Pebbling and Optimal Pebbling of Graphs

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

Joint work with
David P. Bunde, Erin W. Chambers
Daniel Cranston, Kevin Milans

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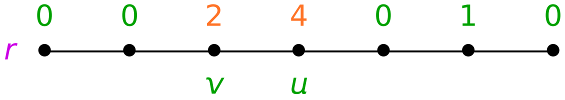

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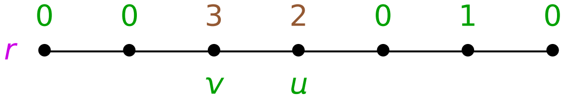

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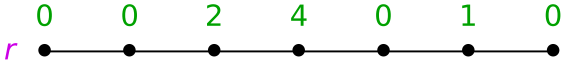

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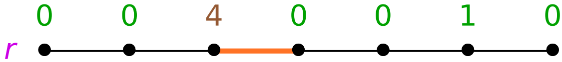

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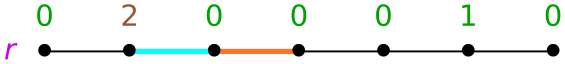

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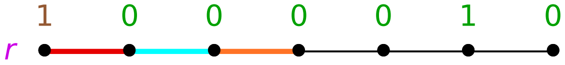

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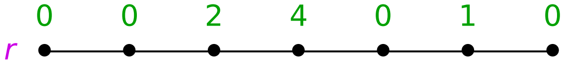

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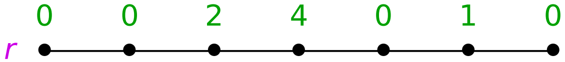

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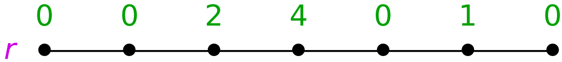

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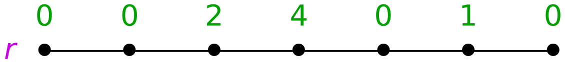

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Chung [1989]: $\Pi(G)$ for paths, trees, $\Pi(Q_k) = 2^k$.

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For $\Pi_{OPT}(G)$: short proofs, bounds for n-vertex graphs, bounds for n-vertex graphs with min-degree k, computation on prisms and ladders, etc.

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Lem. (No-Cycle Lemma; Moews [1992]) If some ordering of a multiset of moves is achievable from D, then some subset without cycles is achievable from D and leaves at least as many pebbles on every vertex. ■

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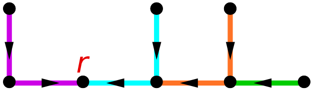

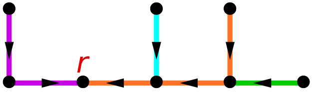

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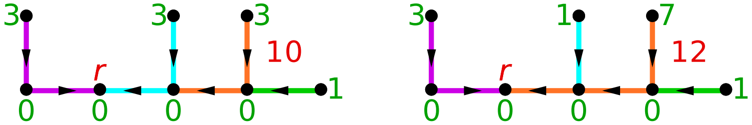

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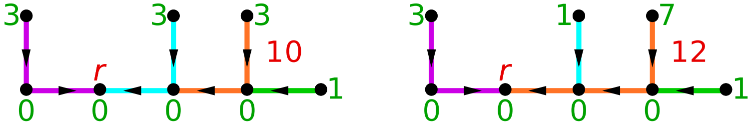

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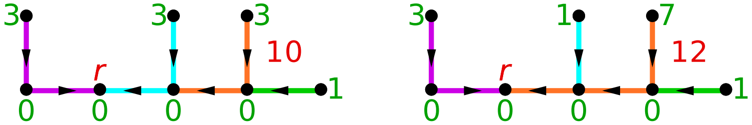

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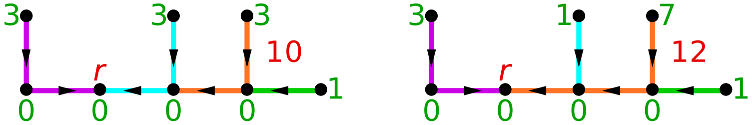

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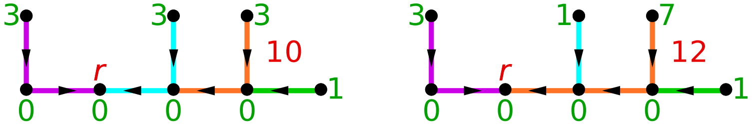

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For optimal $\mathbf{P}$, only moving to r can decrease w.
Each pebble adds ≥ 1, so $|D| > f(\mathbf{P}) \Rightarrow w(D) > f(\mathbf{P})$
$\Rightarrow \exists P_i$ with $w(P_i) > 2^{l(P_i)} \Rightarrow \exists$ move! ■

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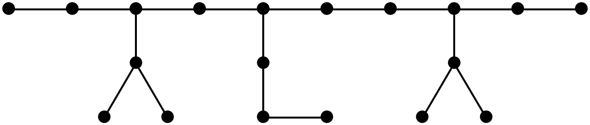

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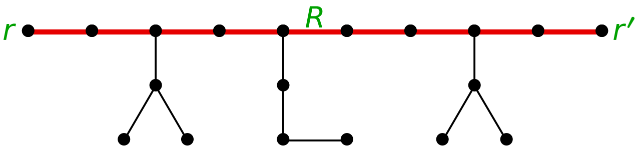

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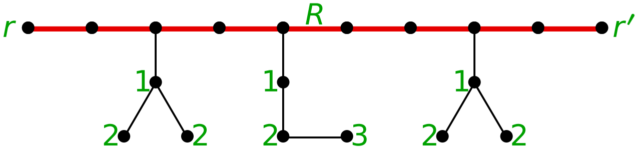

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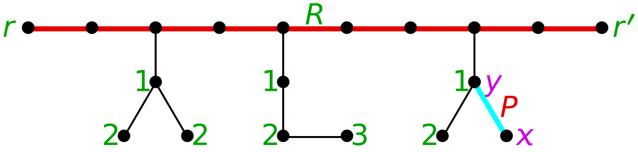

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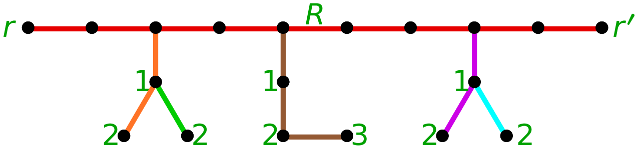

By induction on $|V(T)|$, this partition maximizes $f(\mathbf{P})$. ■

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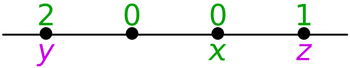

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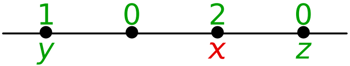

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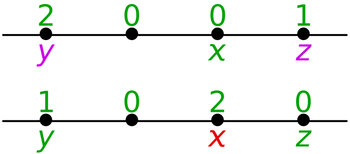

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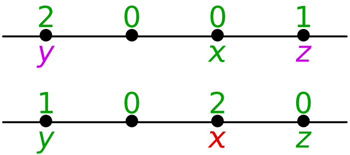

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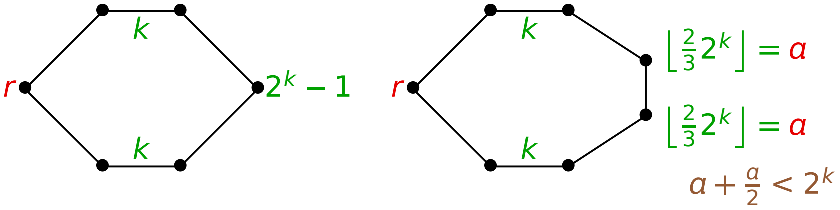

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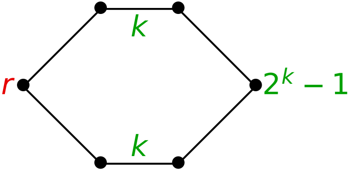

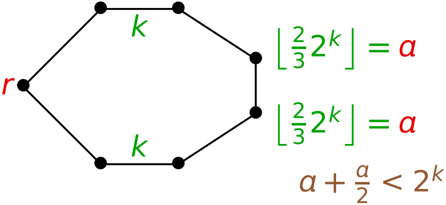

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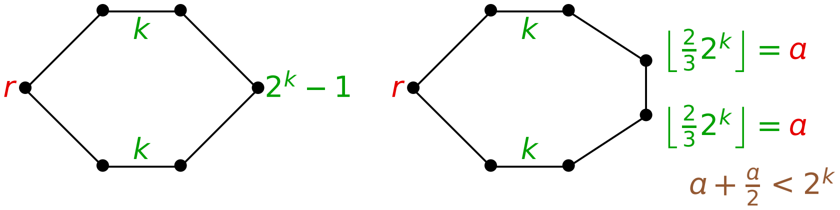

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n even: get 2^{k-1} pebbles within distance $k-1$ of r.

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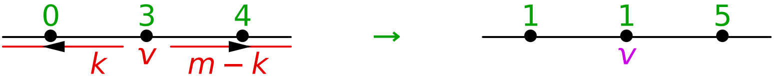

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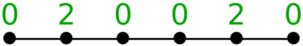

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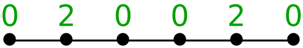

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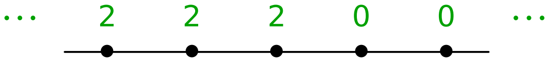

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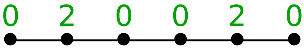

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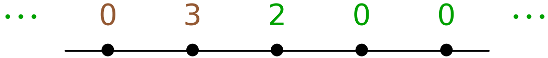

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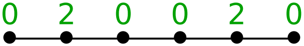

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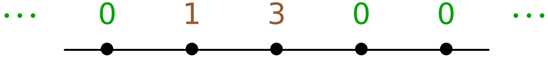

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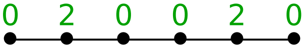

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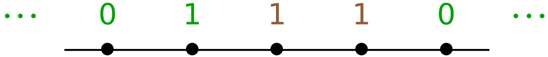

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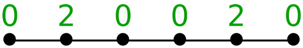

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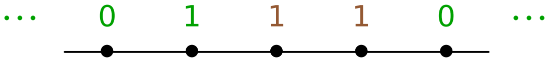

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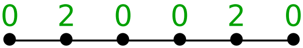

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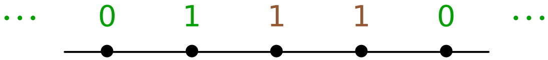

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Cor. For a connected n-vertex G, $\Pi_{OPT}(G) \le \lceil 2n/3 \rceil$.

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- Hurlbert–Munyan [2006]: Cover pebbling for Q_k is 3^k.

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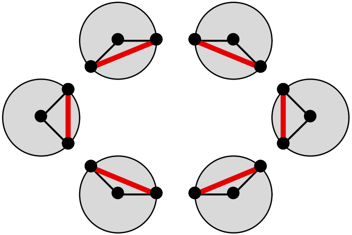

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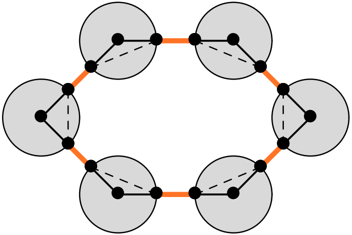

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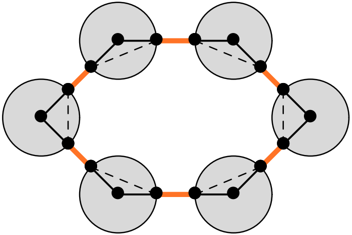

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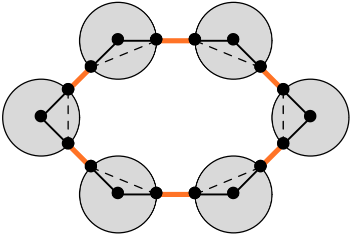

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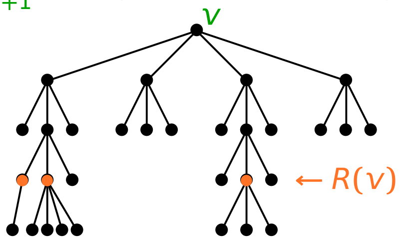
Better yet:



If $|R(v)| < 8$, put 8 on v and one on each vertex of $R(v)$.
If $|R(v)| \geq 8$, put 16 on v. ■

Special Families (Cartesian products)

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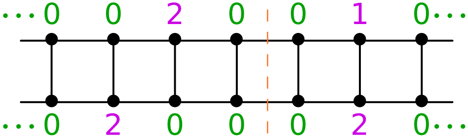

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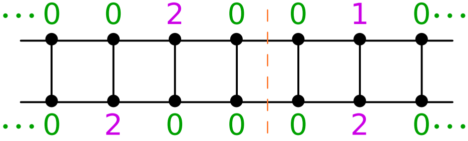

Lower bound: More collapsing and smoothing ideas, 2-fold solvability, four pages. ■

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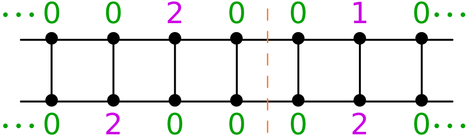

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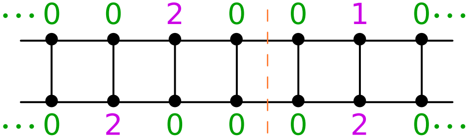

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Ques. For $G \in \mathbf{G}_{n,k}$, what is the best coefficient c_k for $\Pi_{OPT}(G) \leq c_k \frac{n}{k+1}$? It is between 2 and 4.