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THE p-INTERSECTION NUMBER OF A COMPLETE BIPARTITE GRAPH AND ORTHOGONAL DOUBLE COVERINGS OF A CLIQUE

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The *p-intersection graph* of a collection of finite sets $\{S_i\}_{i=1}^n$ is the graph with vertices $1,\dots,n$ such that i, j are adjacent if and only if $|S_i \cap S_j| \geq p$. The *p-intersection number* of a graph G, herein denoted $\theta_p(G)$, is the minimum size of a set U such that G is the p-intersection graph of subsets of U. If G is the complete bipartite graph $K_{n,n}$ and $p \geq 2$, then $\theta_p(K_{n,n}) \geq (n^2+(2p-1)n)/p$. When $p=2$, equality holds if and only if K_n has an *orthogonal double covering*, which is a collection of n subgraphs of K_n, each with $n-1$ edges and maximum degree 2, such that each pair of subgraphs shares exactly one edge. By construction, K_n has a simple explicit orthogonal double covering when n is congruent modulo 12 to one of $\{1,2,5,7,10,11\}$.

1. Introduction

We study a model of graph representation using intersections of finite sets. Given a graph G with vertices V, a collection of sets $\{S_v : v \in V\}$ is an *intersection representation* of G if $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$. We also say that G is the *intersection graph* of the collection of sets.

When using finite sets, we define the *intersection number* of G to be the minimum t such that G has an intersection representation using subsets of a t-element set U. Erdős, Goodman, and Pósa [6] showed that the intersection number of G equals the minimum number of cliques needed to cover the edges of G; the cliques correspond to elements of U.

One way to generalize intersection graphs is to generate edges only when the intersection of the corresponding sets is "large enough". This has been studied when the representing sets are intervals by assigning "tolerances" to vertices, generating an edge uv if the length of $S_u \cap S_v$ is at least some function of the tolerances on u and v (see [12, 17, 18]). When using finite sets, we define the *p-intersection graph* by generating the edge uv if and only if $|S_u \cap S_v| \geq p$. The *p-intersection number* of G is minimum t such that G is the p-intersection graph of a collection of subsets of a t-set U.

A special case of this concept arose in the theory of competition graphs. A graph G is the *p-competition graph* of a digraph D if G has the same vertices as D, and $xy \in E(G)$ if and only if x and y have p common successors. In other words, a p-competition graph is a p-intersection graph, where the sets assigned

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AMS subject classification code (1991): 05 C 35, 05 C 70, 90 C 90 are the successor sets (out-neighborhoods) of the vertices of D. The concept of p-intersection graph was introduced in [19] and studied further in [15, 20].

Like the ordinary intersection number, the p-intersection number has an equivalent description using a covering problem. For this reason we use the notation $\theta_p(G)$ for the p-intersection number of G. Given a p-representation of G, we can associate with each element $a \in U$ the vertex set $T(a) = \{v \in V(G) : a \in S_v\}$. A collection of subsets of $V(G)$ generates a p-representation of G precisely when vertices are adjacent in G if and only if they appear in at least p common sets. We call such a collection of vertex subsets a *p-generator* of G. Since a 1-generator consists of cliques in G, a p-generator has also been called a "p-edge clique cover" (see [3, 20, 21, 22]); we prefer the term "p-generator" because the vertex sets need not induce cliques when $p > 1$. These authors study $\theta_p(G)$ under the name"p-edge clique cover number".

Since sets in a p-generator can be repeated, $\theta_p(G) \leq p\theta_1(G)$. Also $\theta_p(G) \leq \theta_{p-1}(G) + 1$, as observed in [3], since adding the full vertex set $V(G)$ to a $p-1$-generator. Nevertheless, θ_p can be smaller that θ_{p-1}. The natural problem is to maximize $\theta_p(G)$ over n-vertex graphs. For $p = 1$, the solution is well-known to be $\lfloor n^2/4 \rfloor$ [6], achieved by $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Experimentation suggests that this graph also maximizes θ_p for n-vertex graphs when $p > 1$, or at least achieves the maximum infinitely often for any fixed p. We pose this as a conjecture. In this paper, we study the value of $\theta_p(K_{n,n})$. By using a counting argument and solving the resulting integer program, we prove $\theta_p(K_{n,n}) \geq (n^2 + (2p-1)n)/p$ when $p \geq 2$, which places a lower bound on the maximum p-intersection number for $2n$-vertex graphs. Earlier, Jacobson [16] used another counting argument to prove $\theta_p(K_{n,n}) \geq \Omega(n^{1.25})$. More recently, Füredi [9] used a result of Frankl and Rödl [7] (described also in [8, p. 190]) to prove that the lower bound of $(n^2 + (2p-1)n)/p$ is asymptotically best possible for $\theta_p(K_{n,n})$. This was proved independently by Eaton, Gould, and Rödl in [5].

Achieving the lower bound exactly for $p = 2$ is equivalent to a graph design problem of independent interest. An *orthogonal double covering* of K_n is a collection of n subgraphs of K_n, each with $n-1$ edges and maximum degree 2, such that each pair of subgraphs shares exactly one edge. In such a configuration, every edge must appear in exactly two subgraphs. We prove that $\theta_2(K_{n,n}) = (n^2 + 3n)/2$ if and only if K_n has an orthogonal double covering. Orthogonal double coverings of K_n have independently been studied under the name "self-orthogonal 0,2-factorizations of $2K_n$"; see [1] for a survey on this and related topics. Various papers contain orthogonal double coverings with special additional properties. For example, [13, 14] contain constructions of orthogonal double coverings in which each subgraph is a cycle of length $n-1$ plus an isolated vertex, for special values of n. In [4], each subgraph was required to be a union of disjoint cliques; here the terminology and motivation was much different, and the maximum degree condition was not imposed. For $n \equiv 1 \bmod 3$, [4] requested orthogonal double coverings in which each subgraph is a collection of disjoint triangles plus one isolated vertex. Solutions when $n \equiv 1, 4 \bmod 12$ were constructed in [4]. This cannot be done when $n = 10$ [23], but otherwise the cases $n \equiv 7$, $10 \bmod 12$ were solved independently in [2] and [10].

In the less restrictive setting of maximum degree 2 (not necessarily near-spanning cycles or triangles), we obtain simple constructions for infinite classes of values. When n is congruent mod 12 to one of 1, 2, 5, 7, 10, 11, we construct an explicit cyclically invariant orthogonal double covering of K_n, thus proving that $\theta_2(K_{n,n})=(n^2+3n)/2$ for these values. Very recently, Ganter, Gronau, and Mullin [11] solved the problem for all values of n. Using known results in design theory plus a few ad hoc examples, they obtained orthogonal double covers in which each subgraph consists of an isolated vertex and a union of disjoint cycles; except for $n\in\{2,3,8\}$, they obtain constructions using cycles of length at most 5.

Concerning larger p, we present just two special constructions that achieve equality in the linear programming bound, for $(p,n)=(3,7)$ and $(p,n)=(4,13)$. These may generalize when $n=p^2-p+1$ and there is a projective plane of order $p-1$.

2. The lower bound

Consider $G=K_{m,n}$ with partite sets X, Y. Let T be a vertex subset appearing in a p-generator of $K_{m,n}$, and suppose that T consists of r vertices in X and s vertices in Y. Among the $\binom{r+s}{2}$ pairs of vertices in T, there are rs good pairs (corresponding to edges of G) and $\binom{r}{2}+\binom{s}{2}$ bad pairs (corresponding to edges of $\overline{G}$). When we sum this over all subsets in a p-generator, we must have at least pmn good pairs and at most $(p-1)(\binom{m}{2}+\binom{n}{2})$ bad pairs. Note that if T has $k=r+s$ vertices, then the number of good pairs is maximized and the number of bad pairs is minimized simultaneously when $r=\lfloor k/2\rfloor$. Because the argument leads to an optimal bound when $m=n$ (for $p=2$) and because our primary interest is the lower bound on the maximal value of θ_p for graphs of fixed order, we henceforth discuss only $K_{n,n}$.

Theorem 1. *If $p\geq 2$, then $\theta_p(K_{n,n})\geq(n^2+(2p-1)n)/p$.*

Proof. For $k\geq 2$, let y_k be the number of vertex subsets of size k in some fixed p-generator of $K_{n,n}$. Using the constraints described above, we obtain a lower bound on $\sum y_k$ by bounding the following linear programming problem: over nonnegative variables $\{y_k\}$, minimize $\sum y_k$ subject to

$$\sum y_k\left[\binom{\lfloor k/2\rfloor}{2}+\binom{\lceil k/2\rceil}{2}\right]\leq 2(p-1)\binom{n}{2}, \tag{1}$$

$$\sum y_k\lfloor k^2/4\rfloor\geq pn^2. \tag{2}$$

Let $\alpha_k=\binom{\lfloor k/2\rfloor}{2}+\binom{\lceil k/2\rceil}{2}$ and $\beta_k=\lfloor k^2/4\rfloor$.

Our linear minimization problem is phrased in a canonical form: minimize $y\cdot b$ such that $yA\geq c$ and $\{y_i\}\geq 0$. For such a problem, there is a canonical dual maximization problem: maximize $c\cdot x$ such that $Ax\leq b$ and $\{x_j\}\geq 0$. If y, x are both feasible solutions to their respective problems, then we have $yb\geq yAx\geq cx$. Hence $\theta_p(K_{n,n})\geq z$ is implied by any feasible solution to the dual that has value z. We present such a solution.

In our problem, b is a column vector of 1's, and $c=(-(p-1)n(n-1),pn^2)$. The dual problem has two variables x_1, x_2 corresponding to the constrains (1) and

(2), and it has a constraint for each minimization variable y_k. The problem is to maximize $x_2pn^2 - x_1(p-1)n(n-1)$ such that $x_1, x_2 \geq 0$ and $x_2\beta_k = x_1\alpha_k \leq 1$ for $k \geq 2$. We choose $x_1 = (2p-1)/[p(p-1)]$ and $x_2 = 2/p$. The value of this solution is $2n^2 - (2p-1)n(n-1)/p = (n^2 + (2p-1)n)/p$, so it suffices to show that this solution is feasible. Consider constraint k, and let $r = \lceil k/2 \rceil$. We have $\alpha_k = (r-1)/r\beta_k$ and $\beta_k = r(k-r)$. Substituting in these values and those of x_1, x_2, we must show $2r(k-r) - (2p-1)(p-1)^{-1}(r-1)(k-r) \leq p$, or $(2p-1-r)(k-r) \leq p(p-1)$. Since $k-r = \lfloor k/2 \rfloor$, we have $k-r \leq r$, and it suffices to show $(2p-1-r)r \leq p(p-1)$. Since $(2p-1-r)+r = p+(p-1)$, this holds for every value of r not satisfying $p-1 < r < p$, which covers all integers. ∎

To show that the bound of the theorem above is the best possible bound from the linear program and is potentially achievable, we exhibit a solution to the minimization problem that has the same value as the solution (x_1, x_2) given above to the maximization problem. The argument given for feasibility of the dual solution also shows that the kth constraint holds with equality if and only if k is $2p$ or $2p-2$. The complementary slackness theorem of linear programming then implies that a solution to the min problem that has the same value must have $y_k = 0$ for $k \notin \{2p, 2p-2\}$. (In actuality, the dual solution above was obtained from the min solution below using this method.)

This leads us to set all $y_k = 0$ except $y_{2p-2} = pn$ and $y_{2p} = (n^2(p-1)^2n)/p$. We show below that the constraints (1), (2) hold with equality and the solution is thus feasible. The value of this solution is $(n^2 + p^2n - (p-1)^2n)/p = (n^2 + (2p-1)n)/p$. However, the nonnegativity constraint for y_{2p} requires $n \geq (p-1)^2$, so for $n < (p-1)^2$ the bound of the theorem is probably not best possible. Indeed, Füredi [9] (and independently Eaton, Gould, Rödl [5]) has proved the alternative lower bound $\theta_p(K_{n,n}) \geq (n+p-1)^2/p$, which exceeds $(n^2 + (2p-1)n)/p$ when $n < (p-1)^2$.

With the values specified, the left side of (1) becomes $pn[2\binom{p-1}{2}] + (n^2 - (p-1)^2)n[2\binom{p}{2}] = n(p-1)[p(p-2) + n - (p-1)^2]$. Substracting the right side leaves $n(p-1)[p(p-2) + 1 - (p-1)^2] = 0$. For (2), the left side is $pn(p-1)^2 + (n^2 - (p-1)^2n)p^{-1}p^2 = pn[(p-1)^2 + n - (p-1)^2] = pn^2$, which equals the right side.

This solution to the minimization problem narrows the search for constructions achieving equality in the bound.

3. Equivalent design problem for $p = 2$

The solution to the linear programs places considerable restriction on 2-generators of $K_{n,n}$ having size $(n^2 + 3n)/2$, which we call *perfect* 2-generators. A perfect 2-generator must consist of exactly $2n$ 2-sets and $n(n-1)/2 = \binom{n}{2}$ 4-sets, each set having half its vertices on each side. We reduce the existence of such a collection of vertex sets to an equivalent graph design problem, for which we exhibit solutions for special congruence classes of n mod 12.

A *orthogonal double covering* of (the labeled complete graph) K_n is a collection of n subgraphs, each consisting of $n-1$ edges, such that the maximum degree of each subgraph is two and each pair of subgraphs shares exactly one edge. The use of the word "double" in this term arises from the following lemma.

Lemma 1. *If $G_1,\dots,G_n$ are $n-1$-edge subgraphs of K_n such that each pair shares exactly one edge, then each edge appears in exactly two subgraphs.*

Proof. Counting by pairs of subgraphs, there are $\binom{n}{2}$ edge-intersections between subgraphs. Let us count this also by edges of K_n. If e appears in $t(e)$ subgraphs, then e contributes $\binom{t(e)}{2}$ edge-intersections to the total. We have $\binom{n}{2}=\sum_e t(e)[t(e)-1]/2$ and $\sum_e t(e)=n(n-1)/2$. Since the sum of squared numbers with fixed sum is minimized precisely when they are equal, the summation has value at least $\binom{n}{2}$, with equality if and only if each edge is in two subgraphs. ∎

Now we reduce the perfect 2-generator problem to the orthogonal double covering problem.

Theorem 2. *$K_{n,n}$ has a perfect 2-generator if and only if K_n has a orthogonal double covering.*

Proof. Suppose $K_{n,n}$ with partite sets X, Y has a perfect 2-generator $\mathbf{T}$. As noted above, each set in $\mathbf{T}$ must be half in X and half in Y. The $\binom{n}{2}$ 4-sets generate $\binom{n}{2}$ pairs in X. No pair of vertices in X can appear in two sets of $\mathbf{T}$, so each pair appears in exactly one 4-set of $\mathbf{T}$. The same argument holds for Y, so the 4-sets establish a bijection f between the pairs in X and the pairs in Y. Furthermore, since the total count of good pairs is exactly $2n^2$ from the $2n$ 2-sets $\binom{n}{2}$ 4-sets, no $x\in X$ and $y\in Y$ can appear together in three sets of $\mathbf{T}$.

Let X', Y' denote the complete graphs with vertex sets X, Y. Let E_x denote the edges of X' incident to $x\in X$. Consider the edges $f(E_x)$ in Y'. If any three edges of $f(E_x)$ are incident to a single vertex $y\in Y$, then x, y appear together in three of the 4-sets in $\mathbf{T}$. Hence $\{f(E_x):x\in X\}$ is a collection of n subgraphs of Y', each having $n-1$ edges and maximum degree 2. Furthermore, by construction each edge $e\in E(Y')$ is in exactly two of these subgraphs, corresponding to the endpoints of $f^{-1}(e)$. Indeed, the stronger condition of orthogonality holds, for if e_1, $e_2\in f(E_{x_1})\cap f(E_{x_2})$, then both $f^{-1}(e_1)$ and $f^{-1}(e_2)$ are incident to x_1 and x_2.

Conversely, given an orthogonal double covering of Y', we can label the subgraphs arbitrarily by the elements of X and define $g(e)$ for $e\in E(Y')$ to be the edge of X' whose endpoints are the two labels on e. The $\binom{n}{2}$ 4-sets in the perfect 2-generator are then the sets obtained by taking the endpoints of e and the endpoints of $g(e)$, for each e. The properties of double covering imply that each good pair xy appears at most twice. Since there are $2n(n-1)$ good pairs, the 2-generator is completed by adding $2n$ 2-sets. ∎

4. Upper bound constructions for $p=2$

For $p=2$, we seek constructions of orthogonal double coverings of K_n. In fact, these abound, but it is not easy to describe a construction that works for all n. One desirable property for the subgraphs in the covering is cyclic invariance.

Let the vertices of K_n correspond to the integers modulo n, and let the *length* of edge ij be $|j-i|$. Let the kth *displacement class* of $E(K_n)$ be the edges of length k; there are $\lceil (n-1)/2 \rceil$ displacement classes of size n, and if n is even there is one class of size $n/2$. We seek solutions that are invariant under cyclic (rotational) symmetry. This means we need only seek one "fundamental" subgraph of K_n. It must have maximum degree 2, and it must have 2 edges of each displacement class, except one edge of class $n/2$ if n is even. In addition, let the kth *delay* of such a subgraph be the rotational distance between its two edges of length k; we take the convention that the delay for length $n/2$ is $n/2$ if n is even. A fundamental subgraph rotates to form an orthogonal double covering if and only if the delays are the distinct integers $1, \ldots, \lfloor n/2 \rfloor$. In particular, the i th rotation and j th rotation share an edge of length k if and only if the delay for the kth displacement class is $|j-i|$.

Theorem 3. *If $n \equiv 1,5 \bmod 6$, then K_n has a cyclically symmetric orthogonal double covering, and $\theta_2(K_{n,n}) = (n^2+3n)/2$.*

Proof. Viewing the vertices (congruence classes) as $0, \pm 1, \ldots, \pm(n-1)/2$, let the $n-1$ edges of the fundamental subgraph be $\{(i, 2i)\}$. Note that 0 belongs to no edges. Each other vertex is the "initiator" i of exactly one edge $(i, 2i)$. Maximum degree 2 holds as long as no vertex is the "terminator" of one edge from each direction. This would require $2i \equiv 2(-j) \bmod n$, where $1 \leq i,\ j \leq (n-1)/2$, which is impossible since n is odd.

For $1 \leq i \leq (n-1)/2$, the two edges $(i, 2i)$ and $(-i, -2i)$ have length i, and the delay between the them is $3i$, which can also be viewed as $n-3i$. If the delays are not distinct, then we have $3i = n-3j$, which impossible when n is not divisible by 3. ∎

Theorem 4. *If $n \equiv 2,10 \bmod 12$, then K_n has a cyclically symmetric orthogonal double covering, and $\theta_2(K_{n,n}) = (n^2+3n)/2$.*

Proof. View the vertices (congruence classes) as $\{0, \ldots, n-1\}$. Let d_k denote the delay specified between the two edges of length k; we choose $d_k = n/2 - k$ for $1 \leq k < n/2$ (for $k = n/2$ there is only one edge). Let p_k denote the "starting vertex" for the first edge of length k; we choose $p_k = 3k$. Thus we are choosing the edges $(p_k, p_k + k)$ and $(p_k + d_k, p_k + d_k + k)$. Since we have enforced distinct delays by construction, it suffices to show that every vertex is used at most twice.

Think of $\{p_k\}$, $\{p_k + k\}$, $\{p_k + d_k\}$, and $\{p_k + d_k + k\}$ as columns of numbers, with row indexed by k. Call these columns A, B, C, D, respectively. Note that row $n/2$ has only the two numbers $3n/2$ and $4n/2$ (i.e., $n/2$ and 0); all others have four numbers. The successive entries in the columns change by -1, 3, 2, 3, respectively. Since there are at most $n/2$ entries in a column and $n/2$ is not divisible by 3, no column has a repeated entry.

Given our choices for d_k and p_k, the formulas for row k in the four columns are $3k$, $4k$, $2k + n/2$, $3k + n/2$, respectively. We claim no value appears in column A and column D; otherwise, we have $3i = 3j + n/2$, but $n/2$ is not a multiple of 3. Similarly, no value appears in column B and column C; otherwise, we have $4i = 2j + n/2$, but $n/2$ is not a multiple of 2. Hence each vertex appears at most once in columns A and D and once in columns B and C, and the construction has the desired properties. ∎

Discovery of a cyclically invariant orthogonal double covering is equivalent to finding distinct values d_k and suitable values p_k such that the resulting array of four columns contains each congruence class at most twice. By shifting the congruence classes by $n/2$ and looking at the other edge of each class as the starting edge, the fundamental subgraph generated by Theorem 4 can also be obtained by using $d_k = k+n/2$ and $p_k = 2k$. The fundamental subgraph in Theorem 3 can be obtained by $d_k = n-3k$ and $p_k = k$.

5. Constructions for larger p

We briefly mention some ideas for larger p. We seek to achieve the bound $(n^2+(2p-1)n)/p$. If the sets in the p-generator are viewed as blocks, the linear program says that a construction achieving the bound must have $[n^2-n(p-1)^2]/p$ blocks of size $2p$ and pn blocks of size $2p-2$.

Consider the case $n=p^2-p+1$; the requirement becomes n blocks of size $2p$ and pn blocks of size $2p-2$. Let $X=x_1,\ldots,x_n$ and $Y=y_1,\ldots,y_n$, by the counting argument, we must have each pair x_ix_j with $i\neq j$ appearing together in exactly $p-1$ blocks, and each pair x_iy_i appearing together in exactly p blocks. It seems likely that this structure exists when there exists a projective plane of order $p-1$; we have completed the construction for $p=3$ and $p=4$.

Suppose there exists such a plane with points $\{1,\ldots,n\}$. The plane consists of n lines, each containing exactly p points, such that every point is on p lines, every pair of lines has one common point, and every pair of points appears together on one line. Form copies of the plane on X and on Y; given a line L in the plane, we have corresponding lines $\{x_i : i\in L\}$ and $\{y_i : i\in L\}$. We form n blocks of size $2p$ by taking, for each line L, the union of the lines in X and Y corresponding to L. This contributes one copy of each bad pair, p copies of each good pair x_iy_i, and one copy of each good pair x_iy_j for $i\neq j$.

The blocks of size $2p-2$ must have $p-1$ elements in each of X and Y. Let $\mathbf{S}$ be the collection of sets obtainable by deleting one element from a line in L, and let $\mathbf{S}(X)$, $\mathbf{S}(Y)$ be the corresponding collections from X and Y. The pn "half-blocks" from X are the elements of $\mathbf{S}(X)$; those from Y are the elements of $\mathbf{S}(Y)$. Since each bad pair appears together in one line, this contributes exactly $p-2$ copies of each bad pair, regardless of how the half-blocks are matched. It remains only to match up the half-blocks such that each pair x_iy_i never appears in matched half-blocks, and each pair x_iy_j with $i\neq j$ appears together in exactly $p-1$ matched half-blocks. Since there are $pn(p-1)^2$ appearances of XY pairs in any matching and we have requested $n(n-1)(p-1)$ pairs, this exhausts the pairs exactly.

It remains only to define a permutation σ of $\mathbf{S}$ such that $S\cap\sigma(S)=\emptyset$ for all $S\in\mathbf{S}$ and for each ordered pair $i,\ j\in[n]$ there are exactly $p-1$ elements S in $\mathbf{S}$ such that $i\in S$ and $j\in\sigma(S)$.

We have done this for $p=3$ and $p=4$, though the constructions we give below are described more explicitly. In particular, the lines of a projective plane can be described cyclically using difference sets. A *perfect cyclic difference set* of order $p-1$ is a collection of p congruence classes mod p^2-p+1 such that the $p(p-1)$ pairwise difference are incongruent. (Of course, 0 is the class omitted.) Such sets exist whenever there is a projective plane of order $p-1$, and the n translates of the difference set give the lines of a projective plane. Examples include $\{0,1,3\}$ for $p=3$ and $\{0,1,3,9\}$ for $p=4$.

We can seek a cyclically invariant σ, such that $\sigma(S')$ is the translate of $\sigma(S)$ when S' is the translate of S. The result for the blocks of size $2p-2$ will be pn blocks in p classes of size n that are close under cyclic permutation of the indices. For $p=3$ with elements modulo 7, such classes are generated by $\{(05 \mid 23),(23 \mid 14),(14 \mid 05)\}$, for example. Here the first two elements are chosen from X, the last two from Y. For $p=4$, with elements modulo 13, we list below various ways to generate these cyclic classes (with $T=10$, $E=11$, $W=12$), but not using the technique described above. Our blocks of size $2p-2$ come in p cyclic classes and consists of matched half-blocks with the desired covering properties on the pairs, but the half-blocks are not obtained by deleting elements from the lines generated by 0139. Of course, because the covering multiplicities are symmetric, the lines of any projective plane can be used for the blocks of size $2p$.

$$\{(014 \mid 8EW),\ (016 \mid 257),\ (035 \mid 268),\ (046 \mid 9T2)\}$$

$$\{(034 \mid 569),\ (016 \mid 48T),\ (035 \mid EW4),\ (046 \mid E13)\}$$

$$\{(034 \mid 569),\ (056 \mid W14),\ (025 \mid TW3),\ (026 \mid 49T)\}$$

$$\{(014 \mid 8EW),\ (056 \mid 9E2),\ (025 \mid 167),\ (026 \mid 358)\}$$

The conjecture that this can be done when a projective plane exists was made also by Füredi [9]. He obtained the exact value of $\theta_p(K_{n,n})$ for $n=(p-1)^2$ when a projective plane of order $p-1$ exists, using two copies of the blocks in the affine plane (and more generally for higher dimensions and multipartite graphs). With the affine plane, the desired coverage of pairs occurs more simply.

References

[1] B. Alspach, K. Heinrich, and G. Liu: Orthogonal factorizations of graphs, in: *Contemporary Design Theory: A collection of Surveys* (J. Dinitz and D. Stinson, eds.), (Wiley 1992), 13–40.

[2] F. E. Bennet, and L. Wu: On minimum matrix representation of closure operations, *Discrete Appl. Math.* **26** (1990), 25–40.

[3] R. C. Brigham, R. D. Dutton, and F. R. McMorris: On the relationship between p-edge and p-vertex clique covers, *Vishwa Intl. J. Graph Theory* **1** (1992), 133–140.

[4] J. Demetrovics, Z. Füredi, and G. O. H. Katona: Minimum matrix representation of closure operations, *Discrete Appl. Math.* **11** (1985), 115–128.

[5] N. Eaton, R. J. Gould, and V. Rödl: On p-intersection representations, preprint.

[6] P. Erdős, A. Goodman, and L. Pósa: The representation of a graph by set intersections, *Canad. J. Math.* **18** (1966), 106–112.

[7] P. Frankl, and V. Rödl: Near perfect coverings in graphs and hypergraphs, *Europ. J. Comb.* **6** (1985), 317–326.

[8] Z. Füredi: Matchings and covers in hypergraphs, *Graphs and Combinatorics* **4** (1988), 115–206.

[9] Z. Füredi: A random representation of the complete bipartite graphs, submitted.

[10] B. GANTER, and H.-D. O. F. GRONAU: On two conjectures of Demetrovics, Füredi, and Katona on partitions, *Discrete Math.* **88** (1991), 149–155.

[11] B. GANTER, H.-D. O. F. GRONAU, and R. C. MULLIN: On orthogonal double covers of K_n, *Ars Combinatoria* **37** (1994), 209–221.

[12] M. C. GOLUMBIC, C. L. MONNA, and W. T. TROTTER: Tolerance graphs, *Disc. Appl. Math.* **9** (1984), 157–170.

[13] K. HEINRICH, and G. NONAY: Path and cycle decompositions of complete multigraphs, *Annals Discr. Math* **27** (1985), 275–286.

[14] F. HERING: Block designs with cyclic block structure, *Annals Discr. Math.* **6** (1980), 201–214.

[15] G. ISAAK, S.-R. KIM, T. A. MCKEE, F. R. MCMORRIS, and F. S. ROBERTS: 2-competition graphs, *SIAM J. Discr. Math.* **5** (1992), 524–538.

[16] M. S. JACOBSON: On the p-edge clique cover number of complete bipartite graphs, *SIAM J. Discr. Math.* **5** (1992), 539–544.

[17] M. S. JACOBSON, F. R. MCMORRIS, and H. M. MULDER: Tolerance intersection graphs, in: *Graph Theory, Combinatorics, and Applications,* Vol 2 (Proc. 6th Intl. Conf. Kalamazoo, 1988). Y. Alavi et al, eds. (Wiley 1991), 705–723.

[18] M. S. JACOBSON, F. R. MCMORRIS, and E. R. SCHEINERMAN: General results on tolerance intersection graphs, *J. Graph Theory*, **15** (1991), 573–577.

[19] S.-R. KIM: Competition graphs and scientific laws for food webs and other systems, Ph. D. Thesis, Rutgers Univ., 1988.

[20] S.-R. KIM, T. A. MCKEE, F. R. MCMORRIS, and F. S. ROBERTS: p-competition graphs, DIMACS Technical Report 89–19, see also: p-competition numbers, *Discrete Appl. Math.* **46** (1993), 87–92; p-competitions graphs, *Linear Alg. Appl.* (to appear).

[21] G. E. MAJOR: p-edge clique coverings of graphs, M. A. Thesis, Univ. of Louisville, 1990.

[22] G. E. MAJOR, and F. R. MCMORRIS: p-edge clique coverings of graphs, *Congressus Num.* **79** (1990), 143–145.

[23] A. RAUSCHE: On the existence of special block designs. *Rostock. Math. Kolloq* **35** (1988), 13–20.

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