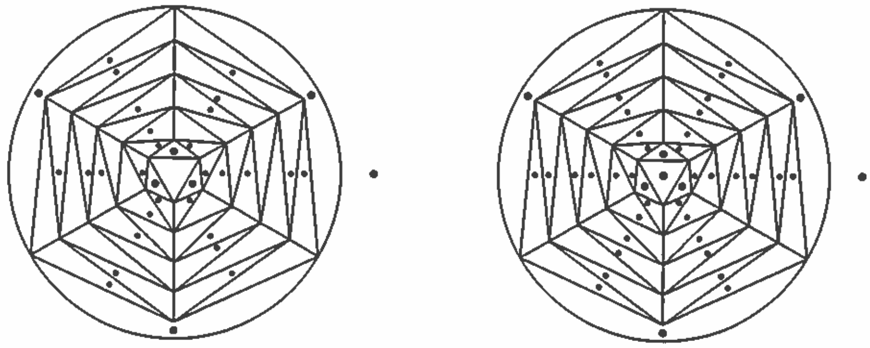

FIGURE 3.

For $k = 9$, suppose $n = 18r + 10$. Start with the graph G_B defined here for $m = 9r + 3$ vertices. Add vertices to the three inner triangles other than $a_{0,1}a_{1,1}a_{2,1}$. However, add vertices to all four outer triangles. For the successive $p - 2 = 3r - 1$ triangles in a wedge, starting from the center (low index), we describe a binary string that specifies adding a vertex to the jth triangle in the wedge if and only if the jth entry in the sequence is a 1. Note that at this point each $a_{i,1}$ has degree 7 and each $b_{i,1}$ has degree 6; however, the vertices on the outer edges of the wedges have degree 8 and those immediately before them have degree 6. In the wedge formed by the vertices of $\{a_{i,j}\}$ and $\{b_{i+1,j}\}$, we start with 11 and then alternate 010 and 011. In the wedge formed by the vertices of $\{a_{i,j}\}$ and $\{b_{i-1,j}\}$, we start with 10 and then alternate 011 and 010. The three successive faces at corresponding positions in two adjacent wedges contribute to the degree of a single vertex, except that the edge vertices are incident to only one face in each wedge and the next vertex to only two. Since the sum of the two sequences is $21(021)^{r-1}$, each vertex receives a contribution of 3 to raise its degree to 9, except that the vertices on the inner and outer ends of the wedges receive 2 and 1, respectively, as desired. We have added vertices to $7 + 9r$ faces, so $n = 18r + 10$. Again this construction matches the bound.

For $k = 10$, suppose $n = 21r + 11$; the construction is simpler than that for $k = 9$. Start with G_B for $m = 9r + 3$, as for $k = 9$. Add vertices to all four inner and all four outer faces, so the degrees of vertices on the wedges are now $8, 6, \ldots, 6, 8$. Within each wedge, use the sequence $11(011)^{r-1}$. The sum of the contributions from two adjacent wedges is $22(022)^{r-1}$, and each vertex of B receives the needed contribution. We have added vertices to $8 + 12r$ faces for $n = 21r + 11$, and the construction matches the bound. □

For $k = 11$, the problem is a bit more difficult, because the upper bound on the sum of the $m(n, 11)$ largest degrees is a bit larger than the maximum sum of the degree exceeding 11. We begin with the construction.

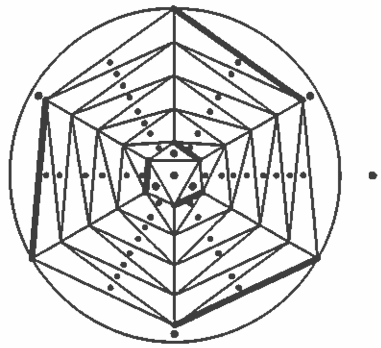

FIGURE 4.

LEMMA 3.1. *For $n = 24r + 14$ and $r \geq 1$, $f(n, 11) \geq D(n, m(n, 11)) - 6$.*

PROOF. Begin with our previous base graph with six spokes on $m = 9r + 3$ vertices, as shown in Figure 4. Add vertices to all four inner and all four outer faces, so the degrees of vertices on the wedges are now $8, 6, \ldots, 6, 8$. In the wedge formed by the vertices of $\{a_{i,j}\}$ and $\{b_{i+1,j}\}$, we use the sequence $11(111)^{r-1}$. In the wedge formed by the vertices of $\{a_{i,j}\}$ and $\{b_{i-1,j}\}$, we use the sequence $11(011)^{r-1}$. The sum of the contributions from two adjacent wedges is $22(122)^{r-1}$, and each vertex of B receives the needed extra 5 neighbors except for the innermost 6 and outermost 6 vertices which have degree only 10. To bring them to degree 11 add a vertex to each of the darkened edges $a_{i,1}b_{i-1,1}$ and $a_{i,\lceil (r+1)/2 \rceil}b_{i-1,\lfloor (r+1)/2 \rfloor}$ of G_B. We have added $8 + 3(5r - 1) + 6$ vertices for a total of $n = 24r + 14$. □

This construction fails to meet the bounds of the previous section, because it adds six vertices to edges before all faces have received vertices. Nevertheless, it is optimal, and we can improve the upper bound by six to match it. The proof of this is surprisingly long.

When we are maximizing the sum of degrees above threshold instead of the m largest vertex degrees, the proof in Lemma 2.1 that G_B is a triangulation is no longer valid. Fortunately, the first two paragraphs of the proof remain valid, and we may assume that G and G_B are connected. Instead of knowing that G_B is a triangulation, the following inequality will suffice.

LEMMA 3.2. *Let G be a connected simple plane graph with m vertices, e edges, and f faces. Let R be a specified set of vertices in G, whose intersections with the face boundaries of G have sizes $l_1 \geq l_2 \geq \ldots \geq l_f$. If $1 \leq r \leq f$, then $2e + \sum_{i=1}^{r} l_i \leq 2(3m - 6) + 3r$.*

PROOF. If G has a face of length exceeding 3, then we can increase the left side of the inequality by adding a triangular chord to such a face. Hence the left side is maximized when G is a triangulation, in which case it equals the right side. □

LEMMA 3.3. *Let G be a simple n-vertex planar graph, $B = \{v \in G : d(v) \geq 11\}$, $m = |B|$, and $T = \sum_{v \in B} d(v)$. Then*

$$T \leq \begin{cases} 2n - 16 + 6m & 3 \leq m < \frac{1}{3}(n-2) \\ 3n - 18 + 3m & m \geq \frac{1}{3}(n-2) \end{cases}$$

PROOF. As already remarked, we may assume G_B is connected. Let e be the number of edges in G_B, and let r be the number of faces of G_B that contain vertices of G. The value of T is $2e$ plus the contribution from the $n-m$ vertices in $S = V(G) - B$. By Lemma 2.2, we do best by placing vertices in the r faces of G_B containing the most vertices of B (since we are not assuming G_B is 2-connected, this need not be the same as the s longest faces). Let the ith largest number of vertices on a face of G_B be l_i, and suppose the corresponding face contains s_i vertices of G. Using Lemma 2.2 to bound the contribution of edges between S and B, and then invoking Lemma 3.2, we have

$$T \leq 2e + \sum_{i=1}^{r}(l_i + 2(s_i - 1)) \leq 2(3m-6) + 3r + 2(n-m) - 2r = 2n + 4m - 12 + r$$

Since a planar graph has at most $2m-4$ faces, $r \leq 2m - 4$. Also $r \leq n - m$, but we argue that at least 6 vertices of $G - B$ must be added to edges of G_B and thus $r \leq n - m - 6$. To see this, let $d'(v)$ denote the degree of v in G_B, and let $U = \{v \in G_B : d'(v) < 6\}$. If we add vertices to all the faces of G_B, vertices in U still have degree at most 10 and must have a vertex added to an incident edge. If $|U| \geq 12$, then at least 6 vertices must be added to edges, since each contributes 1 to at most two vertices in U. If $U \leq 12$, then the number of added edges must be at least half of $\sum_{v \in U}(11 - 2d'(v)) = 11|U| - 2\sum_{v \in B} d'(v) + 2\sum_{v \in B-U} d(v) \geq 11|U| - 2(6m-12) + 2(6(m - |U|)) = 24 - |U| \geq 12$

We obtain the bounds claimed by plugging in these estimates for r in two cases, depending on which of $\{2m-4, n-m-6\}$ is smaller. □

THEOREM 3.5. *Let G be an n-vertex planar graph, $B = \{v \in G : d(v) \geq 11\}$, $m = |B|$, and $T = \sum_{v \in B} d(v)$. Then*

$$m \leq \begin{cases} \frac{2n-16}{5} & 16 \leq n < 38 \\ \frac{3n-18}{8} & n \geq 38 \end{cases}$$

PROOF. First note that both bounds are bigger than 3, so there is nothing to prove unless $m \geq 4$. Using the bound in the last lemma and the trivial fact $11m \leq T$, we have

$$11m \leq \begin{cases} 2n - 16 + 6m & 3 \leq m < \frac{1}{3}(n-2) \\ 3n - 18 + 3m & m \geq \frac{1}{3}(n-2) \end{cases}$$

Hence whenever $3 \leq m < \frac{1}{3}(n-2)$ we have the bound $m \leq \frac{2n-16}{5}$, and whenever $m \geq \frac{1}{3}(n-2)$ we have the bound $m \leq \frac{3n-18}{8}$.

If $n < 38$ and $m \geq \frac{1}{3}(n-2)$, then the second bound says $m \leq \frac{3n-18}{8}$. Together these imply $\frac{1}{3}(n-2) \leq \frac{3n-18}{8}$, which is equivalent to $n \geq 38$ and contradicts the hypothesis. Hence we must have the first bound when $n < 38$.

If $n \geq 38$ and $m \geq \frac{1}{3}(n-2)$, then the second bound $m \leq \frac{3n-18}{8}$ applies as claimed, so suppose $m < \frac{1}{3}(n-2)$. As noted above, $n \geq 38$ is equivalent to $\frac{1}{3}(n-2) \leq \frac{3n-18}{8}$, so in this case we obtain $m \leq \frac{3n-18}{8}$ (again). □

COROLLARY 3.2.

$$f(n,11) \leq \begin{cases} 2n-16+6\lfloor\frac{2n-16}{5}\rfloor & \text{if } 16 \leq n < 38 \\ 3n-12+3\lfloor\frac{3n-12}{8}\rfloor & \text{if } n \geq 38 \end{cases}$$

with equality for $n = 24r + 14 \geq 38$.

PROOF. Let G be an n-vertex planar graph, let $B = \{v \in V(G) : d(v) \geq 11\}$, $m = |B|$, and $T = \sum_{v \in B} d(v)$. Our lemma gives a bound on T which is monotone increasing in m, so we can obtain a bound on T by using the bound on m obtained in the last theorem. Note that $n < 38$ is equivalent to $\frac{2n-16}{5} \leq \frac{1}{3}(n-2)$. Hence in the first case the last theorem implies $m \leq \frac{2n-16}{5} \leq \frac{1}{3}(n+4)$, and then $T \leq 2n-16+6m \leq 2n-16+6\lfloor\frac{2n-16}{5}\rfloor$ from the previous lemma.

This leaves the case $n \geq 38$, where the last theorem gives $m \leq \frac{3n-18}{8}$. If $\frac{3n-18}{8} \geq \frac{1}{3}(n-2)$, we obtain the bound claimed for T. Because our bound for T is monotonic in m it also holds if $m < \frac{1}{3}(n-2)$. The bound is achieved by the construction in Lemma 3.1. □

REFERENCES

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