Perfection Thickness of Graphs

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Abstract

We determine the order of growth of the worst-case number of perfect subgraphs needed to cover an n-vertex graph.

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For a graph G, let $f(G)$ denote the minimum number of perfect subgraphs of G whose union is G. Also let $f(n) = \max\{f(G): |V(G)| = n\}$. Tuza [6] proved that $c_1 \lg n/(\lg\lg n) \le f(n) \le c_2 \lg n$ for some constants c_1 and c_2, where lg denotes base-2 logarithm. We prove that $\frac{1}{2+\epsilon}\lg n \le f(n) \le \lceil \lg n \rceil$.

It is well-known that the minimum number of bipartite subgraphs needed to cover a graph G is $\lceil \lg \chi(G) \rceil$ [2,4,5]. Thus $f(n) \le \lceil \lg n \rceil$. We obtain the lower bound by applying this observation to triangle-free graphs, in which all perfect subgraphs are bipartite.

Erdős [3] proved that for every $\epsilon > 0$ there exist triangle-free k-chromatic graphs with order at most $k^{2+\epsilon}$ (actually at most $c(k \log k)^2$). For such a graph G with order $n \le k^{2+\epsilon}$, we have $f(G) = \lceil \lg k \rceil \ge \frac{1}{2+\epsilon}\lg n$.

Naturally we ask, does $\lim_{n\to\infty} f(n)/\lg n$ exist?

We remark that the same bounds hold for the worst-case number of perfect graphs needed to decompose n-vertex graphs, since our arguments use only coverings by bipartite graphs, which form a hereditary family. Also, the order of growth of the minimum order of a triangle-free k-chromatic graph is now known, lying between two constant multiples of $k^2 \lg k$ (see [1, p61]). This yields $f(n) \geq \frac{1}{2}(\lg n - \lg\lg n - O(1))$.

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