Permutation Bigraphs and Interval Containments

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Abstract

A bipartite graph with partite sets X and Y is a *permutation bigraph* if there are two linear orderings of its vertices so that xy is an edge for $x \in X$ and $y \in Y$ if and only if x appears later than y in the first ordering and earlier than y in the second ordering. We characterize permutation bigraphs in terms of representations using intervals. We determine which permutation bigraphs are interval bigraphs or indifference bigraphs in terms of the defining linear orderings. Finally, we show that interval containment posets are precisely those whose comparability bigraphs are permutation bigraphs, via a theorem showing that a directed version of interval containment provides no more generality than ordinary interval containment representation of posets.

Keywords: permutation graph, chain graph, interval containment poset, interval bigraph, indifference bigraph.

1 Introduction

A *permutation graph* is an undirected graph G representable by two orderings of $V(G)$ so that vertices are adjacent if and only if they appear in opposite order in the two orderings. Permutation graphs have been heavily studied ([3, 10, 26] summarize early work); there are fast recognition algorithms ([7, 16, 19]), and many NP-complete optimization problems for general graphs admit fast algorithms for permutation graphs ([5, 17, 20], etc.).

The definition has several well-known equivalent restatements. The *comparability graph* of a partially ordered set (poset) P has a vertex for each element, with x and y adjacent if $x \leq y$ or $y \leq x$ in P. A poset has *dimension 2* if its elements admit two linear orderings such that $x \leq y$ in P if and only if x precedes y in both orderings. To prove the equivalences in Theorem 1.1, reversing one of the orderings used yields (d)⇔(b), (d)⇒(c), and (a)⇔(b), with (b)⇒(a) because intervals can be expanded simultaneously to have a common point without changing containments. The implication (c)⇒(d) is more subtle.

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Theorem 1.1. [8, 9, 11] *For a graph G, the following conditions are equivalent:*
(a) G is a permutation graph;
(b) G is the containment graph of a family of intervals in $\mathbb{R}$;
(c) Both G and its complement are transitively orientable (that is, comparability graphs);
(d) G is the comparability graph of a poset of dimension at most 2.

The subclass of permutation graphs consisting of those that are bipartite has also been studied, with structure studied initially in [27, 28], optimization problems in [1, 4, 12, 13], and enumeration in [24]. We study a different class of bipartite graphs that properly contains the bipartite permutation graphs.

Definition 1.2. An *X,Y-bigraph* is a bipartite graph with *bipartition* into parts X and Y. A *permutation bigraph* is an X,Y-bigraph G that can be represented by two vertex orderings L and π so that $xy \in E(G)$ for $x \in X$ and $y \in Y$ if and only if x follows y in L and x precedes y in π. We call (L, π) a *permutation model* for G.

If the first ordering L is viewed as numbering the vertices from 1 through n, then (X, Y) becomes a partition of $\{1, \ldots, n\}$, and the edges of G correspond to inversions in the second ordering π such that the first (larger) element of the inversion comes from X. Thus, instead of specifying (L, π), we can just specify a single ordering π on $[n]$ plus the subset of $[n]$ occupied by X, where $[n] = \{1, \ldots, n\}$. The ordering L is then just the reference numbering of the vertices. Similarly, a permutation graph is specified by a single ordering π when we view the first ordering as a reference numbering.

The permutation bigraph on (X, Y) with model (L, π) is a subgraph of the permutation graph with that model, keeping as edges only some of the inversions in π. The *bipermutation bigraph* on (X, Y) with model (L, π) keeps as edges all inversions between elements of X and Y no matter which part contributes the first element; the first "bi" indicates the symmetric treatment of X and Y. This X,Y-bigraph arises by just deleting all edges within X or within Y from the permutation graph with model (L, π). It turns out that the more restricted concept of permutation bigraph is more fruitful.

Interchanging the roles of X and Y while keeping (L, π) the same changes the permutation bigraph generated by the model. The graph whose edges are the inversions where the element of Y comes first also has a permutation model with respect to the ordered partition (X, Y); just reverse both orderings. Thus the bipermutation bigraph on (X, Y) with model (L, π) is the union of the permutation bigraph with model (L, π) and the permutation bigraph generated by the model in which each of L and π is reversed.

Our initial characterizations of permutation bigraphs parallel those of permutation graphs and are quite easy. We need several other families of graphs.

Definition 1.3. An *interval containment bigraph* is an X,Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if the interval for y contains the interval for x. A *chain graph* is an X, Y-bigraph in which the neighborhoods of one partite set (and hence also the other) form a chain under inclusion. A *circular-arc graph* is the intersection graph of a family of arcs on a circle. The *intersection* of two graphs G and H with $V(G) = V(H)$ is the graph $G \cap H$ with vertex set $V(G)$ given by $E(G \cap H) = E(G) \cap E(H)$.

Theorem 1.4. *For an X, Y-bigraph graph B, the following conditions are equivalent:*
(a) B is a permutation bigraph;
(b) B is the intersection of two chain graphs with the same bipartition;
(c) B is an interval containment bigraph;
(d) B is bipartite and its complement is a circular-arc graph.

In Section 2, we prove Theorem 1.4 and provide examples relating permutation bigraphs to other classes. In particular, two well-studied subclasses of permutation bigraphs are the following.

Definition 1.5. An *interval bigraph* is an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if their intervals intersect. An *indifference bigraph* is an interval bigraph having an interval representation in which all intervals have the same length.

An interval X, Y-bigraph arises from an interval graph with vertex set $X \cup Y$ by deleting the edges within X and within Y, just as bipermutation bigraphs arise from permutation graphs. Sen, Das, Roy, and West [21] proved that the interval bigraphs are the intersections of two chain graphs whose union is a complete bipartite graph with the same bipartition. Thus by Theorem 1.4 every interval bigraph is a permutation bigraph. Steiner [28] proved that the indifference bigraphs are just the bipartite permutation graphs. In Section 3 we characterize the permutation models for interval bigraphs and for indifference bigraphs. For a permutation σ of a set $X \cup Y$ of vertices, let σ^* denote the reverse permutation.

Theorem 1.6. *A permutation bigraph is an interval bigraph if and only if it has a permutation model (L, π) such that the permutation bigraph with model (L^*, π^*) has no edges.*

Theorem 1.7. *A permutation bigraph B is an indifference bigraph if and only if it has a permutation model (L, π) such that the model (L^*, π^*) generates no edges and each partite set appears in the same order in π as in L.*

Finally, in Section 4 we relate permutation bigraphs to posets. Our most difficult result is a direct proof of a characterization of posets whose comparability digraphs are interval containment digraphs. The result was proved originally by Sen, Sanyal, and West [22], but there the hard direction relied on a result of Bouchet [2] characterizing the dimension of posets. The proof here uses only a simpler and better known theorem of Cogis [6], plus our results from Section 2. For the statement, we need other structures associated with posets.

Definition 1.8. An *interval containment digraph* is a digraph D representable by giving intervals S_w and T_w to each vertex w so that $uv \in E(D)$ if and only if $S_u \subseteq T_v$. An *interval containment poset* is a poset P representable by giving each element z a real interval I_z so that $x \leq y$ in P if and only $I_x \subseteq I_y$ (trivially equivalent to dimension 2). The *comparability digraph* $D(P)$ of a poset P has an edge uv whenever $u \leq v$ in P. With X and Y being two copies of the elements of P, the *comparability bigraph* $B(P)$ is the X,Y-bigraph such that $u_x \in X$ and $v_y \in Y$ are adjacent if and only if $u \leq v$ in P.

Theorem 1.9. [22] *A poset P is an interval containment poset if and only if its comparability digraph is an interval containment digraph.*

Necessity in Theorem 1.9 follows by letting S_w and T_w both equal the interval assigned to w in an interval containment model of P. The difficult part is showing that the flexibility of allowing S_w and T_w to differ does not permit representations of any additional posets.

Corollary 1.10. *For a poset P, the following conditions are equivalent:*
(a) The comparability graph of P is a permutation graph (an interval containment graph).
(b) P is an interval containment poset (a poset of dimension 2).
(c) The comparability digraph of P is an interval containment digraph.
(d) The comparability bigraph of P is an interval containment bigraph (permutation bigraph).

Proof. (a)⇔(b) is in Theorem 1.1. (c)⇔(d) is Lemma 4.1. (b)⇔(c) is Theorem 1.9, proved as (b)⇒(c) in Lemma 4.2 and (c)⇒(b) in Theorem 4.5. □

2 Characterizations of Permutation Bigraphs

As stated initially in Theorem 1.4, we characterize permutation bigraphs in terms of interval containment bigraphs, circular-arc graphs, and chain graphs. By analogy with the use of "permutation model" for the pair (L, π) of permutations representating a permutation bigraph, we also use *interval model* and *arc model* for the sets of intervals or arcs representing an interval containment bigraph or circular-arc graph, respectively.

Although it has other definitions in graph theory and stochastic processes, the term "chain graph" is used as in Definition 1.3 in the algorithmic community, starting at least from [15] in 1998. The term *Ferrers bigraph* (analogous to *Ferrers digraph* used since [6] in 1979) is also natural, since when the rows (for X) and columns (for Y) of the adjacency matrix are put in order by neighborhood size, the positions with value 1 form a Ferrers diagram. The boundary between entries equaling 1 and entries equaling 0 moves from one corner to the opposite corner, crossing each row for an element of X and each column for an element of Y in a specified order. This vertex ordering by itself specifies a chain graph.

Definition 2.1. An X, Y*-ordering* σ of a chain graph with bipartition (X, Y) is a vertex ordering such that the neighborhood of any $x \in X$ is the set of vertices of Y preceding it in σ, and the neighborhood of any $y \in Y$ is the set of vertices of X following it in σ.

The reverse of an X, Y-ordering of a chain graph is a Y, X-ordering of it. A chain graph does not have a unique X, Y-ordering when two vertices of one part have no vertices of the other part between them. For short proofs of equivalence of various characterizations of chain graphs in the language of Ferrers bi/digraphs, see [29]. The orderings of chain graphs yield short proofs of the elementary characterizations of permutation bigraphs. For (d)⇒(a), we use an old result of Spinrad [25] used also by Hell and Huang [14].

Lemma 2.2 ([25])**.** *If X and Y are disjoint cliques covering the vertices of a circular-arc graph G whose complement is bipartite, then G has an arc model with points a and b such that the arcs for X contain a, the arcs for Y contain b, and no arc contains both.*

Theorem 2.3 (Theorem 1.4)**.** *For a graph B, the following conditions are equivalent:*
(a) B is a permutation bigraph;
(b) B is the intersection of two chain graphs with the same bipartition;
(c) B is an interval containment bigraph;
(d) B is bipartite and its complement $\overline{B}$ is a circular-arc graph.

Proof. Let B be an X, Y-bigraph. Adopt the convention that always $x \in X$ and $y \in Y$.

(a)⇔(b): Given a permutation model (L, π) for B, use L as the X, Y-ordering for one chain graph and π as the Y, X-ordering for another. Now $xy \in E(B)$ if and only if x and y are adjacent in both chain graphs. The transformation is reversible.

(b)⇒(c): Given chain graphs F_1 and F_2 with bipartition (X, Y) such that $F_1 \cap F_2 = B$, let τ be the Y, X-ordering of F_1, and let σ be the Y, X-ordering of F_2. To each vertex v, assign the interval $[-a_v, b_v]$, where a_v and b_v are the positions of v in τ and σ, respectively. Now $[-a_x, b_x] \subseteq [-a_y, b_y]$ if and only if $xy \in E(F_1) \cap E(F_2)$.

(c)⇒(d): Enlarging intervals by the same amount does not change containments; to an interval model for B whose intervals have a common point a. Draw this model on a circle; some point b lies in no arcs. Complement the arcs for elements of Y within the circle. Now all arcs for Y contain b and all arcs for X contain a, yielding cliques. Also, the resulting arcs for x and y are disjoint if and only if in the original interval model the interval for y contains the interval for x, yielding an arc model for $\overline{B}$.

(d)⇒(a): Lemma 2.2 yields an arc model for $\overline{B}$ with a and b as specified. Complementing the arcs for Y allows the model to be cut at b and drawn on the line, yielding intervals that all contain a. The interval for y now contains that for x if and only if the original arcs for x and y were disjoint. Let L be the left-endpoint ordering and π the right-endpoint ordering of the vertices (both from left to right); (L, π) is a permutation model for B. □

The argument for the equivalence of (a) and (c) also shows that B is a bipermutation bigraph if and only if the vertices can be assigned intervals so that $x \in X$ and $y \in Y$ are adjacent precisely when their intervals are ordered by inclusion, in either order.

A permutation bigraph may have many permutation representations. Indeed, when all of Y precedes all of X in L, every permutation π with all of X before all of Y yields a permutation model of the complete bipartite graph $K_{|X|,|Y|}$. Also, interchanging and reversing L and π in a permutation model yields another model for the same graph.

As we have mentioned, we can specify an n-vertex permutation graph by giving just one permutation, letting the other permutation be the identity permutation on $[n]$. To specify a permutation bigraph in this way, one must also give the partition of $[n]$ into X and Y. We may present this partition along with π by putting underbars on the elements of Y and overbars on the elements of X, or simply underbars on Y.

Example 2.4. *The* 6*-cycle* C_6 *is a bipermutation bigraph, but permutation bigraphs contain no chordless cycles of length at least* 6*.* Consider the partition of [6] with $X = \{\overline{1}, \overline{4}, \overline{6}\}$ and $Y = \{\underline{2}, \underline{3}, \underline{5}\}$. Let $\pi = (\overline{4}, \underline{2}, \overline{6}, \underline{5}, \overline{1}, \underline{3})$. In the bipermutation model it does not matter which partite set occurs first in an inversion, so the resulting bipermutation bigraph is a 6-cycle with vertices $\underline{2}, \overline{1}, \underline{5}, \overline{6}, \underline{3}, \overline{4}$ in order.

The characterization by containment graphs (Theorem 1.1b) forbids chordless cycles of length more than 4 from permutation graphs. That in itself does not imply that such even cycles are not permutation bigraphs, since not all permutation bigraphs are bipartite permutation graphs (Example 3.2). Nevertheless, the characterization by interval containment bigraphs (Theorem 1.4b) does similarly forbid such cycles from permutation bigraphs.

Consider such an interval model for C_6, where the vertices of the cycle in order are $x_1, y_1, x_2, y_2, x_3, y_3$. Let X_i and Y_i denote the intervals for x_i and y_i, respectively. We have $X_2 \subseteq Y_1 \cap Y_2$. Since $X_1 \subseteq Y_1$ and $X_1 \not\subseteq Y_2$, we may assume that X_1 and Y_1 extend to the left of Y_2 (and X_2), and similarly X_3 and Y_2 extend to the right of Y_1 (and X_2). Since Y_3 must contain X_1 and X_3, we now have Y_3 containing X_2, which is forbidden.

The argument extends to forbid longer chordless even cycles. □

3 Interval Bigraphs and Indifference Bigraphs

In discussing interval bigraphs (Definition 1.5), we rely on the characterization by Sen, Das, Roy, and West [21]. It was proved in the language of Ferrers bigraphs and adjacency matrices; here we restate it in the language of chain graphs. We write $K_{X,Y}$ for the complete bipartite graph with bipartition (X, Y).

Theorem 3.1 ([21])**.** *An* X, Y*-bigraph is an interval bigraph if and only if it is the intersection of two chain graphs with bipartition* (X, Y) *whose union is* $K_{X,Y}$*.*

Example 3.2. *Indifference bigraphs, interval bigraphs, and permutation bigraphs.* By Theorem 3.1 and Theorem 1.4, every interval bigraph is a permutation bigraph. The families are distinct: [21] exhibits an intersection of two chain graphs with the same bipartition (that is, a permutation bigraph) that is not an interval bigraph.

By Theorem 1.4, the permutation bigraphs are also the bipartite graphs whose complements are circular-arc graphs. This also characterizes interval bigraphs among permutation bigraphs, since Hell and Huang [14] showed that interval bigraphs are the bipartite graphs having an arc model with no two arcs covering the entire circle.

Indifference bigraphs (Definition 1.5) form a proper subclass. Lin and West [18] gave what can be interpreted as a forbidden subgraph characterization for indifference bigraphs within the class of interval bigraphs: An interval bigraph is an indifference bigraph if and only if its adjacency matrix does not contain any of the submatrices below or their transposes.

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

From Steiner's characterization of bipartite permutation graphs, stated below, we conclude also that all bipartite permutation graphs are permutation bigraphs. □

Theorem 3.3 (Steiner [28]). *A graph is a bipartite permutation graph if and only if it is an indifference bigraph.*

In this section we characterize the permutation models that yield interval bigraphs and those that yield indifference bigraphs. Recall that σ^* denotes the reverse of an ordering σ.

Theorem 3.4 (Theorem 1.6). *An X,Y-bigraph B is an interval bigraph if and only if it has a permutation model (L,π) such that the permutation bigraph B^* with model (L^*,π^*) has no edges.*

Proof. Let B be an interval bigraph, and fix $x \in X$ and $y \in Y$.

By Theorem 3.1, $B = F_1 \cap F_2$, where F_1 and F_2 are chain graphs with bipartition (X, Y) whose union is $K_{X,Y}$. Let L be the X, Y-ordering of F_1, and let π be the Y, X-ordering of F_2. By the proof of (b)⇒(a) in Theorem 2.3, (L, π) is a permutation model for B.

Since $xy \in E(K_{X,Y}) = E(F_1) \cup E(F_2)$, we have y before x in L or x before y in π. Hence x precedes y in L^* or follows y in π^*, which is equivalent to $xy \notin E(B^*)$.

Conversely, if B has such a permutation model (L, π), then using L as the X, Y-ordering of a chain graph F_1 and π as the Y, X-ordering of a chain graph F_2 expresses B as the intersection of two chain graphs with bipartition (X, Y) whose union is $K_{X,Y}$, by the same reasoning about the occurrence of edges. □

Within the adjacency matrix of an X, Y-bigraph B, we refer to the submatrix induced by the rows for X and the columns for Y as *the matrix* of B. Sen and Sanyal [23] characterized the matrices of indifference bigraphs (in the language of digraphs). A *monotone consecutive arrangement* of a 0, 1-matrix consists of independent permutations of the rows and the columns and a labeling of each 0-entry as R or C such that every entry above or rightward of an R is R and every entry below or leftward of a C is C. Thus the 1-entries are consecutive in each row and in each column, and the ends of these intervals of 1-entries behave monotonically across the columns or down the rows, as illustrated by the matrix below.

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{matrix}$$

Theorem 3.5 (Sen and Sanyal [23])**.** *A bipartite graph is an indifference bigraph if and only if its matrix has a monotone consecutive arrangement (MCA).*

An MCA expresses an X, Y-bigraph as the intersection of two chain graphs with bipartition (X, Y) whose union is $K_{X,Y}$, so such a graph is an interval bigraph. To characterize indifference bigraphs among permutation bigraphs, we translate the conditions for an MCA into conditions on the chain graphs and hence on the resulting permutation model.

Lemma 3.6. *The matrix of an X, Y-bigraph has an MCA if and only if it is the intersection of chain graphs F_1 and F_2 with bipartition (X, Y) such that $F_1 \cup F_2 = K_{X,Y}$ and such that X appears in opposite order in the X, Y-orderings of F_1 and F_2, as does Y.*

Proof. Given an MCA, treat $E(F_1)$ as the set of positions with 1 or C (lower left) and $E(F_2)$ as the set of positions with 1 or R (upper right). Read an X, Y-ordering σ_1 for F_1 along the upper-right boundary of the positions with 1, from upper left to lower right. Similarly, read an X, Y-ordering σ_2 for F_2 along the lower-left boundary of those positions, from lower right to upper left. The orderings give the order in which rows and columns are crossed. Since σ_1 starts in the upper left and σ_2 in the lower right, X appears in opposite order in the two orderings, as does Y.

Conversely, given chain graphs F_1 and F_2 whose union is $K_{X,Y}$ and whose X, Y-orderings have this property, reversing the construction yields an MCA. □

Theorem 3.7 (Theorem 1.7)**.** *An X, Y-bigraph B is an indifference bigraph if and only if it has a permutation model (L, π) such that the model (L^*, π^*) generates no edges and each partite set appears in the same order in π as in L.*

Proof. Given an indifference bigraph B, generate chain graphs F_1 and F_2 from an MCA as in Lemma 3.6. Let L be the X, Y-ordering of F_1, and let π be the Y, X-ordering of F_2. By the proof of Theorem 3.4, (L, π) is a permutation model for B, and (L^*, π^*) generates no edges. By Lemma 3.6, each partite set appears in the same order in L and π.

The argument reverses, since both Theorem 3.4 and Lemma 3.6 are equivalences. □

4 Comparability graphs and Permutation bigraphs

In this section, we relate permutation bigraphs to various aspects of posets. The needed definitions are in Definition 1.8. We will characterize interval containment posets as precisely those whose comparability digraphs have interval containment representations. As mentioned in Corollary 1.10, this implies that the comparability graph of a poset is a permutation graph if and only if its comparability bigraph is a permutation bigraph.

Lemma 4.1. *For a poset P, the comparability digraph D is an interval containment digraph if and only if the comparability bigraph B is an interval containment bigraph, achieved using the same intervals.*

Proof. In $V(B)$, let z_x and z_y be the copies of element $z \in P$ in X and Y. We have

$$u_x v_y \in E(B) \quad \Leftrightarrow \quad u \le v \text{ in } P \quad \Leftrightarrow \quad uv \in E(D).$$

Interval assignments correspond by letting S_z and T_z for $z \in V(D)$ be the intervals assigned to z_x and z_y in $V(B)$, respectively. The assignment is an interval model for B if and only if it is an interval model for D, since the condition for both is $(S_u \subseteq T_v) \Leftrightarrow (u \le v \text{ in } P)$. □

Lemma 4.2. *If P is an interval containment poset, then the comparability digraph of P is an interval containment digraph. Equivalently, if the comparability graph G of a poset P is a permutation graph, then the comparability bigraph B of P is a permutation bigraph.*

Proof. Given an interval containment model of P with interval I_w assigned to element w, letting $S_w = T_w = I_w$ expresses the comparability digraph as an interval containment digraph; we are given $x \le y$ in P if and only if $I_x \subseteq I_y$, which is now equivalent to $S_x \subseteq T_y$.

The second claim can also be shown directly, but that is a bit more cumbersome. The claim is equivalent to the first because Theorem 1.1 makes P is an interval containment poset if and only if G is a permutation graph, Lemma 4.1 makes the comparability graph D an interval containment digraph if and only if B is an interval containment bigraph, and by Theorem 1.4 the interval containment bigraphs are the permutation bigraphs. □

Lemma 4.2 proves necessity in Theorem 1.9. The converse is not immediate, because there is no immediate way to reverse the transformation when $S_w \ne T_w$. In [22], sufficiency was proved indirectly using the result of Bouchet [2] that the order dimension of a poset equals the Ferrers dimension of its comparability digraph. Our argument below uses only the result of Cogis, which we translate into the language of chain graphs and 0,1-matrices.

Definition 4.3. (Cogis [6]) Consider a $0,1$-matrix D. The two zeros in any 2-by-2 permutation submatrix form an *obstruction* or *couple*. The vertex set of the *associated graph* is the set of zeros in D; two vertices are adjacent if and only if they form a couple in D. The *complement* $\overline{D}$ is obtained by subtracting each entry from 1. A set of positions is *chained* if its partition by rows forms an inclusion chain (similarly for columns).

In particular, the set of positions with 1 in the matrix of a chain graph is a chained set. Cogis proved that a bipartite graph with matrix D is the intersection of two chain graphs if and only if the associated graph of D is bipartite. Necessity is immediate; we state the sufficiency part in more detail.

Theorem 4.4. (Cogis [6]) *Let H be the associated graph of a matrix D, and let $\mathbf{I}$ be the set of isolated vertices in H. If H is bipartite, then $H - \mathbf{I}$ has a bipartition $\{\mathbf{R}, \mathbf{C}\}$ such that the sets $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ are chained.*

Proving the converse of Lemma 4.2 amounts to showing that the flexibility of allowing $S_w \neq T_w$ in interval containment models of posets does not permit representing any more posets than those representable with $S_w = T_w$ for every element w.

Theorem 4.5. [22] *If the comparability digraph of a poset P is an interval containment digraph, then P is an interval containment poset.*

Proof. Let P be a poset whose comparability digraph is an interval containment digraph. Since simultaneous expansion of intervals does not change containments, we may take an interval model in which all intervals contain 0. By Lemma 4.1, these intervals also form an interval model of the comparability bigraph B. Let D denote the corresponding matrix, with entry (u_x, v_y) being 1 if and only if $u \leq v$ in P. As in the equivalence of (b) and (c) in Theorem 2.3, The left endpoints and right endpoints give the X, Y-ordering and Y, X-ordering for two chain graphs whose intersection is B.

Since D has Ferrers dimension at most 2, and complements of Ferrers digraphs are Ferrers digraphs, the 0-positions in D can be expressed as the union of two Ferrers digraphs. Since the isolated vertices of $H(D)$ form no couples, they can be included in both Ferrers digraphs. Put the nonisolated vertices into $\mathbf{R}$ or $\mathbf{C}$ when they lie in the first or second Ferrers digraph, respectively. Now $H(D)$ is bipartite, $\{\mathbf{R}, \mathbf{C}\}$ is a bipartition of the subgraph H' of nonisolated vertices, and the positions in $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ form Ferrers digraphs. This is the trivial direction of Cogis' result. We study the resulting coloring in more detail.

(i) If x and y are incomparable in P, then positions (x, y) and (y, x) in the matrix of D contain 0. Since D is reflexive, these positions form a couple. Hence they have distinct colors in any proper 2-coloring of $H(D)$.

(ii) If $x < y$ in P, then $xy \in E(D)$. Hence position (x, y) is 1 and position (y, x) is 0 in the matrix. If position (y, x) forms a couple with (u, v), then positions (u, x) and (y, v) are 1. Now $u \leq x < y \leq v$, so transitivity of P requires $u < v$, and position (u, v) is 1, a contradiction. We conclude that (y, x) is an isolated vertex in $H(D)$.

Let $\mathbf{E}$ be the set of positions containing 1 in the matrix of D. We have shown that $\mathbf{E}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{I}$ partition the positions. Position (x, x) lies in $\mathbf{E}$. For $x \neq y$, we have

$$(x, y) \in \mathbf{E} \Leftrightarrow (y, x) \in \mathbf{I} \text{ and } (x, y) \in \mathbf{R} \Leftrightarrow (y, x) \in \mathbf{C}. \tag{1}$$

We next obtain an ordering of P such that when the rows and the columns of the matrix of D are simultaneously given this ordering, $\mathbf{C} \cup \mathbf{I}$ occupies the lower triangle of positions below the diagonal, while $\mathbf{R} \cup \mathbf{E}$ occupies the diagonal and the positions above it.

Since $\mathbf{C} \cup \mathbf{I}$ is a Ferrers digraph, the rows and columns of D can be ordered to put $\mathbf{C} \cup \mathbf{I}$ in the positions of a Ferrers diagram in the lower left. The complement $\mathbf{R} \cup \mathbf{E}$ then occupies a Ferrers diagram in the upper right that includes the positions of the form (x, x) ("loops").

If the set $\mathbf{C} \cup \mathbf{I}$ contains position (p, q) with $p \leq q$ in the permuted matrix, then the loops for the first q columns appear in the first $p - 1$ rows. By the pigeonhole principle, some row contains two loops, but there is exactly one loop in each row.

Hence $\mathbf{C} \cup \mathbf{I}$ is confined below the diagonal. From (1), $|\mathbf{C} \cup \mathbf{I}| = \binom{n}{2}$, so $\mathbf{C} \cup \mathbf{I}$ consists of all such positions. Now the only set of n positions in $\mathbf{R} \cup \mathbf{E}$ with one in each row and column is the diagonal itself. Hence the loops all appear on the diagonal, putting the rows and columns in the same order. This common ordering is a numbering f of the elements of P by 1 through n such that $x < y$ in P implies $f(x) < f(y)$.

From this ordering, we obtain an interval containment representation of P. Let the right endpoint of the interval I_x for x be $f(x)$. For the left endpoints, we define g mapping P into $\{-1, \ldots, -n\}$. At step i, among the current minimal elements, let the one with the rightmost right endpoint be assigned $-i$ as its left endpoint. Delete this element and continue.

If $x < y$ in P, then because f is a linear extension we have $f(x) < f(y)$. Also, in the assignment of left endpoints, y cannot receive a (negative) left endpoint before x; hence $g(y) < g(x)$, and $I(x) \subseteq I(y)$.

If the representation fails, then there exist x and y incomparable in P with $I_x \subseteq I_y$. Let x be the element with smaller interval in such a pair, and let y be a minimal element among those incomparable to x whose intervals contain I_x. Since $f(x) < f(y)$, the construction procedure requires that y is not a minimal remaining element when $g(x)$ is assigned. Hence y is then above some currently minimal element z. Since x is chosen now in preference to z, we have $f(z) < f(x)$. We have $f(z) < f(x) < f(y)$, but x is incomparable to y and z.

With the given common ordering of rows and column, in which $\mathbf{C} \cup \mathbf{I}$ consists of the positions below the diagonal, $\mathbf{R} \cup \mathbf{E}$ consists of those on the diagonal and above, and the diagonal corresponds to the loops, we have obtained the submatrix below, which contradicts that $\mathbf{R} \cup \mathbf{I}$ is a Ferrers matrix. Hence we have successfully constructed a representation, and P is an interval containment poset. $\square$

$$\begin{array}{cccc} & z & x & y \\ z & 1 & R & 1 \\ x & & 1 & R \\ y & & & 1 \end{array}$$

As noted in the introduction, Theorem 4.5 completes the proof of Corollary 1.10.

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