Extremal graphs with a given number of perfect matchings

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Abstract

Let $f(n,p)$ denote the maximum number of edges in a graph having n vertices and exactly p perfect matchings. For fixed p, Dudek and Schmitt showed that $f(n,p) = n^2/4 + c_p$ for some constant c_p when n is at least some constant n_p. For $p \le 6$, they also determined c_p and n_p. For fixed p, we show that the extremal graphs for all n are determined by those with $O(\sqrt{p})$ vertices. As a corollary, a computer search determines c_p and n_p for $p \le 10$. We also present lower bounds on $f(n,p)$ proving that $c_p > 0$ for $p \ge 2$ (as conjectured by Dudek and Schmitt), and we conjecture an upper bound on $f(n,p)$. Our structural results are based on Lovász's Cathedral Theorem.

1 Introduction

For even n and positive integer p, Dudek and Schmitt [5] defined $f(n,p)$ to be the maximum number of edges in an n-vertex graph having exactly p perfect matchings. Say that such a graph with $f(n,p)$ edges is *p-extremal*. We study the behavior of $f(n,p)$ and the structure of p-extremal graphs.

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Although existence of a perfect matching can be tested in time $O(n^{1/2}m)$ for graphs with n vertices and m edges [14], counting the perfect matchings is #P-complete, even for bipartite graphs [19]. Let $\Phi(G)$ denote the number of perfect matchings in G. Bounds on $\Phi(G)$ are known in terms of the vertex degrees in G. For a bipartite graph G with n vertices in each part and degrees $d_1, \ldots, d_n$ for the vertices in one part, Brègman's Theorem [2] states that $\Phi(G) \le \prod_{i=1}^{n} (d_i!)^{1/d_i}$. Kahn and Lovász (unpublished) proved an analogue for general graphs (other proofs were given by Friedland [6] and then by Alon and Friedland [1]). For a graph G with vertex degrees $d_1, \ldots, d_n$, the Kahn–Lovász Theorem states that $\Phi(G) \le \prod_{i=1}^{n} (d_i!)^{1/2d_i}$. Both results were reproved using entropy methods by Radhakrishnan [16] and by Cutler and Radcliffe [4], respectively. Gross, Kahl, and Saccoman [7] studied $\Phi(G)$ for a fixed number of edges; they determined the unique graphs minimizing and maximizing $\Phi(G)$.

Maximizing the number of edges when $\Phi(G)$ and n are fixed has received less attention. Hetyei proved that $f(n, 1) = n^2/4$ (see [11, Corollary 5.3.14, page 173]). We describe Hetyei's construction inductively in a more general context.

Construction 1.1. The *Hetyei-extension* of G is the graph G' formed from G by adding a vertex x adjacent to all of $V(G)$ and one more vertex y adjacent only to x. Every perfect matching of G' contains xy and a perfect matching of G, so $\Phi(G') = \Phi(G)$. Starting with $G = K_2$, Hetyei-extension yields graphs with one perfect matching for all even orders.

When G has n vertices, $|E(G')| = |E(G)| + n + 1$. Since $(k+2)^2/4 = k^2/4 + k + 1$, we obtain $f(n, 1) \ge n^2/4$ for all even n. (Note that when G has a unique perfect matching M, at most two edges join the vertex sets of any two edges of M; hence $f(n, 1) \le n/2 + 2\binom{n}{2} = n^2/4$.)

More generally, when $\Phi(G) = p$ and $|E(G)| = n^2/4 + c$, the Hetyei-extension of G yields $f(n + 2, p) \ge (n + 2)^2/4 + c$. This observation is due to Dudek and Schmitt [5]. ∎

In light of the observation in Construction 1.1, we let $c(G) = |E(G)| - |V(G)|^2/4$ and call $c(G)$ the *excess of* G. For fixed p, Dudek and Schmitt proved that the maximum excess is bounded by a constant.

Theorem 1.2 (Dudek and Schmitt [5])**.** *For $p \in \mathbf{N}$, there is an integer c_p and a threshold n_p such that $f(n, p) = n^2/4 + c_p$ when $n \ge n_p$ and n is even. Also, $-(p-1)(p-2) \le c_p \le p$.*

Dudek and Schmitt determined c_p and n_p for $1 \le p \le 6$, although the proofs for $p \in \{5, 6\}$ were omitted since they were prohibitively long. They conjectured that $c_p > 0$ when $p \ge 2$. We prove their conjecture in Section 2 by generalizing Hetyei's construction. The construction yields $c_p > 0$ but does not generally give the best lower bounds. We give better lower bounds in Section 7; first we must analyze the structure of extremal graphs.

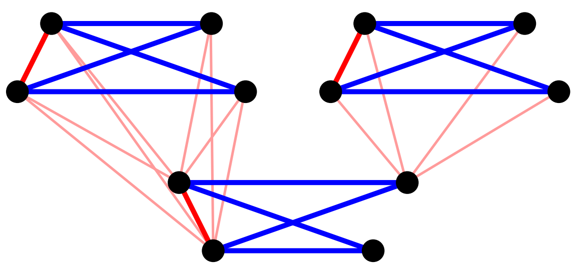
We develop a systematic approach to computing c_p. With this we give shorter proofs for $p \le 6$ and identify the values c_p and n_p for $7 \le p \le 10$. Using structural results from this paper and further computational techniques, c_p and n_p are determined for $p \leq 27$ in [17]. The complete behavior of c_p for larger p remains unknown.

Definition 1.3. Let $\mathcal{F}_p$ denote the family of graphs that are p-extremal and have excess c_p; that is, $\mathcal{F}_p = \left\{G\colon\ \Phi(G) = p \text{ and } |E(G)| = \frac{|V(G)|^2}{4} + c_p\right\}$. Equivalently, $\mathcal{F}_p$ is the set of p-extremal graphs with at least n_p vertices.

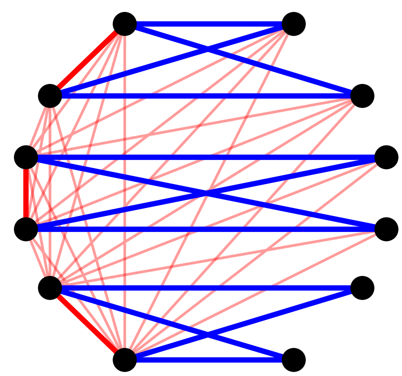
We study the extremal graphs as a subfamily of a larger family.

Definition 1.4. A graph is *saturated* if the addition of any missing edge increases the number of perfect matchings.

Extremal graphs are contained in the much larger family of saturated graphs. Figure 1(a) shows a saturated graph G_1 with 12 vertices, eight perfect matchings, and 27 edges. Although G_1 is saturated, it is not 8-extremal, since the graph G_2 in Figure 1(b) has the same number of vertices and perfect matchings but has 39 edges.



(a) G_1 with 27 edges



(b) G_2 with 39 edges

Figure 1: Two graphs with eight perfect matchings

Lovász's Cathedral Theorem (see [11]) gives a recursive decomposition of all saturated graphs; we describe it in Section 3. In terms of this construction, we describe the graphs in $\mathcal{F}_p$. In Sections 4 and 5, study of the cathedral construction for extremal graphs allows us to reduce the problem of computing c_p to examining a finite (but large) number of graphs.

In Section 6, we extend $f(n, p)$ to odd n and study the corresponding extremal graphs. Section 7 gives constructions for improved lower bounds on c_p. In Section 8, we conjecture an upper bound on c_p that would be sharp for infinitely many values of p. The conjectured bound would be the best possible monotone upper bound, if true. Section 9 mentions several conjectures and discusses a computer search based on our structural results; the search found the extremal graphs for $4 \leq p \leq 10$. Other search techniques are used in [17] to determine c_p for $p \leq 27$.

2 The Excess is Positive

We begin with a simple construction proving the Dudek-Schmitt conjecture that $c_p > 0$.

The *disjoint union* of graphs G and H (with disjoint vertex sets) is denoted $G + H$. The *join* of G and H, denoted $G \vee H$, consists of $G + H$ plus edges joining each vertex of G to each vertex of H. Thus the Hetyei-extension of G is $(G + K_1) \vee K_1$. A *split graph* is a graph whose vertex set is the union of a clique and an independent set.

Definition 2.1. The *Hetyei graph* with $2k$ vertices, produced iteratively in Construction 1.1 from K_2 by repeated Hetyei-extension, can also be described explicitly. It is the split graph with clique $\ell_1, \ldots, \ell_k$, independent set $r_1, \ldots, r_k$, and additional edges $\ell_i r_j$ such that $i \le j$.

The Hetyei graph is the unique extremal graph of order $2k$ with exactly one perfect matching. It has $\frac{(2k)^2}{4}$ edges, so $c_1 = 0$. In the constructions here and in Section 7, the Hetyei graph is a proper subgraph, so the excess is larger.

In a graph having an independent set S with half the vertices, every perfect matching joins S to the remaining vertices. Therefore, to study the perfect matchings in such a graph it suffices to consider the bipartite subgraph consisting of the edges incident to S. In the Hetyei graph, the only perfect matching consists of the edges $\ell_i r_i$ for all $1 \le i \le k$.

For $m \in \mathbf{N}$, let $w(m)$ denote the number of 1s in the binary expansion of m.

Definition 2.2. For $p \ge 2$ and $k = \lceil \log_2(p-1) \rceil + 1$, let $(x_{k-2}, \ldots, x_0)$ be the binary $(k-1)$-tuple such that $p - 1 = \sum_{j=0}^{k-2} 2^j x_j$. The *binary expansion construction* for p, denoted $B(p)$, consists of the Hetyei graph with $2k$ vertices plus the edges $\{\ell_{i+2} r_1 \colon x_i = 1\}$ (see Fig. 2).

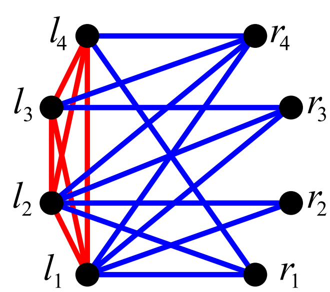


Figure 2: The graph $B(6)$

Theorem 2.3. *If $p \ge 2$, then $\Phi(B(p)) = p$ and $c(B(p)) = w(p-1)$. Thus $c_p \ge w(p-1) \ge 1$.*

Proof. Name the vertices of $B(p)$ as in the Hetyei graph. We construct perfect matchings in $B(p)$ by successively choosing the edges that cover $r_1, \ldots, r_k$. The matching $\{\ell_i r_i \colon 1 \le i \le k\}$

from the Hetyei graph is always present. If r_1 is matched to ℓ_{i+2} instead of to ℓ_1 for some nonnegative i, then for $r_2, \ldots, r_{i-1}$ exactly two edges are available when we choose the edge to cover this vertex. For vertices $r_i, \ldots, r_k$ in order, only one choice then remains. Therefore, each edge of the form $\ell_{i+2}r_1$ lies in 2^{i-2} perfect matchings.

The edge $\ell_{i+2}r_1$ exists if and only if $x_i = 1$ in the binary representation of $p-1$. Thus $\Phi(B(p)) = 1 + \sum_{i=2}^{k} 2^{i-2}x_{i-2} + 1 = 1 + p - 1 = p$. Since $B(p)$ is formed by adding $w(p-1)$ edges to the Hetyei graph, $c(B(p)) = w(p-1)$. ■

3 Lovász's Cathedral Theorem

As we have mentioned, Lovász's Cathedral Theorem characterizes saturated graphs. Since the extremal graphs are saturated, this characterization will be our starting point. Chapters 3 and 5 of Lovász and Plummer [11] present a full treatment of the subject. Another treatment appears in Yu and Liu [20]. A 1-*factor* of a graph G is a spanning 1-regular subgraph; its edge set is a perfect matching. An edge is *extendable* if it appears in a 1-factor.

Definition 3.1. A graph is *matchable* if it has a perfect matching. The *extendable subgraph* of a matchable graph G is the union of all the 1-factors of G. An induced subgraph H of G is a *chamber* of G if $V(H)$ is the vertex set of a component of the extendable subgraph of G.

Every vertex of a matchable graph G is incident to an extendable edge, so the chambers of G partition $V(G)$. Perfect matchings in G are formed by independently choosing perfect matchings in the chambers of G.

Lemma 3.2. *If a matchable graph G has chambers $H_1, \ldots, H_k$, then $\Phi(G) = \prod_{i=1}^{k} \Phi(H_i)$.*

The chambers form the outermost decomposition in Lovász's structure (see Fig. 3). When the extendable subgraph is connected, there is only one chamber and no further breakdown.

Definition 3.3. A graph is *elementary* if it is matchable and its extendable subgraph is connected.

Tutte [18] characterized the matchable graphs. An *odd component* of a graph H is a component having an odd number of vertices; $o(H)$ denotes the number of odd components. An obvious necessary condition for existence of a perfect matching in G is that $o(G-S) \leq |S|$ for all $S \subseteq V(G)$. Tutte's 1-Factor Theorem states that this condition is also sufficient.

Definition 3.4. A *barrier* in a matchable graph G is a set $X \subseteq V(G)$ with $o(G-X) = |X|$.

Lemma 3.5 (Lemma 5.2.1 [11])**.** *If G is elementary, then the family of maximal barriers in G is a partition of $V(G)$, denoted $\mathcal{P}(G)$.*

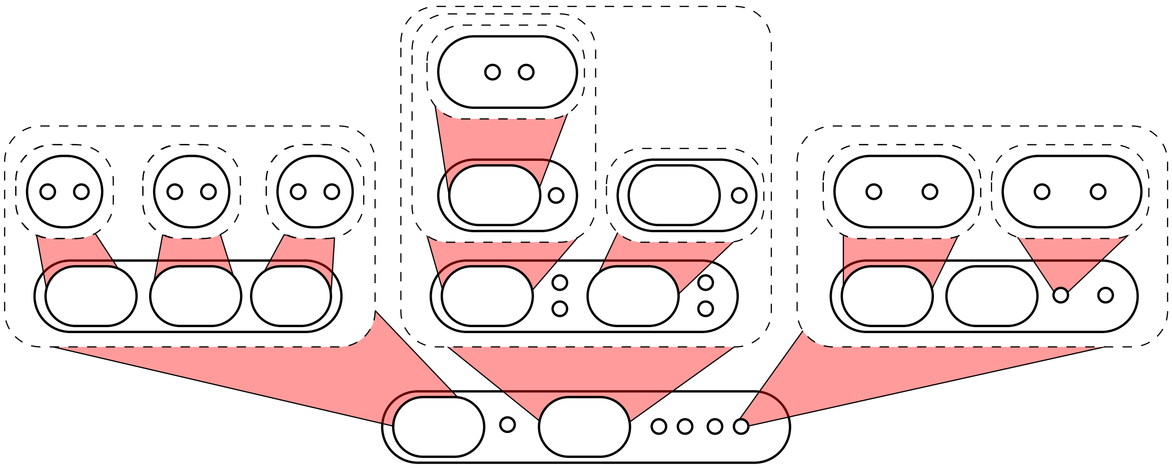


Figure 3: An example cathedral construction.

Construction 3.6 (The Cathedral Construction)**.** A graph G is a *cathedral* if it consists of (1) a saturated elementary graph G_0, (2) disjoint cathedrals $G_1, \ldots, G_t$ corresponding to the maximal barriers $X_1, \ldots, X_t$ of G_0, and (3) edges joining every vertex of X_i to every vertex of G_i, for $1 \le i \le t$. The graph G_0 is the *foundation* of the cathedral. The cathedral G_i may have no vertices when $i > 0$; thus every saturated elementary graph is a cathedral (with empty cathedrals over its barriers).

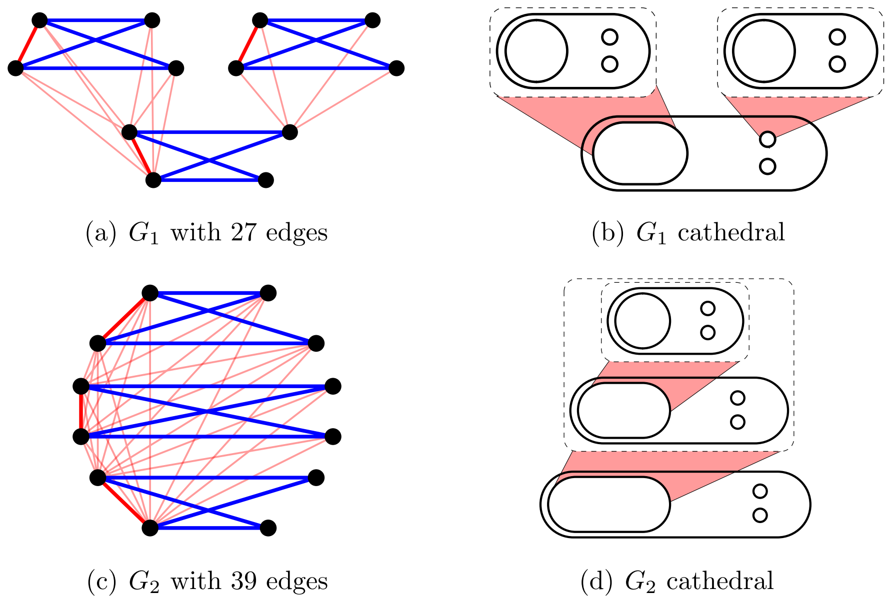
Since the cathedral construction has a cathedral "above" each maximal barrier of G_0, the construction is recursive, built from saturated elementary graphs. Each nonempty subcathedral G_i contains a saturated elementary graph $G_{i,0}$, and each maximal barrier $X_{i,j} \in \mathcal{P}(G_{i,0})$ has a cathedral $G_{i,j}$ over it in G_i. Figure 3 illustrates the cathedral construction. Here cathedrals are indicated by dashed curves (except for the full cathedral). Each foundation is indicated by a solid curve, as are the barriers within it.

Theorem 3.7 (The Cathedral Theorem; Theorem 5.3.8 [11])**.** *A graph G is saturated if and only if it is a cathedral. The foundation G_0 in the cathedral construction of G is unique, and every perfect matching in G contains a perfect matching of G_0.*

Since each perfect matching in a cathedral G contains a perfect matching of G_0, the edges joining G_0 to the cathedrals $G_1, \ldots, G_t$ appear in no perfect matching. Therefore, G_0 is a chamber in G. Recursively, the foundations of the subcathedrals are the chambers of G.

The saturated graphs of Figure 1 are cathedrals having the same chambers (and hence the same number of perfect matchings). Their cathedral structures are shown in Figure 4.

Let $G \in \mathcal{F}_p$ be a p-extremal graph. Since G is extremal, it is saturated, and hence it is a cathedral. Recall that the Hetyei-extension of G is $(G + K_1) \vee K_1$. The complete graph K_2



(a) G_1 with 27 edges (b) G_1 cathedral

(c) G_2 with 39 edges (d) G_2 cathedral

Figure 4: The saturated graphs from Figure 1 and their cathedral structures.

is a saturated elementary graph; its barriers are single vertices, say $\{x_1\}$ and $\{x_2\}$. Letting $G_1 = G$ and $G_2 = \varnothing$, we obtain the Hetyei-extension of G as a cathedral with $G_0 = K_2$.

4 Extremal Graphs are Spires

From the cathedral structures of the two graphs in Figure 4, it is easy to see why G_2 has many more edges. Nesting of cathedrals generates many edges from foundations to the cathedrals over them. We introduce a special term for cathedrals formed in this way.

Definition 4.1. A *spire* is a cathedral in which at most one maximal barrier in the foundation has a nonempty cathedral over it, and that nonempty cathedral (if it exists) is a spire. In particular, every saturated elementary graph is a spire.

In Figure 4, the graph G_2 is a spire, while G_1 is not. By the recursive definition, the chambers of a spire G form a list $(H_0, \ldots, H_k)$ such that each H_i is the foundation of the spire induced by $\bigcup_{j=i}^{k} V(H_j)$, and in H_i with $i < k$ there is a maximal barrier Y_i that is adjacent to the vertices of the spire induced by $\bigcup_{j=i+1}^{k} V(H_i)$. We then say that G is a spire *generated by* $H_0, \ldots, H_k$ over $Y_0, \ldots, Y_k$.

Our first goal is to prove that extremal graphs are spires.

Lemma 4.2. *Every p-extremal graph is a spire such that in each chamber, the maximal barrier having neighbors in later chambers is a barrier of maximum size.*

Proof. Since a p-extremal graph is saturated, it is a cathedral. Let G be a cathedral having nonempty cathedrals G_i and G_j over maximal barriers X_i and X_j in its foundation, with $|X_i| \geq |X_j|$. Let G' be the cathedral obtained from G by removing G_j from the neighborhood of X_j and attaching it instead as a cathedral over a barrier in an innermost chamber of G_i. The cathedrals over the barriers of innermost chambers are empty, so G' is a cathedral.

The chambers of G and G' are isomorphic, so $\Phi(G) = \Phi(G')$, but G' has more edges. We replaced $|X_j| \cdot |V(G_j)|$ edges with $|X_i| \cdot |V(G_j)|$ edges, and also new edges were created incident to an innermost chamber over X_i. We conclude that in a p-extremal graph, only one maximal barrier of the foundation has a nonempty chamber over it. Also, that must be a largest barrier, since otherwise shifting to a larger one increases the number of edges, again without changing the number of perfect matchings. The claim follows by induction. ∎

The number of edges in a spire is maximized by ordering the chambers greedily.

Lemma 4.3. *Let $\{H_0, \ldots, H_k\}$ be saturated elementary graphs. Let $n_i = |V(H_i)|$, and let s_i be the maximum size of a barrier in H_i. Among the spires having $H_0, \ldots, H_k$ as chambers, the number of edges is maximized by indexing the chambers so that $\frac{s_0}{n_0} \geq \cdots \geq \frac{s_k}{n_k}$.*

Proof. For a spire G generated by $H_0, \ldots, H_k$ indexed in a different order, let i be an index such that $\frac{s_i}{n_i} < \frac{s_{i+1}}{n_{i+1}}$. Form a spire G' from G by interchanging H_i and H_{i+1} in the ordering (always the spire after H_j is built over a largest barrier Y_j of H_j).

In G' and G, the edges from $Y_i \cup Y_{i+1}$ to other chambers are the same. Only the edges joining $V(H_i)$ and $V(H_{i+1})$ change. In G, there are $s_i n_{i+1}$ such edges, and in G' there are $s_{i+1} n_i$ of them. By the choice of i, the change increases the number of edges. The number of perfect matchings remains unchanged.

Hence a p-extremal spire has its chambers ordered as claimed. ∎

Note that always $\frac{s_i}{n_i} \leq \frac{1}{2}$ for a chamber H_i in a spire G, since $o(H_i - X_i) = |X_i|$ for any barrier X_i in H_i. We show next that the excess $c(G)$ is subadditive over the chambers.

Lemma 4.4. *If G is a spire generated by $H_0, \ldots, H_k$ over $Y_0, \ldots, Y_k$, with $s_i = |Y_i|$ and $n_i = |V(H_i)|$, then $c(G) \leq \sum_{i=0}^{k} c(H_i)$, with equality if and only if $\frac{s_0}{n_0} = \cdots = \frac{s_{k-1}}{n_{k-1}} = \frac{1}{2}$.*

Proof. Let $m_i = |E(H_i)|$. Counting edges within chambers and from chambers to barriers in earlier chambers, we have $|E(G)| = \sum_{i=0}^{k} m_i + \sum_{0 \leq i < j \leq k} s_i n_j$. Always $s_i \leq \frac{1}{2} n_i$ and $m_i = \frac{1}{4} n_i^2 + c(H_i)$. Thus

$$\frac{n^2}{4} + c(H) \leq \sum_{i=1}^{k} \left[\frac{n_i^2}{4} + c(H_i) \right] + \sum_{0 \leq i < j \leq k} \frac{1}{2} n_i n_j = \frac{1}{4} \left[\sum_{i=0}^{k} n_i \right]^2 + \sum_{i=0}^{k} c_i.$$

Therefore, $c(G) \leq \sum_{i=0}^{k} c(H_i)$, with equality if and only if $\frac{s_i}{n_i} = \frac{1}{2}$ for $i < k$. ∎

5 Extremal Chambers

We now know how to combine chambers in the best way, so it remains to determine which chambers should be used. A chamber is a saturated elementary graph, meaning that its extendable subgraph has just one component. We will bound the size of a saturated elementary graph with n vertices by bounding separately the *extendable* edges (those in perfect matchings) and the *free* edges (those in no perfect matching).

When G is elementary, the maximal barriers partition $V(G)$. Since each barrier matches to vertices outside it in any perfect matching, all edges within barriers are free. Also, adding such edges does not increase the number of perfect matchings. Thus in a saturated graph, the barriers are cliques. To bound the number of free edges, the crucial fact is that in a saturated elementary graph, the only free edges are those within barriers (proved in Lemma 5.2.2.b of Lovász and Plummer [11]).

Lemma 5.1. *If G is a saturated elementary n-vertex graph with ℓ maximal barriers, then G has at most $q\binom{\ell-1}{2} + \binom{r+1}{2}$ free edges, where $q = \left\lfloor \frac{n-\ell}{\ell-2} \right\rfloor$ and $r = n - \ell - q(\ell - 2)$.*

Proof. Let $x_1, \ldots, x_\ell$ be the sizes of the barriers, so $\sum_{i=1}^{\ell} x_i = n$. Since each barrier is a clique, there are exactly $\sum_{i=1}^{\ell} \binom{x_i}{2}$ free edges. The sizes of the barriers are further restricted because deleting a barrier of size x_i must leave x_i odd components. Since the other barriers are cliques, deleting a barrier leaves at most $\ell - 1$ components. Thus $1 \le x_i \le \ell - 1$ for all i.

If $a \le b$, then $\binom{a-1}{2} + \binom{b+1}{2} > \binom{a}{2} + \binom{b}{2}$ (shifting a vertex from an a-clique to a b-clique increases the number of edges). Subject to the constraints we have specified, the number of free edges is thus bounded by greedily choosing as many of $x_1, \ldots, x_\ell$ to equal $\ell - 1$ as possible, given that at least one unit must remain for each remaining variable. Let q be the number of values equal to $\ell - 1$. Among the remaining values, whose total is less than $\ell - 1$, all values should be 1 except for one. After allocating 1 to each of these $\ell - q$ values, a total of r remains, where $0 \le r < \ell - 2$. Thus $n = q(\ell - 1) + (\ell - q) + r$, which we write as $n - \ell = q(\ell - 2) + r$.

The specified choice of q and r satisfies all the conditions, and the bound on the number of free edges is then as claimed. ∎

We show next that the bound in Lemma 5.1 is maximized when all barriers except one are singletons, producing $\ell = 1 + n/2$.

Corollary 5.2. *A saturated elementary n-vertex graph has at most $\frac{n^2}{8} - \frac{n}{4}$ free edges.*

Proof. The proof of Lemma 5.1 describes how to maximize $\sum_{i=1}^{\ell} \binom{x_i}{2}$ subject to $1 \le x_i \le \ell - 1$. Since barriers in saturated graphs are cliques, the number of odd components left by deleting a barrier is at most the number of other barriers, but it must equal the size of the barrier deleted. Hence each barrier has size at most $n/2$, which yields $\ell \le n/2+1$.

Thus $2 \le \ell \le n/2+1$. Since $0 \le r < \ell - 2$, we have $\binom{r+1}{2} \le r(\ell-1)/2$ (with equality only when $r=0$). Hence

$$q\binom{\ell-1}{2} + \binom{r+1}{2} \le \frac{q(\ell-1)(\ell-2)+r(\ell-1)}{2} = \frac{(\ell-1)(n-\ell)}{2}.$$

The upper bound is maximized at $(\ell-1)=(n-1)/2$, among integers when $\ell \in \{n/2, n/2+1\}$. The value there is $\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}\right)$, which is the claimed bound. ∎

Next consider the extendable edges. Deleting the edges within barriers yields a graph in which every edge is extendable. Such graphs are called *1-extendable*, which motivates our name for extendable edges (the term *matching-covered* has also been used for 1-extendable graphs). Since the extendable edges form a 1-extendable graph, we seek a bound on the size of 1-extendable graphs with n vertices. All such graphs are 2-connected, and 2-connected graphs are precisely those constructed by ear decompositions. The 1-extendable graphs have special ear decompositions that yield a bound on the number of edges, described by the "Two Ears Theorem" of Lovász.

Definition 5.3. Let G be a 1-extendable graph. A *graded ear decomposition* of G is a list $G_0, \ldots, G_k$ of 1-extendable graphs such that $G_k = G$, each $G - V(G_i)$ is matchable, and each G_i for $i > 1$ is obtained from G_{i-1} by adding disjoint ears of odd length. A graded ear decomposition of G is *non-refinable* if no other graded ear decomposition of G contains it.

Theorem 5.4 (Two Ears Theorem; Lovász and Plummer [10]; see also Section 5.4 of [11])**.** *Every 1-extendable graph has a non-refinable graded ear decomposition in which each subgraph arises by adding at most two ears to the previous one (starting with any single edge).*

For example, such a decomposition of K_4 starts with any edge, adds one ear to complete a 4-cycle, and then adds both remaining edges as ears. Both ears must be added in the last step, because adding just one of them does not produce a 1-extendable graph.

Lovász and Plummer [11, page 178] remark that long graded ear decompositions are desirable, because $\Phi(G) \ge k+1$ when G has a graded ear decomposition $G_0, \ldots, G_k$. We explain and use this fact in our next lemma.

Lemma 5.5. *For $p \ge 2$, a 1-extendable graph G with $\Phi(G) = p$ has at most $2p-4+n$ edges.*

Proof. Let $G_0, \ldots, G_k$ be an ear decomposition as guaranteed by Theorem 5.4. Since the decomposition is non-refinable, G_1 is an even cycle, so $\Phi(G_1) = 2$.

Let $m = |E(G)|$. The number of edges added at each step after G_1 is at most two more than the number of vertices added. Hence $m \le n + 2(k-1)$. It suffices to show that $k - 1 \le p - 2$. To do this, it suffices to prove that $\Phi(G_i) > \Phi(G_{i-1})$ for $i \ge 2$.

Every added ear in a graded ear decomposition has odd length and hence an even number of internal vertices. These can be matched along the ear. Since G_i arises from G_{i-1} by adding one ear of odd length or two *disjoint* ears of odd length, every perfect matching in G_{i-1} extends to a perfect matching in G_i. In addition, since G_i is also required to be 1-extendable, it has a perfect matching using an initial edge of an added ear; such a matching is not counted by $\Phi(G_{i-1})$. ∎

Theorem 5.6. *For $p \ge 2$, an elementary graph with n vertices and exactly p perfect matchings has at most $\frac{n^2}{8} + \frac{3n}{4} + 2p - 4$ edges.*

Proof. Add the maximum number of extendable edges from Lemma 5.5 to the maximum number of free edges from Corollary 5.2. ∎

Since the coefficient on the quadratic term in this edge bound is $\frac{1}{8}$, while the leading coefficient for p-extremal graphs will be $\frac{1}{4}$, large extremal graphs will not be elementary. This enables us to limit the search for extremal elementary graphs.

Corollary 5.7. *Fix $p \ge 2$. If G is an elementary graph with n vertices, p perfect matchings, and $\frac{n^2}{4} + c_p$ edges, then $n^2 - 6n - 16p + 8c_p + 32 \le 0$. Thus $n \le 3 + \sqrt{16p - 8c_p - 23}$.*

Recall that $n_p = \min\{n\colon\ f(n,p) = \frac{n^2}{4} + c_p\}$. We can bound this threshold using the fact that all the chambers in a spire are elementary graphs.

Corollary 5.8. *For $p \ge 2$, let N_p be the largest even number bounded by $3+\sqrt{16p - 8c_p - 23}$. Every elementary graph in $\mathcal{F}_p$ has at most N_p vertices, and $n_p \le \max\left\{\sum_{i=0}^{k} N_{p_i}\colon\ \prod_i p_i = p\right\}$.*

Proof. By Corollary 5.7, all elementary graphs with n vertices and $\frac{n^2}{4} + c_p$ edges have at most $3 + \sqrt{16p - 8c_p - 23}$ vertices, and the number of vertices must be even.

Let $G \in \mathcal{F}_p$ be a spire generated by $H_0, \ldots, H_k$. Set $p_i = \Phi(H_i)$. We have observed that $p = \prod_{i=0}^{k} p_i$. Since each H_i is elementary, it has at most N_{p_0} vertices, so G has at most $\sum_{i=1}^{k} N_{p_i}$ vertices. Taking the maximum over all factorizations bounds n_p. ∎

The lower bound $c_p \ge -(p-1)(p-2)$ given by Dudek and Schmitt [5] implies $N_p \in O(p)$. The construction in Theorem 2.3 shows that c_p is nonnegative. Together with Corollary 5.8, this yields $N_p \in O(\sqrt{p})$. With N_q known for $q < p$, this reduces the determination of the exact value of c_p for a given p to a search over a finite set of graphs.

We close this section by summarizing the results of this and the previous section. The outcome is a systematic approach to classifying all graphs in $\mathcal{F}_p$.

Theorem 5.9. *For an n-vertex graph G in $\mathcal{F}_p$,*

1. *G is a spire with chambers $H_0, \ldots, H_k$ built over barriers $Y_0, \ldots, Y_k$.*
2. *Each Y_i is a barrier of maximum size in H_i.*
3. *If $0 \le i < j \le k$, then $\frac{|Y_i|}{|V(H_i)|} \le \frac{|Y_j|}{|V(H_j)|}$.*
4. *Letting $p_i = \Phi(H_i)$, there are at most N_{p_i} vertices in H_i, and $c(H_i) \le c_{p_i}$.*
5. *$\Phi(G) = p = \prod_{i=0}^{k} p_i$ and $c(G) = c_p \le \sum_{i=1}^{k} c(H_i)$.*
6. *If $p_i = 1$, then $H_i \cong K_2$.*

6 Graphs with an Odd Number of Vertices

Since graphs with an odd number of vertices do not have perfect matchings, we generalize $f(n,p)$ to odd n using near-perfect matchings. In this section, n is odd.

Definition 6.1. An *near-perfect matching* in a graph is a matching that covers all but one vertex. Let $\tilde{\Phi}(G)$ denote the number of near-perfect matchings in G. Let $\tilde{f}(n,p)$ denote the maximum number of edges in an n-vertex graph with p near-perfect matchings.

The computation of $\tilde{f}(n,p)$ almost reduces to the computation of $f(n,p)$.

Theorem 6.2. *If n is odd and larger than n_p, then*

$$\tilde{f}(n,p) = f(n-1,p) = \frac{(n-1)^2}{4} + c_p.$$

Proof. Since $n > n_p$, we may choose $G \in \mathcal{F}_p$ with $n-1$ vertices. Adding an isolated vertex to G produces a graph with p near-perfect matchings and $f(n-1,p)$ edges. Thus $f(n-1,p) \le \tilde{f}(n,p)$.

Let H be an n-vertex graph having $\tilde{f}(n,p)$ edges and p near-perfect matchings. Adding a new vertex adjacent to every vertex in H produces a graph H' having p perfect matchings and $\tilde{f}(n,p)+n$ edges (there is a one-to-one correspondence between near-perfect matchings in H and perfect matchings in H').

Thus $\tilde{f}(n,p) = |E(H')| - n \le f(n+1,p) - n$. By Theorem 1.2, $n > n_p$ implies $f(n+1,p) = f(n-1,p) + n$. We conclude that $\tilde{f}(n,p) \le f(n-1,p)$, so equality holds. ∎

Not only is the numerical value of $\tilde{f}(n,p)$ determined by the even case, but also the extremal graphs correspond to extremal graphs in the even case, using the bijection in the proof of Theorem 6.2.

Definition 6.3. Let $\tilde{\mathcal{F}}_p$ be the set of graphs G having $\frac{(|V(G)|-1)^2}{4} + c_p$ edges and exactly p near-perfect matchings.

Corollary 6.4. *For each graph $H \in \tilde{\mathcal{F}}_p$, there is a graph $G \in \mathcal{F}_p$ and a vertex $u \in V(G)$ such that u is adjacent to $V(G) - \{u\}$ and $H \cong G - u$.*

Not every graph in $\mathcal{F}_p$ has a dominating vertex, so there are n-vertex graphs in $\mathcal{F}_p$ that do not arise in this simple way from $(n-1)$-vertex graphs in $\tilde{\mathcal{F}}_p$. The graph $\overline{3K_2}$ has eight perfect matchings (each of the 12 edges appears in two perfect matchings, and each perfect matching has three edges). With $n = 6$, we have $n^2/4 + 3$ edges. We will see that $c_8 = 3$, so $\overline{3K_2} \in \mathcal{F}_p$, but the graph has no dominating vertex. On the other hand, when $n > n_p$, Hetyei-extension of an n-vertex graph in $\mathcal{F}_p$ yields a graph in $\mathcal{F}_p$ with $n+2$ vertices that does have a dominating vertex.

7 Constructive Lower Bounds

In this section, we refine the binary expansion construction $B(p)$ of Theorem 2.3 to give improved lower bounds for c_p. Because the barrier is large in $B(p)$, it can be used to increase the excess while multiplying the number of perfect matchings. Recall that $w(m)$ is the number of 1s in the binary expansion of m.

Proposition 7.1. *If p_1 and p_2 are integers with $p_1, p_2 \geq 2$, then $c_{p_1 p_2} \geq c_{p_1} + w(p_2 - 1)$.*

Proof. Let G be a n-vertex graph having $\frac{n^2}{4} + c_{p_1}$ edges and exactly p_1 perfect matchings. Let $H = B(p_2)$, in which the clique is a barrier containing exactly half of the vertices. Let G' be the saturated graph formed by making G a tower above this barrier in H.

By Lemma 4.4, $c(G') = c(G) + c(H) = c_1 + w(p-1)$. By Lemma 3.2, G' has $p_1 \cdot p_2$ distinct perfect matchings. Therefore, $c_{p_1 p_2} \geq c_{p_1} + w(p_2 - 1)$. ∎

Corollary 7.2. *If p properly divides p', then $c_{p'} > c_p$.*

The binary expansion construction yields $c_p \geq \log_2 p$ when p is a power of 2. However, when $p-1$ is a power of 2, it yields only $c_p \geq 1$. To combat this deficiency, we develop further lower bounds using graphs where $|E(G)|$ and $\Phi(G)$ are easy to compute. These constructions properly contain the Hetyei graphs, so the excess is positive. Unfortunately, not every p can be realized as $\Phi(G)$ using these constructions.

Definition 7.3. A *Hetyei list* is a nondecreasing list $d_1, \ldots, d_k$ of positive integers such that $d_i \geq i$ for all i and $d_k = k$. The *nested-degree graph* generated by a Hetyei list $(d_1, \ldots, d_k)$, denoted $\mathrm{Deg}(d_1, \ldots, d_k)$, is the supergraph of the Hetyei graph of order $2k$ in which the edge $\ell_i r_j$ exists if and only if $i \leq d_j$.

Theorem 7.4. *If $G = \mathrm{Deg}(d_1, \ldots, d_k)$ for a Hetyei list $d_1, \ldots, d_k$, then G has a barrier of size k and $\Phi(G) = \prod_{i=1}^{k}(d_i + 1 - i)$,*

Proof. Since $\{r_1, \ldots, r_k\}$ is an independent set, every perfect matching pairs its vertices with $\{\ell_1, \ldots, \ell_k\}$. Also, $\{\ell_1, \ldots, \ell_k\}$ is a barrier of size k in G.

To compute $\Phi(G)$, choose edges to cover vertices in the order $r_1, \ldots, r_k$. When covering r_i, there are $i - 1$ previously matched vertices in $\{\ell_1, \ldots, \ell_k\}$. Since $\bigcup_{j=1}^{i-1} N(r_i) \subseteq N(r_i)$, there are $d_i - i + 1$ choices for the edge to cover r_i. Since $d_i \geq i$ for all i, the process completes a perfect matching in $\prod_{i=1}^{k}(d_i + 1 - i)$ ways. ∎

When a graph G has a barrier B with half its vertices, the edges in perfect matchings form a bipartite graph with partite sets B and $V(G) - B$, and $G - B$ has no edges. Ostrand [15] proved that if a bipartite graph G has a perfect matching, and $d_1, \ldots, d_k$ is the nondecreasing list of degrees of the vertices in one partite set, then $\Phi(G) \geq \prod_{i=1}^{k} \max\{1, d_i - i + 1\}$ (Hwang [9] gave a simple proof). When the list d is a Hetyei list, the corresponding nested-degree graphs achieve equality in the lower bound.

Example 7.5. If $G = \mathrm{Deg}(k, \ldots, k)$, then $\Phi(G) = k!$ and $c(G) = \binom{k}{2}$. By Stirling's approximation, $c_p \geq \Omega\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ when $p = k!$.

Definition 7.6. Let $(d_1, \ldots, d_k)$ be a Hetyei list and $\{e_1, \ldots, e_m\}$ be a set of disjoint pairs in $\{1, \ldots, k\}$. The resulting *generalized nested-degree graph*, denoted $\mathrm{Gen}\,(d_1, \ldots, d_k; e_1, \ldots, e_m)$, consists of the nested-degree graph $\mathrm{Deg}(d_1, \ldots, d_k)$ plus each edge $r_i r_j$ such that $\{i, j\} = e_t$ for some t.

The *double factorial* of an integer n, denoted $n!!$, is the product of the integers in $\{1, \ldots, n\}$ with the same parity as n. As an empty product, by convention $(-1)!!$ equals 1.

Theorem 7.7. *For a set $\{e_1, \ldots, e_m\}$ of disjoint pairs in $\{1, \ldots, k\}$, let $\mathcal{P}$ denote the family of all subsets of $\{r_i r_j \colon \{i, j\} \in \{e_1, \ldots, e_m\}\}$. If $G = \mathrm{Gen}\,(d_1, \ldots, d_k; e_1, \ldots, e_m)$, then*

$$\Phi(G) = \sum_{M \in \mathcal{P}} (2|M| - 1)!! \prod_{r_i \notin V(M)} (d_i - |\{j < i \colon r_j \notin V(M)\}|).$$

Also, if $m \geq 1$, then G has no barrier of size k.

Proof. Every perfect matching in G contains some subset M of $\{r_i r_j \colon \{i, j\} \in \{e_1, \ldots, e_m\}\}$. To complete a matching, cover the remaining vertices in $\{r_1, \ldots, r_k\}$ in increasing order of subscripts by selecting neighbors in $\{\ell_1, \ldots, \ell_k\}$. The number of ways to do this is $\prod_{r_i \notin V(M)}(d_i - |\{r_j \notin V(M) \colon j < i\}|)$, as in the proof of Theorem 7.4. Finally, the $2|M|$ remaining unmatched vertices form a clique and can be matched in $(2|M| - 1)!!$ ways. ∎

Theorem 7.4 is the special case of Theorem 7.7 for $m = 0$. When m is small, there are not many subsets of $\{e_1, \ldots, e_m\}$, and computing $\Phi(G)$ is feasible.

Example 7.8. When $m = \binom{k}{2}$ and $d_i = k$ for all i, the generalized nested-degree graph is K_{2k}, with $(2k-1)!!$ perfect matchings. Thus $c_{(2k-1)!!} \geq k^2 - k$. This yields the lower bound $c_p \geq \Omega\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ when $p = (2m-1)!!$ for some m.

Examples 7.5 and 7.8 provide our best asymptotic lower bounds but apply only for special values. The generalized nested-degree construction is our most efficient method for finding lower bounds when k and m are small In Section 9, we discuss the results of computer search over small cases of these constructions to find explicit lower bounds on c_p when p is small.

8 A Conjectured Upper Bound

Dudek and Schmitt conjectured that the complete graph K_{2t} is p-extremal for $p = (2t-1)!!$, giving $c_p = t^2 - t$. We generalize this to conjecture an upper bound for all p. First, a lemma provides motivation. In light of the proof, we call it the "Star-Removal Lemma".

Lemma 8.1. *If $p, k, t \in \mathbf{N}$ satisfy $k \leq 2t$ and $p = k(2t-1)!!$, then $c_p \geq t^2 - t + k - 1$.*

Proof. Let G be the graph obtained from K_{2t+2} by removing $2t+1-k$ edges with a common endpoint x. The vertex x has k neighbors; after choosing one, the rest of the graph is isomorphic to K_{2t}. Thus $\Phi(G) = k(2t-1)!!$. The number of edges in G is $\binom{2t+2}{2} - (2t+1-k)$, which equals $\frac{(2t+2)^2}{4} + t^2 - t + k - 1$. Hence $c_p \geq t^2 - t + k - 1$. ∎

To reduce the number of perfect matchings from $(2t+1)!!$ to $k(2t-1)!!$, only $2t-1-k$ edges were removed; with each edge deleted, $(2t-1)!!$ perfect matchings were lost. This seems to be the most edge-efficient way to remove perfect matchings, which suggests a conjecture.

Conjecture 8.2. *For $p \in \mathbf{N}$, if integers k and t are defined uniquely by $k(2t-1)!! \leq p < (k+1)(2t-1)!!$ with $k \leq 2t$, then $c_p \leq C_p$, where $C_p = t^2 - t + k - 1$.*

The conjecture matches the lower bound in Lemma 8.1 when $p = k(2t-1)!!$. It also matches the value of c_p for $p \leq 6$ as computed in [5]. In Section 9, we verify that C_p also equals c_p for $7 \leq p \leq 10$, and we give empirical evidence that the bound holds for all p.

9 Exact Values for Small p

To confirm the values of c_p for $p \leq 6$, we used McKay's `geng` program [12, 13] to generate all graphs on 10 vertices. We checked that none of these graphs have exactly p perfect matchings while achieving larger excess. This yields a proof, since $N_p \leq 10$ for $p \leq 6$ and the smallest graph in $\mathcal{F}_p$ has at most N_p vertices.

For $p \leq 10$, we have $N_p \leq 12$. Generating all graphs on 12 vertices presently is infeasible for us; instead, we use the following lemma.

Lemma 9.1 (Dudek–Schmitt [5, Lemma 2.4])**.** *If $p \geq 2$, then $c_p \leq 1 + \max\{c_q\colon\ q < p\}$,*

If we know all previous values of c_p, and we construct an n-vertex graph G with $\Phi(G) = p$ and $|E(G)| = \frac{n^2}{4} + C$, where $C = \max\{c_q\colon\ q < p\}$, then we only need to check graphs with $\frac{n^2}{4} + C + 1$ edges to see whether one has exactly p perfect matchings. Thus our proof of the next theorem is by computer search. It yields the values in Table 1.

p	1	2	3	4	5	6	7	8	9	10
c_p	0	1	2	2	2	3	3	3	4	4
n_p	2	4	4	6	6	6	6	6	6	6
N_p		4	6	8	8	10	10	12	12	12
	[5]						Theorem 9.2			

Table 1: Excess c_p (at n_p), bound N_p on extremal chambers.

Theorem 9.2. *$c_7 = 3$, $c_8 = 3$, $c_9 = 4$, and $c_{10} = 4$.*

Proof. Explicit constructions in Fig. 5 give the lower bounds; we will subsequently describe how these constructions arise.

For the first two upper bounds, Lemma 9.1 yields (a) $c_7 \leq 4$ and (b) if $c_7 = 3$, then $c_8 \leq 4$. Thus to show $c_7 = c_8 = 3$ it suffices to examine graphs with 12 vertices and $\frac{12^2}{4} + 4$ edges. Using `geng`, we generated these and found none with exactly seven or eight perfect matchings, so $c_7 = c_8 = 3$.

By Lemma 9.1, $c_9 \leq 4$, and then similarly $c_{10} \leq 5$. To test equality, it suffices to study graphs with 12 vertices and $\frac{12^2}{4} + 5$ edges. Using `geng`, we enumerated these and found no graph with exactly ten perfect matchings, so $c_{10} \leq 4$. ∎

For small p, the chambers in the p-extremal graphs are instances of the general constructions we have provided in earlier sections. Below we characterize all p-extremal graphs for $p \leq 10$. Fig. 5 shows the smallest instances of the classes of graphs in these characterizations. The edge-colorings indicate the decomposition into chambers. Blue edges are extendable; when the subgraph of blue edges is connected, the graph is elementary. Red edges indicate the maximal barriers in chambers. Faint edges join these barriers to the spires over them when the graph is not elementary, in which case the factorization of p should be apparent; recall that $\Phi(G)$ is the product of $\Phi(H_i)$ when the chambers of G are $H_0, \ldots, H_k$.

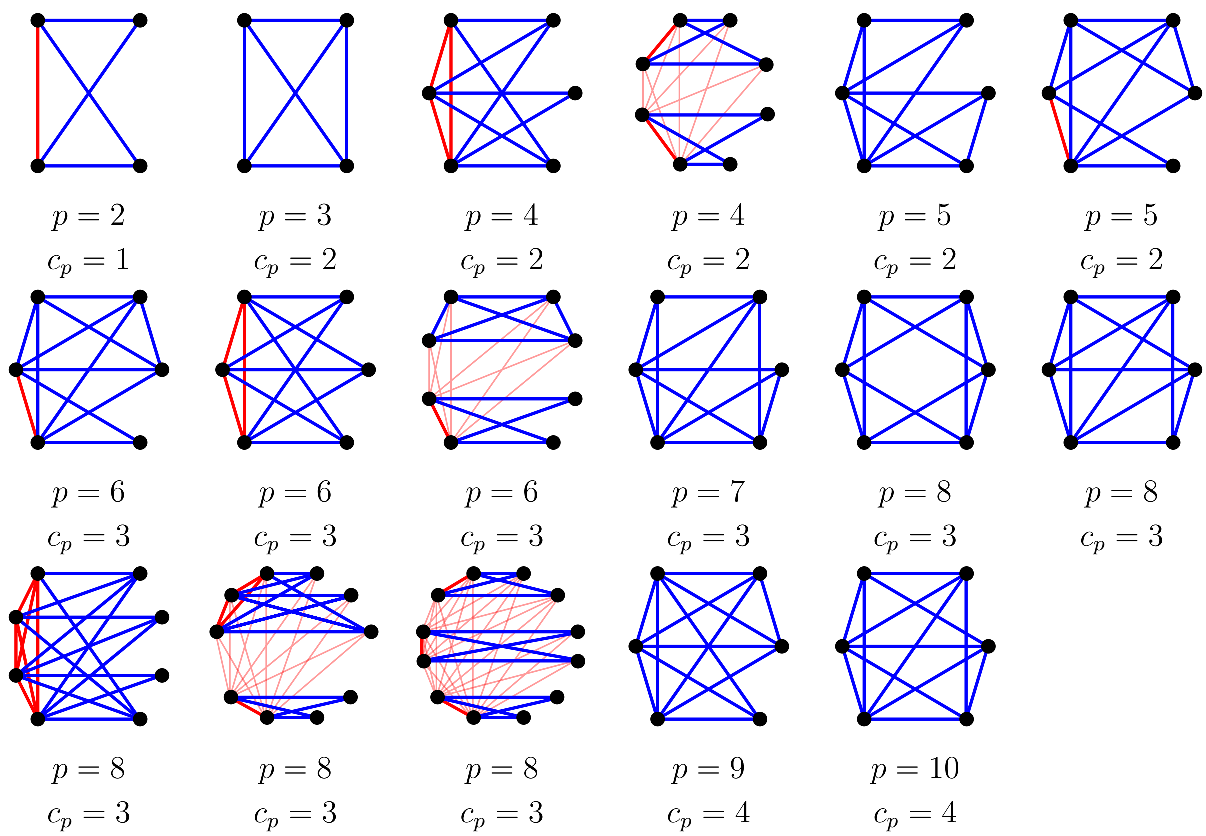


Figure 5: The smallest p-extremal configurations, for $2 \le p \le 10$

Hetyei characterized the 1-extremal graphs. Dudek and Schmitt determined c_p for $p \le 6$ but provided proof only for $p \le 4$ and characterized the extremal graphs only for $p \le 3$. We restate these characterizations in the language of the Cathedral Theorem. The "top" of a spire is the chamber last in the list of chambers describing the cathedral decomposition; it is at the other end from the foundation. When G is edge-transitive, G^- denotes the graph obtained by deleting one edge from G.

Theorem 9.3 (Hetyei)**.** *For even n with $n \ge 2$, the unique 1-extremal graph has $\frac{n^2}{4}$ edges and is a spire whose chambers all equal K_2.*

Theorem 9.4 ([5])**.** *For even n with $n \ge 4$, the 2-extremal graphs have $\frac{n^2}{4} + 1$ edges and are spires whose chambers are $(n-4)/2$ copies of K_2 and one copy of K_4^-, taken in any order.*

Theorem 9.5 ([5])**.** *For even n with $n \ge 4$, the unique 3-extremal graph has $\frac{n^2}{4} + 2$ edges and is a spire whose chambers are $(n-4)/2$ copies of K_2 and one copy of K_4 at the top.*

Note that K_2 and K_4^- have a barrier containing half of the vertices, while K_4 does not; hence the chambers for $p = 2$ appear in any order, while K_4 is at the top when $p = 3$.

The characterizations of the p-extremal graphs for $4 \le p \le 10$ all use the same method and involve the computer search used to prove Theorem 9.2. Instead of repeating the observations for each proof, we outline them here and just state the resulting characterizations.

Outline of Characterization Proofs. A p-extremal graph G is a spire of chambers $H_0, \ldots, H_k$ (Lemma 4.2), and $c(G) \le \sum_i c(H_i)$ (Lemma 4.4). The number of perfect matchings in G equals $\prod_i \Phi(H_i)$ (Lemma 3.2). Hence to know the p-extremal graphs it suffices to know the p_j-extremal chambers for all p_j that are factors of p and compare the numbers of edges in the spires corresponding to factorizations of p.

The chambers in spires are elementary graphs. Every p-extremal elementary graph has at most N_p vertices, where N_p is the largest even number bounded by $3 + \sqrt{16p - 8c_p - 23}$ (Corollary 5.8). A p-extremal elementary graph with fewer than N_p vertices extends to a p-extremal graph with N_p vertices by Hetyei-extension (repeatedly adding K_2 as a chamber at the beginning of the spire), so the p-extremal chambers are found within the graphs on N_p vertices. The q-extremal chambers for $q < p$ are already known from previous searches.

When searching graphs with N_p vertices for p-extremal chambers, we limit the search to specific numbers of edges. A p-extremal graph with N_p vertices has $\frac{1}{4}N_p^2 + c_p$ edges. By Lemma 9.1, $c_p \le C + 1$, where $C = \max_{q<p} c_q$. Hence we begin by searching graphs with N_p vertices and excess $C + 1$, looking for those having exactly p perfect matchings. The search moves to excess C if none are found with excess $C + 1$. In the results for $p \le 10$, graphs with N_p vertices and p perfect matchings were always found having excess C or $C + 1$, so there was no need to search further.

At this point the q-extremal chambers are known for all factors q of p, and hence the complete description of p-extremal graphs can be given. The chambers in a p-extremal spire are q_i-extremal elementary graphs, where $\prod q_i = p$. However, a spire with q_i-extremal chambers may have too few edges to be $\prod q_i$-extremal (for example, the spire with chambers K_4 and K_4 has nine perfect matchings but is not 9-extremal).

The order of chambers in a spire does not affect the number of perfect matchings, but it does affect the number of edges. To have the most edges, the chambers must be listed in decreasing order of the fractions of their vertices occupied by their largest barrier (Lemma 4.3). Spires for which these fractions are equal (such as K_2 and K_4^- having barriers with half their vertices) may be listed in any order. ∎

See Fig. 5 for the smallest instances of the classes of graphs in these characterizations.

Theorem 9.6. *For even n with $n \ge 6$, the 4-extremal graphs have $\frac{n^2}{4} + 2$ edges and are spires whose chambers are*

a) $\frac{n-6}{2}$ copies of K_2 and one copy of $B(4)$ in any order, or

b) $\frac{n-8}{2}$ copies of K_2 and two copies of K_4^- in any order.

Theorem 9.7. *For even n with $n \geq 6$, the* 5*-extremal graphs have* $\frac{n^2}{4} + 2$ *edges and are spires whose chambers are* $\frac{n-6}{2}$ *copies of* K_2 *plus one* 6*-vertex graph at the top that is* $\mathrm{Gen}\,(2, 2, 3; \{1, 2\})$ *or* $\mathrm{Gen}\,(2, 3, 3; \{2, 3\})) - \ell_2 r_2$.

Theorem 9.8. *For even n with $n \geq 6$, the* 6*-extremal graphs have* $\frac{n^2}{4} + 3$ *edges and are spires whose chambers are*
a) $\frac{n-6}{2}$ *copies of* K_2 *and one copy of* $\mathrm{Gen}\,(2, 3, 3; \{2, 3\})$ *at the top,*
b) $\frac{n-6}{2}$ *copies of* K_2 *and one copy of* $\mathrm{Deg}(3, 3, 3)$ *in any order, or*
c) $\frac{n-8}{2}$ *copies of* K_2 *and one copy of* K_4^- *in any order, plus one copy of* K_4 *at the top.*

Theorem 9.9. *For even n with $n \geq 6$, the unique* 7*-extremal graph has* $\frac{n^2}{4} + 3$ *edges and is a spire whose chambers are* $\frac{n-6}{2}$ *copies of* K_2 *and one* $\mathrm{Gen}\,(2, 2, 3; \{1, 2\}, \{1, 3\})$ *at the top.*

Theorem 9.10. *For even n with $n \geq 6$, the* 8*-extremal graphs have* $\frac{n^2}{4} + 3$ *edges and are spires whose chambers are a)* $\frac{n-6}{2}$ *copies of* K_2*, plus one copy of* $\overline{3K_2}$ *at the top,*
b) $\frac{n-6}{2}$ *copies of* K_2*, plus one copy of* $\mathrm{Gen}\,(1, 3, 3; \{1, 2\}, \{1, 3\}, \{2, 3\})$ *at the top,*
c) $\frac{n-8}{2}$ *copies of* K_2 *and one copy of* $B(8)$ *in any order,*
d) $\frac{n-10}{2}$ *copies of* K_2*, one copy of* K_4^-*, and one copy of* $B(8)$ *in any order, or*
e) $\frac{n-12}{2}$ *copies of* K_2 *and three copies of* K_4^- *in any order.*

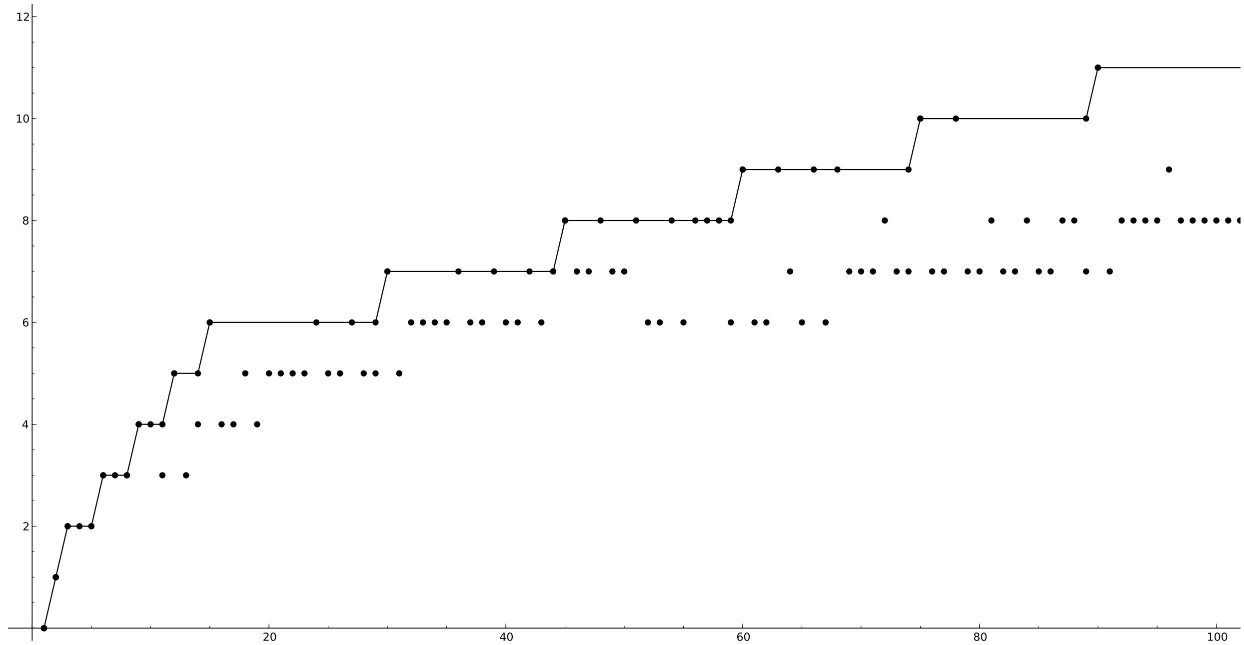
Theorem 9.11. *For even n with $n \geq 6$, the unique* 9*-extremal graph has* $\frac{n^2}{4} + 4$ *edges and is a spire whose chambers are* $\frac{n-6}{2}$ *copies of* K_2 *and one* $\mathrm{Gen}\,(3, 3, 3; \{2, 3\})$ *at the top.*

Theorem 9.12. *For even n with $n \geq 6$, the unique* 10*-extremal graph has* $\frac{n^2}{4} + 4$ *edges and is a spire whose chambers are* $\frac{n-6}{2}$ *copies of* K_2 *and one* $\mathrm{Gen}\,(2, 3, 3; \{1, 2\}, \{1, 3\})$ *at the top.*

Moving beyond $p = 11$, note that $N_{11} = 14$. Unfortunately, the number of graphs with 14 vertices and suitable number of edges is beyond the capacity of our computer resources to determine c_{11} by this method.

In Figure 6, we present the lower bounds on c_p found by searching all graphs of order 10 to find chambers and forming spires from these chambers and chambers arising from the generalized nested degree construction on 12, 14, and 16 vertices with $k \in \{5, 6, 7, 8\}$ and $m \in \{4, 3, 2, 1\}$. The upper line is the conjectured upper bound C_p from Conjecture 8.2, defined as $t^2 - t + k - 1$, where t and k are determined by $k(2t-1)!! \leq p < (k+1)(2t-1)!!$ with $k \leq 2t$. As the plot shows, we have found no construction that violates the upper bound, and sometimes it equals the excess of the best construction found so far.

Using the structural theorems in this paper, a subsequent paper [17] develops a method for searching all 1-extendable graphs with at most p perfect matchings and then adding edges to find dense elementary graphs. The 1-extendable graphs are built using graded ear



(a) $1 \leq p \leq 100$

Figure 6: Lower bounds on c_p and conjectured upper bound C_p.

decompositions. Since all 1-extendable graphs are explored, adding free edges generates all spires with at most p perfect matchings, thus determining c_p. This method produces the values of c_p for $p \leq 27$ and provides the elementary graphs that attain the maximum excess. The resulting values of c_p appear in Table 2. The conjectured upper bound C_p is not always sharp; more surprising is that c_p is not monotone.

p	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
N_p	14	14	14	16	16	16	16	18	18	18	18	20	20	20	20	20	22
C_p	4	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6
c_p	3	5	3	4	6	4	4	5	4	5	5	5	5	6	5	5	6
n_p	8	6	8	8	6	8	8	8	8	8	8	8	8	8	8	8	8

Table 2: New values of N_p, C_p, c_p, and n_p.

The method we used to determine the graphs in $\mathcal{F}_p$ for $p \leq 10$ is feasible only for small p. Several natural questions arise from these computational results. The data suggest that there is always at least one p-extremal graph whose subgraph of extendable edges is connected. If this is true, then it could help to guide searches, since when $n \geq n_p$ some p-extremal graph would have a chamber other than K_2 only at the top (this must happen when p is prime).

Conjecture 9.13. *For $p \in \mathbf{N}$, there exist a p-extremal graph that is an elementary graph.*

For p-extremal spires that consist of copies of K_2 and one p-extremal chamber, the value of c_p may be small. The examples we have of $c_p < c_{p-1}$ occur when p is a power of a prime, at $p \in \{11, 13, 16, 19, 25\}$. With greater variety of factorizations available, there are more ways to form spires with exactly p perfect matchings and hence more ways for c_p to be large. The data suggest the following.

Conjecture 9.14. *For $p \in \mathbf{N}$, always $c_p \geq \max\{\sum c_i\{p_i\} \colon \prod p_i = p\}$.*

A different conjecture is suggested by consider the values of c_q after c_p. Let $m_p = \min\{c_q \colon q \geq p\}$; how does this sequence behave? We know only that $m_p \geq 1$, by Theorem 2.3. However, if there are finitely many Fermat primes (primes of the form $2^k + 1$, see [8]), then $m_p \geq 2$ for sufficiently large p, by Corollary 7.2. This is too difficult a task for such a small payoff, especially since a much stronger statement seems likely.

Conjecture 9.15. $\lim_{p\to\infty} m_p = \infty$.

Figure 6 provides strong support for this conjecture. If it holds, then what is the asymptotic behavior of m_p? Is it $\Omega\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$, matching the conjectured upper bound? Or, is it smaller, such as $\Omega(\log_2 p)$, the current lower bound when p is a power of 2?

Finally, it would be interesting to characterize $\mathcal{F}_p$ or at least compute c_p for some infinite family of values of p.

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References

[1] N. Alon and S. Friedland, The maximum number of perfect matchings in graphs with a given degree sequence. *Electron. J. Combin.* 15 (2008), Paper N13, 2 pp.

[2] L. M. Brègman, Some properties of nonnegative matrices and their permanents, *Soviet Math. Dokl.* 14 (1973), 945–949.

[3] B. Cuckler and J. Kahn, Entropy bounds for perfect matchings and Hamiltonian cycles, to appear.

[4] J. Cutler and A. J. Radcliffe, An entropy proof of the Kahn–Lovász Theorem. *Electron. J. Combin.* 18(1) (2011), Paper P10, 9 pp.

[5] A. Dudek and J. Schmitt, On the size and structure of graphs with a constant number of 1-factors (2010), submitted.

[6] S. Friedland, An upper bound for the number of perfect matchings in graphs, arXiv: 0803.0864v1, 6 March 2008.

[7] D. Gross, N. Kahl, and J. T. Saccoman, Graphs with the maximum or minimum number of 1-factors. *Discrete Math.* 310 (2010), 687–691.

[8] R. K. Guy, *Unsolved problems in number theory*, Third edition. Problem Books in Mathematics, (Springer-Verlag, 2004).

[9] S.-G. Hwang, A note on system of distinct representatives. *Kyungpook Math. J.* 35 (1996), 513–516.

[10] L. Lovász and M. D. Plummer, On bicritical graphs. in *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, Colloq. Math. Soc. Janos Bolyai* 10, (North-Holland, 1975), 1051–1079.

[11] L. Lovász and M. D. Plummer, *Matching Theory. North-Holland Mathematics Studies* 121, *Annals of Discrete Mathematics* 29, (North-Holland Publishing Co.; Akadémiai Kiadó, 1986), xxvii+544 pp.

[12] B. D. McKay, Isomorph-free exhaustive generation. *J. Algorithms* 26 (1998), 306–324.

[13] B. D. McKay, nauty User's Guide (v. 2.4), Dept. Computer Science, Austral. Nat. Univ. (2006).

[14] S. Micali and V. V. Vazirani, An $O(V^{1/2}E)$ algorithm for finding maximum matching in general graphs, *Proc. 21st Symp. on Foundations of Computer Science* (1980), 17–27.

[15] P. A. Ostrand, Systems of distinct representatives, II. *J. Math. Anal. Appl.* 32 (1970), 1–4.

[16] J. Radhakrishnan, An entropy proof of Brègman's theorem. *J. Combin. Theory Ser. A* 77 (1997), 161–164.

[17] D. Stolee, Generating p-extremal graphs, preprint.

[18] W. T. Tutte, The factorization of linear graphs. *J. London Math. Soc.* 22 (1947), 107–111.

[19] L. G. Valiant, The complexity of computing the permanent. *Theoret. Comput. Sci.* 8 (1979), 189–201

[20] Q. R. Yu and G. Liu, *Graph factors and matching extensions.* (Higher Education Press, Beijing; Springer-Verlag, Berlin, 2009), xii+353 pp.