Parity and Strong Parity Edge-Colorings of Graphs

Douglas B. West*

Definition. A *parity walk* in an edge-colored graph is a walk along which each color is used an even number of times ([2]). Let $p(G)$ denote the least number of colors in a *parity edge-coloring* of G, which is a coloring having no parity path. Let $\widehat{p}(G)$ denote the least number of colors in a *strong parity edge-coloring* of G, which is a coloring having no open parity walk.

Note $\widehat{p}(G) \geq p(G) \geq \chi'(G)$, and they are monotone under taking subgraphs.

Theorem 1. *A tree T embeds in the k-dimensional hypercube Q_k if and only if $p(T) \leq k$.*

Corollary 2. *A graph G is a subgraph of Q_k if and only if G has a parity k-edge-coloring in which every cycle is a parity walk.*

Corollary 3. *For G connected, $p(G) \geq \lceil \lg |V(G)| \rceil$, with equality for paths and even cycles.*

Theorem 4. *If n is odd, then $p(C_n) = \widehat{p}(C_n) = \lceil \lg n \rceil + 1$.*

THEOREM 5 ([3])**.** $\widehat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$.

The upper bound is by *canonical coloring*: assign binary vectors to vertices and their sum to edges. The lower bound involves binary vector spaces (parity vectors of closed walks).

Conjecture 6. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ *for all n.* [Equaling $\widehat{p}(K_n)$; proved for $n \leq 16$.]

Conjecture 7. $p(K_{n,n}) = \widehat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$ *for all n. More generally, $\widehat{p}(K_{r,s}) = r \circ s$ for all r and s, where*

$$r \circ s = \min_{k \in \mathbb{N}} \left\{ 2^k \left(\left\lceil \frac{r}{2^k} \right\rceil + \left\lceil \frac{s}{2^k} \right\rceil - 1 \right) \right\}.$$

This would strengthen Yuzvinsky's Theorem [8, 7, 5] on sums of sets of binary vectors.

Conjecture 8. $p(G) = \widehat{p}(G)$ *for every bipartite graph G.*

*Departments of Mathematics, Zhejiang Normal University & University of Illinois, dwest@illinois.edu.

Question 9. *What is the maximum of* $\widehat{p}(G)$ *when* $p(G) = k$*?* [If G is obtained by merging a vertex of K_3 with an endpoint of P_8, then $\widehat{p}(G) = 5 > 4 = p(G)$. All our examples with $\widehat{p}(G) > p(G)$ have difference 1 and contain odd cycles.]

Question 10. *What are the extremal values of* $p(G)$ *among graphs (or trees) with various subsets of* $\Delta(G)$, $\mathrm{diam}(G)$, $|V(G)|$, *or* $|E(G)|$ *fixed?*

Question 11. *Which connected graphs* G *satisfy* $p(G) = \lceil \lg n(G) \rceil$*?* $\widehat{p}(G) = \lceil \lg n(G) \rceil$*?*

Question 12 (Mubayi)**.** *Does there exist a parity edge-coloring of* K_{2^k} *with* $(1 + o(1))2^k$ *colors that is "far" from the canonical coloring?*

Question 13. *For what graphs* G *and* H *does equality hold in* $\widehat{p}(G \square H) \leq \widehat{p}(G) + \widehat{p}(H)$*?*

Definition. A *nonrepetitive edge-coloring* [1] is an edge-coloring in which no pattern repeats immediately along a path. A *conflict-free coloring* [6] is an edge-coloring in which every path uses some color exactly once; it is an *edge-ranking* [4] if on every path the highest-indexed color appears once. Let $\pi'(G)$, $c(G)$, $r(G)$ be the least numbers of colors needed.

Note $r(G) \geq c(G) \geq p(G) \geq \pi'(G) \geq \chi'(G)$.

Question 14. *What is the maximum of* $c(G)$ *when* $p(G) = k$*? Over trees with* $p(G) = k$*?*

Question 15. *What is the maximum of* $p(T)$ *or* $\widehat{p}(T)$ *when* T *is an* n*-vertex tournament?*

References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, and O. Riordan, Nonrepetitive colorings of graphs. *Random Structures Algorithms* 21 (2002), 336–346.

[2] D.P. Bunde, K. Milans, D.B. West, H. Wu; Parity and strong parity edge-coloring of graphs. *Proc. 38th SE Intl. Conf. Combinatorics, Graph Theory and Computing, Congr. Numer.* 187 (2007), 193–213.

[3] D.P. Bunde, K. Milans, D.B. West, H. Wu; Optimal strong parity edge-coloring of complete graphs. *Combinatorica* 28 (2008), 625–632.

[4] A. V. Iyer, H. D. Ratliff, and G. Vijayan, On an edge ranking problem of trees and graphs. *Discrete Appl. Math.* 30 (1991), 43–52.

[5] G. Károlyi, A note on the Hopf–Stiefel function. *Europ. J. Combin.* 27 (2006), 1135–1137.

[6] J. Pach and G. Tóth, Conflict-free colorings. In *Discrete and computational geometry*, Algorithms Combin. 25 (Springer, 2003), 665–671.

[7] A. Plagne, Additive number theory sheds extra light on the Hopf–Stiefel $\circ$ function. *L'Enseignement Math.* 49 (2003) 109–116.

[8] S. Yuzvinsky, Orthogonal pairings of Euclidean spaces. *Michigan Math. J.* 28 (1981), 109–119.