Pebbling and Optimal Pebbling in Graphs

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Abstract

Given a distribution of pebbles on the vertices of a graph G, a *pebbling move* takes two pebbles from one vertex and puts one on a neighboring vertex. The *pebbling number* $\Pi(G)$ is the least k such that for every distribution of k pebbles and every vertex r, a pebble can be moved to r. The *optimal pebbling number* $\Pi_{OPT}(G)$ is the least k such that some distribution of k pebbles permits reaching each vertex.

Using new tools (such as the "Squishing" and "Smoothing" Lemmas), we give short proofs of prior results on these parameters for paths, cycles, trees, and hypercubes, a new linear-time algorithm for computing $\Pi(G)$ on trees, and new results on $\Pi_{OPT}(G)$. If G is connected and has n vertices, then $\Pi_{OPT}(G) \leq \lceil 2n/3 \rceil$ (sharp for paths and cycles). Let $a_{n,k}$ be the maximum of $\Pi_{OPT}(G)$ when G is a connected n-vertex graph with $\delta(G) \geq k$. Always $2\left\lceil \frac{n}{k+1} \right\rceil \leq a_{n,k} \leq 4\left\lceil \frac{n}{k+1} \right\rceil$, with a better lower bound when k is a nontrivial multiple of 3. Better upper bounds hold for n-vertex graphs with minimum degree k having large girth; a special case is $\Pi_{OPT}(G) \leq 16n/(k^2+17)$ when G has girth at least 5 and $k \geq 4$. Finally, we compute $\Pi_{OPT}(G)$ in special families such as prisms and Möbius ladders.

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1 Introduction

Graph pebbling is a model for the transmission of consumable resources. Initially, pebbles are placed on the vertices of a graph G according to a *distribution* D, a function $D\colon V(G) \to \mathbb{N} \cup \{0\}$. A *pebbling move* from a vertex v to a neighbor u takes away two pebbles at v and adds one pebble at u. Before the move, v must have at least two pebbles. A *pebbling sequence* is a sequence of pebbling moves.

Given a distribution and a "root" vertex r, the task is to put a pebble on r. A distribution D is *r-solvable* (and r is *reachable* under D) if r has a pebble after some (possibly empty) pebbling sequence starting from D. For a graph G, let $\Pi(G, r)$ be the least k such that every distribution of k pebbles on G is r-solvable. A distribution D is *solvable* if every vertex is reachable under D. The *pebbling number* of a graph G, denoted $\Pi(G)$, is the least k such that every distribution of k pebbles on G is solvable; note that $\Pi(G) = \max_{r\in V(G)} \Pi(G, r)$. The *optimal pebbling number* of G, denoted $\Pi_{OPT}(G)$, is the least k such that some distribution of k pebbles is solvable.

Graph pebbling originated in efforts of Lagarias and Saks to shorten a result in number theory. Surveys by Hurlbert [7, 8] describe this history, early results, and recent directions. At an early stage, the following generalization became useful (see Chung [4]). A distribution D is *m-fold r-solvable* (and r is *m-reachable* under D) if r has at least m pebbles after some (possibly empty) pebbling sequence. A distribution D is *m-fold solvable* if every vertex is m-reachable under D.

Moews [11] developed several useful tools for computing pebbling numbers. (An unpublished longer version of [11] appears on his webpage [13].) We call the first of these tools the *Weight Argument*, which we express here for m-fold solvability. Given a root r and distribution D, let a_t be the total number of pebbles on vertices at distance t from r. A pebbling move cannot increase the sum $\sum_{t\geq 0} a_t 2^{-t}$. Therefore, m-fold r-solvability of D requires the *weight inequality* $\sum_{t\geq 0} a_t 2^{-i} \geq m$.

Another key tool is that when each pebbling move is represented by a directed edge from the vertex losing pebbles to the vertex gaining a pebble, no directed cycle is needed. If r is reachable using moves containing a cycle, then also r is reachable using a proper subset of these moves. In particular, if a distribution is r-solvable, then r is reachable without moving a pebble in both directions along any edge.

To make this precise, say that a directed multigraph H is *orderable* under a distribution D if some linear ordering σ of the edges of H is a valid list of pebbling moves starting from D. For such D and H, the *balance* of a vertex v is $d_H^-(v) + D(v) - 2d_H^+(v)$, where $d_H^-(v)$

and $d^+_H(v)$ are the indegree and outdegree of v under H. When H is orderable under D (by σ), the balance of any vertex v is nonnegative, since it is the number of pebbles at v after applying σ. The *No-Cycle Lemma* states that if H is orderable under D, then it has an acyclic subgraph H' that is orderable under D and gives balance to each vertex at least as large as does H. The lemma was proved in [3] and in [11] and has a short proof in [10].

The pebbling number is known exactly for some graphs. Moews [13] observed that a distribution on a path rooted at its end is solvable if and only if the weight inequality holds. Thus $\Pi(P_n) = 2^{n-1}$ for the n-vertex path P_n, since each pebble contributes weight at least $2^{-(n-1)}$ (Chung [4] earlier stated the value of $\Pi(P_n)$.) The n-vertex cycle C_n is more complicated; Pachter, Snevily, and Voxman [14] proved that $\Pi(C_{2k}) = 2^k$ and $\Pi(C_{2k+1}) = 2\lfloor 2^{k+1}/3\rfloor + 1$. For the k-dimensional hypercube Q_k, Chung [4] proved that $\Pi(Q_k) = 2^k$. For a tree, Chung [4] showed how to calculate the pebbling number from the set of decompositions into paths. For general graphs, Milans and Clark [10] showed that recognizing $\Pi(G) \le k$ is a Π_2^{P}-complete problem, meaning that it is complete for the class of languages whose complements are recognizable in polynomial time by nondeterministic machines.

Study of the optimal pebbling number began with the result of Pachter, Snevily, and Voxman [14] that $\Pi_{OPT}(P_n) = \lceil 2n/3\rceil$. Moews [12] proved that $(4/3)^k \le \Pi_{OPT}(Q_k) \le (4/3)^{k+O(\log k)}$ and proved a related result for Π_{OPT} on cartesian product graphs. Milans and Clark [10] proved that computing Π_{OPT} is NP-hard on arbitrary graphs.

In this paper, we present several new results (mostly on optimal pebbling) and new proofs for some previously-known results. We use lemmas that restrict the form of pebble distributions that need to be considered. Our methods use a precise version of the following intuition. For distributions with k pebbles, the hardest ones to make solvable are concentrated on one or two vertices, while the easiest ones are spread over many vertices. Thus to determine $\Pi(G)$ we consider concentrated distributions (via the "Squishing Lemma"), while to determine $\Pi_{OPT}(G)$ we consider "smooth" distributions. These simplifications are particularly helpful when studying paths and cycles or graphs that can be reduced to them.

For the pebbling number, we give an alternative proof for its computation on a tree T from decompositions into paths (Chung [4], Moews [11]), and we give a linear-time algorithm for computing $\Pi(T)$. Also, we give short proofs of the results of Pachter, Snevily, and Voxman [14] that $\Pi(C_{2k}) = 2^k$ and $\Pi(C_{2k+1}) = 2\lfloor 2^{k+1}/3\rfloor + 1$.

For optimal pebbling, the "Smoothing Lemma" implies that for each graph a solvable distribution of minimum size exists with at most two pebbles on each vertex of degree at most 2. This leads to a simpler proof of the result of Pachter, Snevily, and Voxman [14] that $\Pi_{OPT}(P_n) = \lceil 2n/3 \rceil$ and a proof that $\Pi_{OPT}(C_n) = \lceil 2n/3 \rceil$. Recently, Friedman and Wyels [6] found a short derivation of $\Pi_{OPT}(P_n)$ different from ours, and like us they adapted it to compute $\Pi_{OPT}(C_n)$.

We also show that $\Pi_{OPT}(T) \le \lceil 2n/3 \rceil$ for every n-vertex tree T, which immediately yields $\Pi_{OPT}(G) \le \lceil 2n/3 \rceil$ for every connected n-vertex graph G, and we give a short proof of the result of Moews [12] that $\Pi_{OPT}(Q_k) \ge (4/3)^k$.

Let $\mathbf{G}_{n,k}$ be the family of connected n-vertex graphs with minimum vertex degree at least k, and let $a_{n,k} = \max_{G \in \mathbf{G}_{n,k}} \Pi_{OPT}(G)$. Czygrinow [5] observed that $a_{n,k} \le 4\frac{n}{k+1}$. We construct graphs to show that $a_{n,k} \ge 2\left\lfloor \frac{n}{k+1} \right\rfloor$ for $n > k \ge 2$ and that $a_{n,k} \ge (2.4 - \frac{24}{15k+5} - \frac{6k}{5n})\frac{n}{k+1}$ when $n \ge k+3$ and k is a nontrivial multiple of 3 (that is, $k = 3j$ with $j \ge 2$). These results use another lower-bound technique, the simplest version of which is that if G is obtained from H by collapsing sets of vertices into single vertices, then $\Pi_{OPT}(H) \ge \Pi_{OPT}(G)$.

We obtain tighter bounds when we further restrict G to have girth (minimum cycle length) at least $2t+1$. Suppose that $k = 3$ and $t \ge 4$ or that $k \ge 4$ and $t \ge 2$. Letting $c_k(t) = 1 + k\sum_{i=1}^{t}(k-1)^{i-1}$ and $c'(t) = (4^t - 2^{t+1})\frac{t}{t-1}$, we prove that $\Pi_{OPT}(G) \le 4^t n/(c_k(t) + c'(t))$ for $G \in \mathbf{G}_{n,k}$. When $k \ge 4$ and G has girth at least 5, this yields $\Pi_{OPT}(G) \le \frac{16n}{k^2+17}$. We also show that $\Pi_{OPT}(C_m \,\square\, K_2) = \Pi_{OPT}(P_m \,\square\, K_2) = m$ when $m \ge 3$ (except that $\Pi_{OPT}(P_5 \,\square\, K_2) = 6$), where $\square$ denotes cartesian product (see Section 6). The same bound holds also for the graph consisting of a $2m$-cycle with chords added joining opposite vertices (the so-called “Möbius ladder”). Except for $C_3 \,\square\, K_2$, these graphs all have girth 4.

We discuss the pebbling numbers of trees and cycles in Sections 2 and 3, respectively. The final three sections treat optimal pebbling number. In addition to the results mentioned above, we pose the question of whether every connected n-vertex graph with minimum degree at least 3 has optimal pebbling number at most $\lceil n/2 \rceil$.

2 Pebbling Number of Trees

For a tree T, Chung [4] sketched a proof of a formula for $\Pi(T, r)$ in terms of path decompositions, as a special case of a more general result computing the minimum t such that all distributions of size t are m-fold r-solvable. This inductive proof considers the components of $T - r$, rooted at the neighbors of r. Moews [11] presents another proof using various auxiliary results. In this section, we give a new short proof using only a weight function argument and then show how to perform the computation in linear time.

A partition of the edge set of a tree is a *path partition* if each set in the partition is a (directed) path when all edges are directed toward a root r. The *length list* of a path partition is the list of lengths of its paths, in nonincreasing order. Path partition $\mathcal{L}$ *majorizes* path partition $\mathcal{L}'$ if the length list of $\mathcal{L}$ is larger than that of $\mathcal{L}'$ in the first position where they differ (distinct lists with equal sum differ in some position where both are nonzero). Majorization is a linear (lexicographic) order on length lists, but distinct path partitions may have the same length list. A path partition with root r is *r-optimal* if it is not majorized by any other path partition with root r. It is *optimal* if it is not majorized by any other path partition with any root. We use *leaf* to refer to a vertex of degree 1 in any graph.

Theorem 2.1 (Chung [4], Moews [11]). *If the length list of an r-optimal path partition of tree T with root r is $(l_1, \ldots, l_m)$, then*

$$\Pi(T, r) = \left(\sum_{i=1}^{m} 2^{l_i} \right) - m + 1.$$

Proof. We have noted that r-solvability never requires moving pebbles in both directions along an edge. Thus in T we may direct all edges toward r and assume that pebbles move only toward r. Let $\mathcal{L}$ be an r-optimal path partition of T, with length list $(l_1, \ldots, l_m)$.

Lower Bound. We construct a non-r-solvable distribution with $\sum_{i=1}^{m} \left(2^{l_i} - 1\right)$ pebbles. If some path in $\mathcal{L}$ starts at a nonleaf vertex, then another path ends there, and they combine to produce a path partition majorizing $\mathcal{L}$. Hence in $\mathcal{L}$ each path begins at a leaf. For each path of length l_i in $\mathcal{L}$, put $2^{l_i} - 1$ pebbles on the starting leaf. Now no pebble can be the first pebble to reach the end of the path in $\mathcal{L}$ on which it starts, so no pebble can reach r.

Upper Bound. Let $M = \sum_{i=1}(2^{l_i} - 1)$. We show that a distribution D with $|D| > M$ is r-solvable, using a weight function based on $\mathcal{L}$. Let P_i be the path in $\mathcal{L}$ corresponding to length l_i. Let $a_{i,t}$ be the number of pebbles D has on P_i at distance t from the end. Let $w_i(D) = 2^{l_i} \sum_{t=1}^{l_i} a_{i,t} 2^{-t}$, and let $w(D) = \sum_{i=1}^{m} w_i(D)$.

We claim that a pebbling move cannot decrease the weight w unless it reaches r. By definition, a pebble on vertex v contributes weight only to the path containing the edge leaving v. Hence a move along P_i that does not reach its end does not change the weight. For a move to vertex v at the end of P_i, where the edge leaving v is in P_j, the weight decreases by $2 \cdot 2^{l_i - 1}$ and increases by $2^{l_j - t}$, where t is the distance from v to the end of P_j. If $l_i > l_j - t$, then P_i can replace the beginning of P_j to produce a path partition majorizing $\mathcal{L}$, contradicting the r-optimality of $\mathcal{L}$. Hence $l_i \le l_j - t$, and the weight does not decrease.

Since every pebble contributes at least 1 to $w(D)$ and initially $|D| > M$, the total weight starts above M. We have shown that it doesn't decrease unless we move a pebble to r. While

$w > M$, the pigeonhole principle yields a path $P_i \in \mathcal{L}$ such that $w_i \geq 2^{l_i}$, which guarantees two pebbles on some non-terminal vertex of P_i. Hence we can make a move along P_i.

Each move loses a pebble, so the process must terminate. It cannot terminate until the weight decreases, which happens only when a pebble moves to r, so every sequence of moves eventually brings a pebble to r. □

The theorem easily yields $\Pi(T)$. We use $V(G)$ for the vertex set of a graph G.

Corollary 2.2 ([4, 11])**.** *If an optimal path partition of tree T has lengths $l_1, \ldots, l_m$, then*

$$\Pi(T) = \left(\sum_{i=1}^{m} 2^{l_i} \right) - m + 1.$$

Proof. Appending a 0 to a length list does not change the formula on the right, since $2^0 - 1 = 0$. This and the convexity of exponentiation imply that the formula is maximized by an optimal path partition, which is r-optimal for some r. Also $\Pi(T) = \max_{r \in V(T)} \Pi(T, r)$. Hence the claim follows from Theorem 2.1. □

The difficulty in applying Corollary 2.2 is in finding an optimal path partition. Given a root, a natural idea is to select a longest path greedily and iterate. Although this works, it disconnects the tree, leaving awkward bookkeeping details. The inductive proof is simpler if we peel away shorter paths first. A *peripheral vertex* in a tree is an endpoint of a longest path. A *branch vertex* in a tree is a vertex of degree at least 3. An *x, y-path* in a graph is a path with endpoints x and y.

Theorem 2.3. *There is a linear-time algorithm to compute the pebbling number of trees. In particular, if r is an endpoint of a longest path in T, then $\Pi(T, r) = \Pi(T)$, and any longest path to r can be chosen as a path in an r-optimal path partition.*

Proof. In a tree, the vertices at greatest distance from a vertex x are endpoints of a longest path. Hence a single breadth-first search from an arbitrary vertex finds a peripheral vertex r. Another breadth-first search from r finds a longest path R, ending at another vertex r'.

With R chosen, another breadth-first search computes distances from R. We find an r-optimal path partition using these distances. The partition will have R as a path, and it will be both r-optimal and r'-optimal. We view all edges off R as directed toward R.

Suppose that R is not all of T. Iteratively, we select a leaf x closest to R among the leaves that remain in the tree. Let y be the branch vertex that is closest to x in T; vertex

y is well-defined. Since R is a longest path, y cannot be r or r'. Let P be the x, y-path in T. Put P into the path partition and delete P from the tree, leaving only the endpoint y. When the remaining tree is just R, it becomes the last path in the partition. (We can pause the computation of distances from R each time a leaf is found and extract P then.)

We prove, by induction on the number of vertices outside R, that the path P deleted at each step lies in an r-optimal path partition of the tree remaining at that step. By the majorization criterion, the path P' containing x in an r-optimal path partition $\mathcal{L}$ contains all of P. If P' continues past y, then some path Q in $\mathcal{L}$ ends at y. Since Q starts at a leaf, Q is at least as long as P (if Q starts in $\{r, r'\}$, this holds because R is a longest path; otherwise, it holds by the choice of x).

Let Q' be the union of Q and the part of P' after y. Let $\mathcal{L}'$ be the partition obtained from $\mathcal{L}$ by replacing P' and Q with P and Q'. Now P and Q' are shortest and longest, respectively, among $\{P, P', Q, Q'\}$. If Q is longer than P, then $\mathcal{L}'$ majorizes $\mathcal{L}$, a contradiction. Otherwise, $\mathcal{L}'$ and $\mathcal{L}$ have the same length list.

Thus, P occurs in some r-optimal path partition $\mathcal{L}$. The rest of $\mathcal{L}$ is an r-optimal path partition of the remaining tree T'. Distances from R are the same in T' as in T. By the induction hypothesis, the remainder of the algorithm produces an r-optimal path partition of T' that contains R. It combines with P to yield the desired path partition of T.

We show next that the procedure produces the same length list from each peripheral vertex. When r and r' are the endpoints of a longest path R, the r-optimal partition produced is also r'-optimal, since the computation is the same when viewed from r' (distances from R are the same). Since lexicographic order is a linear order, all r-optimal path partitions have the same length list. Therefore, to prove that all peripheral vertices have the same optimal length list, it suffices to show that we can move from each peripheral vertex to any other via traversals of longest paths.

Every longest path in a tree contains the center of the tree. If the path joining two peripheral vertices is not a longest path, then each is an endpoint of a longest path to some common other peripheral vertex. Hence one can move from one peripheral vertex to any other by at most two instances of "move to the opposite end of a longest path".

Since we find an r-optimal path partition containing a longest path, the length list of a globally optimal path partition must include the longest path length. Hence $\Pi(T)$ equals $\Pi(T, r)$ for some peripheral vertex r. Since we obtain the same length list from any such vertex, it suffices to run the algorithm from any longest path R. $\square$

3 Pebbling Number of Cycles

Proving an upper bound on $\Pi(G)$ requires showing that each of a large number of distributions is solvable. The following lemma restricts the distributions that need to be considered. In this paper, a *thread* in a graph G is a path whose vertices have degree 2 in G.

Lemma 3.1 (Squishing Lemma)**.** *Let r be a vertex in a graph G. If $p < \Pi(G, r)$, then there is a non-r-solvable distribution of p pebbles on G such that on each thread not containing r, all pebbles occur on just one vertex or on two adjacent vertices.*

Proof. Let P be a thread in G. If a distribution has pebbles on only one vertex of P or on only two adjacent vertices of P, then we say that P is *squished*.

Let D be a distribution that is not r-solvable. We transform D into a non-r-solvable distribution of the same size such that every thread not containing r is squished. A *squishing move* takes one pebble each from two vertices y and z on a thread P and adds two pebbles to some vertex x between them on P. If P is not squished, then we can perform a squishing move on P. Each squishing move reduces the value of $\sum_p 2^{-b(p)}$, where the sum is over the set of pebbles on P and $b(p)$ is the distance of pebble p from a fixed end of P. Thus a sequence of squishing moves must end by squishing P.

Let D' be obtained from D by a squishing move along a thread P not containing r, moving pebbles from y and z to x. It suffices to show that if D' is r-solvable, then D is r-solvable. Let σ be a pebbling sequence from D' that reaches r. If σ never moves pebbles off x, then σ also reaches r from D. Hence we may assume that σ makes a first move from x to a neighbor, which by symmetry is the neighbor x' in the direction toward y along P.

By the No-Cycle Lemma, we may assume that σ makes no move from x' to x. The two pebbles removed from x to put one pebble on x' thus produce no more benefit in D' than the corresponding one pebble that started on y in D; under D this pebble starts farther than x' in the only direction it can go in D'. Also it cannot hurt to have the extra pebble on z. Thus D also is r-solvable. □

The Squishing Lemma provides a short proof for the pebbling number of C_n. We use *pile* to refer to the set of pebbles on a vertex.

Theorem 3.2 ([14])**.** $\Pi(C_{2k}) = 2^k$ *and* $\Pi(C_{2k+1}) = 2\lfloor 2^{k+1}/3 \rfloor + 1$.

Proof. By symmetry, it suffices to show for fixed r that $\Pi(C_n, r)$ has the claimed value.

Lower Bound. In C_{2k}, a distribution with $2^k - 1$ pebbles on the one vertex at distance k from r is not r-solvable. We show that in C_{2k+1}, a distribution with $\lfloor 2^{k+1}/3 \rfloor$ pebbles on each of the two vertices at distance k from r is not r-solvable. One pile alone cannot move distance k to reach r. If we combine them first, moving half of one pile to the other, then the resulting pile has at most $\frac{2^{k+1}-1}{3} + \frac{1}{2}\frac{2^{k+1}-1}{3}$ pebbles, since 2^{k+1} is not divisible by 3. The sum is less than 2^k, so again the pile cannot reach r.

Upper Bound. A distribution having 2^k pebbles on some path of length k ending at r is r-solvable, since $\Pi(P_{k+1}) = 2^k$. When studying non-r-solvable distributions, the Squishing Lemma allows us to restrict attention to distributions using only one or two adjacent vertices. In C_{2k}, every two adjacent vertices lie together in a path of length k ending at r, and hence all distributions with 2^k pebbles are r-solvable.

The same observation holds in C_{2k+1} except when the two adjacent vertices are the two vertices s and s' at distance k from r. In this case, with all pebbles on $\{s, s'\}$, we move as many as possible from the vertex with fewer pebbles to the vertex with more pebbles. With m pebbles total, the new pile has at least $(3m-2)/4$ pebbles. With $m \geq 2\lfloor 2^{k+1}/3 \rfloor + 1$, we obtain a pile of size at least $\lceil 2^k - \frac{3}{4} \rceil$ at distance k from r, which suffices. □

4 Optimal Pebbling Number

For optimal pebbling numbers, upper bounds are generally easier than lower bounds. For an upper bound, we give a distribution and show that it is solvable. For a lower bound, we must show that every distribution up to a certain size is not solvable.

The Smoothing Lemma plays the role for Π_{OPT} that the Squishing Lemma plays for Π. Again we want to restrict the form of distributions studied to determine the value. We want to make solvability easy, so instead of squishing pebbles on a thread, we spread them out.

When D is a distribution on a graph with a vertex v of degree 2, and v has at least three pebbles in D, a *smoothing move* from v changes D by removing two pebbles from v and adding one pebble at each neighbor of v. The case $m = 2$ below will be used in Section 5.

Lemma 4.1. *Let D be a distribution on a graph G with distinct vertices u and v, where v has degree 2. If $D(v) \geq 3$, and u is m-reachable under D, then u is m-reachable under the distribution D' obtained by making a smoothing move from v.*

Proof. For any pebbling sequence σ starting from D, we form a sequence σ' from D'. If σ

never makes a move from v, then we may set $\sigma' = \sigma$, since at each step there are at least as many pebbles at each vertex other than v when starting with D'.

If σ makes a move from v, then let σ' be the same as σ except that σ' skips the first such move. Having made that move, σ on D produces the same configuration as σ' on D', except that σ' on D' has an extra free pebble on one neighbor of v. We complete σ' using the rest of σ and have the same number of pebbles at each vertex as under σ from D, plus an extra pebble on one neighbor of v. (Since σ' mimics σ, we never use that extra pebble.) □

A distribution D is *smooth* if it has at most two pebbles on every vertex of degree 2 (so no smoothing move is possible). A vertex D is *unoccupied* under D if $D(v) = 0$.

Lemma 4.2 (Smoothing Termination Lemma)**.** *For any distribution on a connected graph G other than a cycle, any sequence of smoothing moves eventually terminates in a smooth distribution. If $G = C_n$ and there is no such termination, then a distribution with no unoccupied vertices is reached.*

Proof. Suppose first that G is not a cycle. Starting from any distribution on G, we show that only finitely many smoothing moves can be made, using a weight function argument. Every vertex v of degree 2 lies in a unique maximal thread P. Let Q be the trail obtained by extending P along the other edge incident to each endpoint of P. If the two new edges reach the same vertex u, then Q is a cycle and we view u as both "ends"; otherwise Q is a path. When Q has length m and v has distance k from some end of Q, then each pebble on v contributes weight $k(m-k)$; it does not matter which end the distance is measured from. Pebbles on a vertex with degree other than 2 contribute weight 0; this agrees with the general formula, since in that case $k = 0$.

A smoothing move from v replaces weight totaling $2k(m-k)$ at v with weight totaling $(k-1)(m-k+1) + (k+1)(m-k-1)$ at its neighbors. The global total declines by 2. It remains nonnegative, so we must reach a distribution with no move available.

When G is a cycle, we use induction on the number of unoccupied vertices. If D is not smooth and some vertex u is unoccupied, then we view u as both endpoints of Q as above. Using the same weight argument, with k measured as distance from u, each smoothing move reduces the total weight by 2. Thus eventually the distribution becomes smooth or a pebble moves to u. Since smoothing never uncovers a vertex, moving a pebble to u reduces the number of unoccupied vertices. By the induction hypothesis, continuing the smoothing process produces a distribution that is smooth or leaves no vertex unoccupied. □

We have separated Lemma 4.2 from Lemma 4.3 because it is used again in Lemma 5.6 and Theorem 6.5. We henceforth say that a distribution is *optimal* if it has $\Pi_{OPT}(G)$ vertices and is solvable; that is, it is a solvable distribution of minimum size.

Lemma 4.3 (Smoothing Lemma)**.** *If G is a connected n-vertex graph, with $n \geq 3$, then G has a smooth optimal distribution with all leaves unoccupied.*

Proof. An optimal distribution has at most n pebbles. By Lemma 4.2, applying smoothing produces a smooth optimal distribution, because when $G = C_n$ a distribution with at most n pebbles that occupies every vertex is smooth.

By Lemma 4.1, a smoothing move from v preserves the reachability of vertices other than v. Since a smoothing move from v leaves a pebble at v, it remains reachable also. Therefore, smoothing preserves solvability, and there is a smooth optimal distribution D. To eliminate pebbles from the leaves of such a distribution, consider the cases below at each leaf v. Let u be the neighbor of v, and suppose that $D(u) = j$ and $D(v) = k \geq 1$.

Case 1: $j + k \geq 3$. Modify D by deleting the pebbles on v and adding $k - 1$ pebbles to u instead. The resulting distribution D' is solvable, since $D'(u) \geq 2$ makes v reachable and $D'(u) \geq D(u) + \lfloor D(v)/2 \rfloor$ makes the other vertices reachable. Now $|D'| < |D|$ contradicts the optimality of D.

Case 2: $j + k = 2$. Modify D by putting both pebbles on u. The resulting D' is smooth. The two pebbles enable reaching v, and they help other vertices as much as in D.

Case 3: $(j, k) = (0, 1)$. Move the one pebble to u; again D' is smooth. Because D is u-solvable and cannot use the pebble on v to reach u, we can now move another pebble to u and use the two of them to reach v. □

The Smoothing Lemma yields a short proof that $\Pi_{OPT}(P_n) = \lceil 2n/3 \rceil$ (Pachter, Snevily, and Voxman [14]), and it yields the same value also for cycles. Another short proof was found by Friedman and Wyels [6]. We separate an observation useful in Section 6.

Lemma 4.4. *In a path with a smooth distribution D having at most two pebbles on each endpoint, let v be an unoccupied vertex. If v is an endpoint, then v is not 2-reachable under D. If v is an internal vertex, then no pebbling sequence can move a pebble out of v without using an edge in both directions.*

Proof. The first claim follows from the case $m = 2$ of the Weight Argument, since each vertex has at most two pebbles. For the second, moving a pebble off v without first moving a pebble in from each neighbor would contradict the first claim on a smaller path. □

Theorem 4.5. $\Pi_{OPT}(C_n) = \Pi_{OPT}(P_n) = \lceil 2n/3 \rceil$.

Proof. Let G be C_n or P_n.

Upper Bound. Partition G into $\lfloor n/3 \rfloor$ copies of P_3 and possibly one or two leftover vertices. Put two pebbles on the central vertex of each P_3 and one pebble on each of the leftover vertices (if any exist). This distribution is solvable and has size $\lceil 2n/3 \rceil$.

Lower Bound. We use induction on n, checking $n \le 5$ exhaustively. By Lemma 4.3, it suffices to consider a smooth solvable distribution D with no pebbles on leaves.

By the No-Cycle Lemma, we may assume that the directed edges representing moves in a pebbling sequence to reach a target vertex form edge-disjoint paths, and no edge is used in both directions. Since D is smooth and has no pebbles on leaves, Lemma 4.4 implies that each such path has no unoccupied internal vertex.

Since $\Pi_{OPT}(G) \le \lceil 2n/3 \rceil$ and $n \ge 6$, at least two vertices of G are unoccupied. In fact, there are at least three, since otherwise $n \in \{6, 7, 8\}$, no vertex has two pebbles, and D is not solvable. From three unoccupied vertices, we can choose an unoccupied internal vertex in P_n or nonadjacent unoccupied vertices in C_n; let S be this chosen set.

Since pebbles cannot be sent across an unoccupied vertex, S splits G into two paths, each of which cannot contribute pebbles to help pebble a vertex on the other path. Since the distribution is solvable, each vertex of S can be pebbled; we treat each vertex of S as part of the path that pebbles it, choosing one such path if both can pebble it.

We now have paths of order l and $n - l$ with $1 \le l \le n - 1$, and D breaks into solvable distributions for these two paths. By the induction hypothesis, the number of pebbles in D is at least $\lceil 2l/3 \rceil + \lceil 2(n-l)/3 \rceil$, which is at least $\lceil 2n/3 \rceil$. □

Next we show that the path is a hardest tree for optimal pebbling number. It is far from unique; there are many n-vertex trees with optimal pebbling number $\lceil 2n/3 \rceil$. We write $d(v)$ for the degree of a vertex v, and $N(v)$ for the set of vertices adjacent to v.

Theorem 4.6. *If T is an n-vertex tree, then* $\Pi_{OPT}(T) \le \lceil 2n/3 \rceil$.

Proof. We use induction on n. For $n \le 3$, all trees are paths, which satisfy the bound. In the induction step ($n > 3$), we delete three or more vertices near an end of a longest path in T to obtain a subtree T'. It suffices to add two pebbles to an optimal distribution D' on T' to form a solvable distribution D on T. When we add pebbles to D', all vertices in T' remain reachable, so we need only show that the new vertices can be reached.

Let P be a longest path in T. Let z be an endpoint of P, let y be its neighbor, and let x be the other neighbor of y on P. We consider four cases.

Case 1: $d(y) > 2$. Since P is a longest path, all neighbors of y other than x are leaves. Let $T' = T - y - (N(y) - \{x\})$. Form D from D' by adding two pebbles on y; these make leaf neighbors of y reachable.

Case 2: $d(x) = d(y) = 2$. Let $T' = T - \{x, y, z\}$. Form D from D' by adding two pebbles on y; these make x and z reachable.

Case 3: $d(y) = 2$ and x has a leaf neighbor u. Let $T' = T - \{u, y, z\}$. Form D from D' by adding two pebbles on y. Now y and z are reachable. Also u is reachable by moving a pebble to x using D' on T' and then moving a second pebble to x from y.

Case 4: $d(y) = 2$, $d(x) > 2$, and x has no leaf neighbors. Let u be a neighbor of x outside P. Since P is a longest path, every neighbor of u other than x is a leaf. Let v be a leaf neighbor of u, and let $T' = T - \{v, y, z\}$. If x is 2-reachable under D', then form D by adding two pebbles on x, making v, y, and z reachable under D. If u is 2-reachable under D', then v is reachable; form D by adding two pebbles on y. If neither x nor u is 2-reachable under D', then no pebbling sequence starting with D' uses the edge xu in either direction. Hence from D' we can reach x and u simultaneously. Now form D by adding two pebbles on y, making v, y, and z reachable after moving pebbles to both x and u using D'. □

Corollary 4.7. *If G is a connected n-vertex graph, then $\Pi_{OPT}(G) \le \lceil 2n/3 \rceil$, which is sharp.*

Proof. Since G is connected, it has a spanning tree T, and $T \subseteq G$ yields $\Pi_{OPT}(G) \le \Pi_{OPT}(T)$. Applying Theorem 4.6 to T gives the bound, with equality for P_n. □

Finally, we give a short proof that $\Pi_{OPT}(Q_k) \ge (4/3)^k$. The proof by Moews [12] used a continuous relaxation of pebbling, but the standard weight function and expectation suffice.

Theorem 4.8 (Moews [12])**.** $\Pi_{OPT}(Q_k) \ge (4/3)^k$, *where Q_k is the k-dimensional hypercube.*

Proof. Let D be a solvable distribution on Q_k; we show that $|D| \ge (4/3)^k$. Since D is solvable, the standard weight inequality $\sum_{t \ge 0} a_{r,t} 2^{-t} \ge 1$ holds for each vertex r, where $a_{r,t}$ is the number of pebbles at distance t from r in D.

Select a vertex r in Q_k uniformly at random. Since the weight inequality holds for each r, linearity of expectation yields $\sum_{t \ge 0} 2^{-t} \mathbf{E}[a_{r,t}] \ge 1$. For a fixed pebble under D, the probability it is distance t from r is $\binom{k}{t} 2^{-k}$, since Q_k has 2^k vertices and $\binom{k}{t}$ of them have distance t from this pebble. By linearity of expectation, $\mathbf{E}[a_{r,t}] = |D| \binom{k}{t} 2^{-k}$. Substituting and simplifying now yields $|D| \sum_{t \ge 0} \binom{k}{t} 2^{-t} \ge 2^k$. From the Binomial Theorem, $|D|(1 + \frac{1}{2})^k \ge 2^k$, and hence $|D| \ge (4/3)^k$. □

A variation on pebbling is the *cover pebbling number*: the minimum t such that for every distribution of size t on G, some sequence of pebbling moves produces a distribution with no unoccupied vertex. Hurlbert and Munyan [9] recently proved that the cover pebbling number of the hypercube Q_k is 3^k.

5 Bounds in Terms of Minimum Degree

We have proved that $\Pi_{OPT}(G) \le \lceil 2n/3 \rceil$ for every connected n-vertex graph G, with equality for paths and cycles. One would expect that tighter upper bounds hold for denser graphs. How large can $\Pi_{OPT}(G)$ be when $G \in \mathbf{G}_{n,k}$ (n vertices and minimum degree at least k)?

A *dominating set* in a graph G is a set $S \subseteq V(G)$ such that every vertex not in S has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum size of a dominating set. Placing two pebbles at each vertex of a dominating set yields $\Pi_{OPT}(G) \le 2\gamma(G)$. Thus upper bounds on $\gamma(G)$ yield upper bounds on $\Pi_{OPT}(G)$.

For $G \in \mathbf{G}_{n,k}$, Arnautov [2] and Payan [15] proved that $\gamma(G) \le n\frac{1+\ln(k+1)}{k+1}$; a short probabilistic proof appears in Alon [1]. In a k-regular n-vertex graph, dominating sets have size at least $\frac{n}{k+1}$. For k-regular graphs, where k is fixed and the number n of vertices grows, Alon [1] showed that $\gamma(G)$ may be as large as $(1+o(1))n\frac{1+\ln(k+1)}{k}$. Hence we cannot improve the bound using domination number alone.

Czygrinow [5] communicated to us an easy argument for a better upper bound when $k \ge 3$; we begin by presenting this. The *distance-d neighborhood* of a vertex v in a graph G is the set of all vertices having distance at most d from v. A *distance-d dominating set* is a set of vertices whose distance-d neighborhoods together cover $V(G)$. The case $d = 1$ of the following proposition is folklore in some circles but seems to be unknown in the subject of graph domination. We will use the general result in Section 6.

Proposition 5.1. *If c is the minimum size of a distance-d neighborhood in an n-vertex graph G, then G has a distance-$2d$ dominating set of size at most n/c.*

Proof. We build such a set S. Initially, put one vertex in S. As we proceed, let T consist of all vertices within distance d of S. If T is not a distance-d dominating set, then let v be a vertex that is not within distance d of T. Add v to S; this adds the distance-d neighborhood of v to T, none of which was in T before. Thus T grows by at least c vertices for each vertex added to S. We therefore add at most n/c vertices to S by the time T becomes a distance-d dominating set, at which point S is a distance-$2d$ dominating set. □

Corollary 5.2 (Czygrinow)**.** $\Pi_{OPT}(G) \le \frac{4n}{k+1}$ *when* $G \in \mathbf{G}_{n,k}$.

Proof. Distance-1 neighorhoods have size at least $k+1$, so Proposition 5.1 yields a distance-2 dominating set S of size at most $n/(k+1)$. Put four pebbles at each vertex of S. □

Corollary 5.2 improves the upper bound of $\lceil 2n/3 \rceil$ from Corollary 4.7 when $k \ge 5$. We show that this easy bound is at most a factor of 2 from optimal by an easy construction of n-vertex graphs with minimum degree k and optimal pebbling number $2\left\lfloor \frac{n}{k+1} \right\rfloor$. Later we present a better construction with optimal pebbling number near $2.4\frac{n}{k+1}$.

We begin by introducing another technique for proving lower bounds. A graph H is a *quotient* of a graph G if the vertices of H correspond to the sets in a partition of $V(G)$, and distinct vertices of H are adjacent if at least one edge of G has endpoints in the sets corresponding to both vertices of H. In other words, each set in the partition of $V(G)$ "collapses" to a single vertex of H (we use "collapses" rather than "contracts" because the set in $V(G)$ need not induce a connected subgraph). If H is a quotient of G via the surjective map $\phi\colon V(G) \to V(H)$, and D is a distribution on G, then the *quotient distribution* D_ϕ is the distribution on H defined by $D_\phi(u) = \sum_{v \in \phi^{-1}(u)} D(v)$.

Lemma 5.3 (Collapsing Lemma)**.** *Let H be a quotient of G via ϕ. If a distribution D' on G is obtainable from a distribution D on G via pebbling moves, then in H any vertex v is $D'_\phi(v)$-reachable under D_ϕ. In particular,* $\Pi_{OPT}(G) \ge \Pi_{OPT}(H)$.

Proof. The sequence σ of pebbling moves that produces D' from D in G collapses to a sequence σ_ϕ in H. When a move in σ transfers a pebble from one part to another in the partition under ϕ, the corresponding move is available for σ_ϕ in H (formally by induction on the length of σ). When a move in σ is within a part of the partition under ϕ, we don't need to do anything in σ_ϕ and simply have an extra pebble available on the image of that part.

In particular, the quotient of a solvable distribution D on G is solvable on H, since some vertex in each part of $V(G)$ under ϕ is reachable from D. □

Proposition 5.4. *For $n > k \ge 2$, there is an n-vertex graph G with minimum degree k such that* $\Pi_{OPT}(G) \ge 2\left\lfloor \frac{n}{k+1} \right\rfloor$.

Proof. When $k+1 \le n < 2(k+1)$, deleting edges at one vertex of K_n yields such a graph.

For $n \ge 2(k+1)$, let $r = \left\lfloor \frac{n}{k+1} \right\rfloor$. Let $J_1, \ldots, J_r$ be complete graphs, each with at least $k+1$ vertices, totaling n vertices. Choosing vertices $x_i, y_i \in J_i$, form a "ring of cliques"

G from the disjoint union $J_1 \cup \cdots \cup J_r$ by deleting each edge $x_i y_i$ and adding $y_i x_{i+1}$ instead (treating indices modulo r). Note that G has minimum degree k.

By Lemma 5.3, collapsing $V(J_i) - \{x_i, y_i\}$ into one vertex cannot increase the optimal pebbling number. Doing this in each J_i produces C_{3r}. By Theorem 4.5, $\Pi_{OPT}(G) \geq 2r$. □

When $k = 2$ and $3 \mid n$, the construction in Proposition 5.4 produces C_n, which by Corollary 4.7 and Theorem 4.5 is the extremal graph for all n. For $k = 3$, it provides connected n-vertex graphs with optimal pebbling number asymptotic to $n/2$; still the upper bound is $\lceil 2n/3 \rceil$ (Corollary 4.7). As k grows, the coefficient on n in Proposition 5.4 decreases.

However, for $k > 9$ the optimal pebbling number of our next construction exceeds $2\frac{n}{k+1}$ asymptotically for large n. In particular, there is a sequence of graphs of minimum degree k, with G_n having n vertices, such that $\Pi_{OPT}(G_n)\frac{k+1}{n} \to 2.4 - \frac{24}{5k+15}$ as $n \to \infty$. This limit exceeds 2 when $k > 9$. We present the construction only for $k \equiv 0 \pmod 3$; slightly weaker results hold for general k.

We will apply Lemma 5.3 to a graph that we will contract to a cycle. First we must study 2-reachability of vertices.

Lemma 5.5. *Let D be a distribution on a graph G, and let A be a subset of $V(G)$ such that each vertex in A has a neighbor in A. If each vertex in A is 2-reachable under D, then each vertex in A is 2-reachable under any distribution produced from D by a smoothing move.*

Proof. Let D' be a distribution obtained from D by a smoothing move from v. Note that $D'(v) \geq 1$, by the definition of smoothing. By Lemma 4.1, every vertex of $A - \{v\}$ is 2-reachable under D'. Hence we may assume that $v \in A$.

Let u be a neighbor of v in A, and let σ be a pebbling sequence under D' after which u has two pebbles. If σ has a move out of v, then truncating σ yields a pebbling sequence showing that v is 2-reachable. Otherwise, v retains at least one pebble after executing σ, and then a pebbling move from u to v gives it another. □

Lemma 5.6. *For $n \geq 3$, if at least $n - 1$ vertices are 2-reachable under a distribution D on C_n, then $|D| \geq n$.*

Proof. With at least $n-1$ vertices 2-reachable, every 2-reachable vertex has a 2-reachable neighbor. Letting A be the set of 2-reachable vertices, Lemma 5.5 implies that the hypothesis here is preserved under smoothing. By the Smoothing Termination Lemma, we may assume that D is smooth or has no unoccupied vertex. Since some vertex is 2-reachable, D has two pebbles on some vertex. This completes the proof if D leaves at most one vertex unoccupied. Hence we may choose distinct unoccupied vertices u and v.

Let P and P' be the u, v-paths along the cycle. Since at least $n-1$ vertices are 2-reachable, we may assume that u is 2-reachable. By Lemma 4.4, a pebbling sequence cannot move a pebble out of v without using an edge in both directions. By the No-Cycle Lemma, some pebbling sequence that moves two pebbles to u uses no edge in both directions. Lemma 4.4 also implies that u is not 2-reachable under the restrictions of D to P or P'. Therefore, 2-reachability of u requires moving a pebble to u from each of P and P', independently. Hence each path must have a vertex with two pebbles.

In particular, there is a vertex with two pebbles on each path of occupied vertices joining two unoccupied vertices, and therefore $|D| \geq n$. □

Let $V_1, \dots, V_r$ be pairwise disjoint s-sets of vertices. For $r, s \geq 1$, let $G_{r,s}$ be the graph defined on $V_1 \cup \cdots \cup V_r$ by letting $V_1, \dots, V_r$ be cliques and making each vertex of V_i adjacent to $s-1$ vertices of V_{i+1}, for $1 \leq i \leq r-1$ (the resulting graph is unique, up to isomorphism). For $r \geq 3$, let $H_{r,s}$ be a graph obtained from $G_{r,s}$ by making each vertex of V_r adjacent to $s-1$ vertices of V_1 (nonisomorphic graphs may result). Here we compute the optimal pebbling numbers of these graphs for $s \geq 3$. For $s = 2$ the value is higher; see Theorem 6.6.

Theorem 5.7. *If $r, s \geq 3$, then $\lceil 4r/5 \rceil \leq \Pi_{OPT}(H_{r,s}) \leq \Pi_{OPT}(G_{r,s}) \leq 4 \lceil r/5 \rceil$. The bounds hold also for $G_{1,s}$ and $G_{2,s}$.*

Proof. Since $G_{r,s} \subseteq H_{r,s}$, it suffices to show the upper bound for $G_{r,s}$ and the lower bound for $H_{r,s}$ (and $G_{1,s}$ and $G_{2,s}$).

Upper Bound. By dividing $V_1, \dots, V_r$ into groups of five consecutive sets (the last has fewer sets if $5 \nmid r$) and placing four pebbles on one vertex in a set closest to the center of each group, we obtain a solvable distribution that uses $4 \lceil r/5 \rceil$ pebbles. It is solvable because $s \geq 3$ implies that any two vertices in sets whose indices differ by two have a common neighbor in the intervening set.

Lower Bound. The proof is by induction on r. When $r \leq 4$, the claims are easily checked. For $r \geq 5$, consider an optimal distribution D on $H_{r,s}$. Let $B = \{i\colon V_i \text{ contains no vertex that is 2-reachable under } D\}$, viewing the indices modulo r. If $|B| \leq 1$, then collapsing each V_i to a single vertex yields a distribution on C_r under which at least $r-1$ vertices are 2-reachable, by the Collapsing Lemma (5.3). Now Lemma 5.6 yields $|D| \geq r \geq \lceil 4r/5 \rceil$.

Hence we may assume that $|B| \geq 2$. For $i \in B$, suppose that B contains neither $i-1$ nor $i+1$ (modulo r). Let $u \in V_{i-1}$ and $v \in V_{i+1}$ be 2-reachable vertices. Let j be another index in B. Since $i, j \in B$, we cannot put two pebbles on a vertex in $V_i \cup V_j$, and hence we cannot move a pebble out of $V_i \cup V_j$. Since u and v are separated by $V_i \cup V_j$, we conclude that u and v are 2-reachable simultaneously; that is, nothing used in moving two pebbles to one of them is used in moving two pebbles to the other. Since $s \geq 3$, u and v have a common neighbor w in V_i. Now w is 2-reachable using pebbles from u and v, contradicting $i \in B$.

It follows that for $i \in B$, one of $\{i-1, i+1\}$ also belongs to B. When $i, i+1 \in B$, we call the edges joining V_i and V_{i+1} *useless*. Since we cannot move two pebbles to any vertex in either clique, we cannot move a pebble along an edge joining them. Hence deleting these edges does not affect the solvability of D.

Since for every member of B there is a neighboring index also in B, every V_i for $i \in B$ is incident to a useless set of edges. Hence there are at least $|B|/2$ such useless sets of edges.

If $|B| \geq 3$, then there are at least two useless sets of edges; deleting them leaves a graph whose components are $G_{t,s}$ and $G_{r-t,s}$, and the restrictions of D to those two components are solvable. Applying the induction hypothesis to the two components yields $|D| \geq \lceil 4r/5 \rceil$.

Otherwise, $|B| = 2$. Lemma 5.3 implies that collapsing each clique to a single vertex and collapsing the two resulting vertices indexed by B to a single vertex v yields a distribution on C_{r-1} under which every vertex except v is 2-reachable. Since its size is $|D|$, Lemma 5.6 implies that $|D| \geq r-1 \geq 4r/5$. □

Corollary 5.8. *Let k be a nontrivial multiple of 3. For $n \geq k+3$, there is an n-vertex connected graph G with minimum degree k such that $\Pi_{OPT}(G) \geq (2.4 - \frac{24}{5k+15} - \frac{6k}{5n})\frac{n}{k+1}$. When n is a multiple of $(k/3)+1$, the term $-\frac{6k}{5n}$ can be dropped.*

Proof. Given such n and k, let $s = k/3 + 1$ and $r = \lfloor n/s \rfloor$. Since $k \geq 6$, we have $r, s \geq 3$. The graph $H_{r,s}$ is $3(s-1)$-regular, since each vertex has $s-1$ neighbors in its own clique and in each neighboring clique. If n is a multiple of s, then let $G = H_{r,s}$. Now

$$\frac{\Pi_{OPT}(G)(k+1)}{n} = \frac{\Pi_{OPT}(H_{r,s})(3s-2)}{rs} \geq \frac{4r}{5}\frac{3s-2}{rs} = \frac{12}{5} - \frac{8}{5s} = \frac{12}{5} - \frac{24}{5k+15}.$$

If n is not a multiple of s, then form G by adding to $H_{r,s}$ a set of $n - rs$ vertices whose neighborhoods duplicate neighborhoods of vertices in $H_{r,s}$. Now G has n vertices and minimum degree at least k, and $H_{r,s}$ is a quotient of G. Thus $\Pi_{OPT}(G) \geq \Pi_{OPT}(H_{r,s})$, by Lemma 5.3. Since $n \leq rs + s - 1$, we can change rs to $rs(1 + \frac{s-1}{rs})$ in the denominator of the previous computation. Since $(1 + \frac{s-1}{rs})^{-1} \geq 1 - \frac{s-1}{rs} \geq 1 - \frac{k}{3n-k} \geq 1 - \frac{k}{2n}$, we complete the computation using $\left(\frac{12}{5} - \frac{8}{5s}\right)\left(1 - \frac{k}{2n}\right) \geq \frac{12}{5} - \frac{24}{5k+15} - \frac{6k}{5n}$. □

Let $\mathbf{G}_{n,k}$ be the family of connected n-vertex graphs with minimum degree k. Let $f(k) = \limsup_{n\to\infty} \max_{G\in\mathbf{G}_{n,k}} \frac{\Pi_{OPT}(G)}{n/(k+1)}$. Corollary 5.2 yields $f(k) \le 4$, and Proposition 5.4 and Corollary 5.8 yield $f(k) \ge \max\{2, 2.4 - \frac{24}{15k+5}\}$ when k is a multiple of 3. Given the simplicity of Corollary 5.2, we believe that $f(k)$ is bounded away from 4.

It should be possible to determine $f(3)$. We can prove only $2 \le f(3) \le 8/3$, using the ring-of-cliques construction (Proposition 5.4) and the general upper bound $\Pi_{OPT}(G) \le \lceil 2|V(G)|/3 \rceil$ (Corollary 4.7). Theorem 6.6 provides another construction for the lower bound whenever n is even.

Question 5.9. *Is it true that $\Pi_{OPT}(G) \le \lceil n/2 \rceil$ whenever G is a connected n-vertex graph with minimum degree at least 3? The bound would be sharp for even n.*

When $k = 4$, Corollary 5.8 does not apply, but there may be other constructions needing more pebbles than the $2n/5$ in Proposition 5.4. We present a possible such construction based on the "Sierpinski Triangle". We have not determined the optimal pebbling number for these graphs in $\mathbf{G}_{n,4}$, but we conjecture that its ratio to n approaches $4/9$ as $n \to \infty$.

Example 5.10. Let G_1 be a triangle; its three vertices are its *corners* $\{x, y, z\}$. For $j > 1$, given three copies of G_{j-1} with corner vertices $\{x_i, y_i, z_i\}$ in the ith copy, form G_j by collapsing the pairs $\{z_1, x_2\}$, $\{y_2, z_3\}$, and $\{x_3, y_1\}$. The remaining corner vertices $\{x_1, y_3, z_2\}$ are the corners of G_j. Another way to construct G_j from G_{j-1}, starting with a layout of G_1 in the plane, is to subdivide the edges of each bounded face that is a triangle and add edges forming a triangle on each such set of three new vertices.

For $j > 1$, form H_j from G_j by adding three edges to make the corners pairwise adjacent. Since the corners of G_j have degree 2 and all other vertices of G_j have degree 4, H_j is 4-regular for $j > 1$. Letting $n_j = |V(H_j)| = |V(G_j)|$, we have $n_j = 3n_{j-1} - 3$ for $j > 1$, with $n_1 = 3$, so $n_j = (3^j + 3)/2$.

For $j \ge 3$, we present a solvable distribution on H_j with $2 \cdot 3^{j-2}$ pebbles (there are many such distributions). If this is optimal, then $\Pi_{OPT}(H_j)/n_j \to 4/9$.

In G_j, there are three copies of G_{j-1} and thus 3^{j-3} copies of G_3. The number a_j of vertices of G_j that are corners of copies of G_3 is n_{j-2}, by the alternative construction. Since the corners of G_3 form a distance-2 dominating set of G_3, we have $\Pi_{OPT}(H_j) \le \Pi_{OPT}(G_j) \le 4n_{j-2} = 2 \cdot 3^{j-2} + 6$.

For $j \ge 3$, we can save six more pebbles on H_j. The distance between corners of G_j is 2^{j-1}. In H_j, these corners are pairwise adjacent. Hence the four pebbles on one corner x can satisfy the other corners y and z and the neighbors of y and z. Let P be the shortest

y, z-path. If we delete the pebbles on P, then the unreachable vertices are within distance 1 of P. By putting two pebbles each on the corners of copies of G_2 along P (except for y and z), we have deleted $4(2^{j-3}+1)$ pebbles and added $2(2^{j-2}-1)$ pebbles, saving 6. □

6 Girth and Minimum Degree

Forbidding short cycles restricts graphs in a way that improves upper bounds on the optimal pebbling number. In particular, if G has minimum degree k and girth at least 5, then four pebbles at a vertex v can take care of k^2+1 vertices, because the neighborhoods of the neighbors of v overlap only at v.

Proposition 6.1. *If G is an n-vertex graph with minimum degree k and girth at least $2t+1$, then $\Pi_{OPT}(G) \le 2^{2t}n/c_k(t)$, where $c_k(t) = 1 + k\sum_{i=1}^{t}(k-1)^{i-1}$.*

Proof. When G has minimum degree k and girth at least $2t+1$, every distance-t neighborhood has size at least $c_k(t)$. Proposition 5.1 then applies. □

Note that $c_k(t) > k(k-1)^{t-1} > (k-1)^t$ for fixed k. Thus $2^{2t}/c_k(t) \to 0$ as $t \to \infty$ when $k \ge 6$. A more detailed analysis further improves the upper bound. The idea is to use 2^{2t} pebbles on a vertex of a distance-$2t$ dominating set only when it is used to reach substantially more than the $c_k(t)$ vertices guaranteed in its distance-t neighborhood.

Theorem 6.2. *Let k and t be positive integers with $k \ge 3$ and $t \ge 2$ such that $(k,t) \notin \{(3,2),(3,3)\}$. If G is an n-vertex graph with minimum degree k and girth at least $2t+1$, then $\Pi_{OPT}(G) \le 2^{2t}n/(c_k(t)+c'(t))$, where $c_k(t)$ is defined as above and $c'(t) = (2^{2t}-2^{t+1})\frac{t}{t-1}$. In particular, $\Pi_{OPT}(G) \le 16n/(k^2+17)$ when $t=2$ and $k \ge 4$.*

Proof. We begin with a distance-$2t$ dominating set S of size at most $n/c_k(t)$ as constructed in the proof of Proposition 5.1, where $c_k(t)$ is defined as in Proposition 6.1. As each vertex is added to S in the construction, its distance from all other vertices of S is at least $2t+1$.

To each $v \in S$, we assign a set $R(v)$ of vertices in G; pebbles on v will be used to reach the vertices of $R(v)$, and these sets partition $V(G)$. Grow the sets $R(v)$ simultaneously for all $v \in S$ by a breadth-first search from S; each vertex of G goes into exactly one of these sets when it is reached. Every vertex within distance t of v goes into $R(v)$, since the distance-t neighborhoods of vertices of S are disjoint. As $R(v)$ grows, also grow a spanning tree $T(v)$

of the subgraph induced by $R(v)$; add each new vertex as a leaf whose neighbor is the vertex from which it is reached. Since S is a distance-$2t$ dominating set, the leaves of $T(v)$ have distance at most $2t$ from v in $T(v)$.

Let $R'(v)$ be the set of nonleaf vertices of $T(v)$ that are not within distance t of v. Let $r'(v) = |R'(v)|$. If $r'(v) < 2^{2t} - 2^{t+1}$, then put 2^{t+1} pebbles on v and one pebble on each vertex of $R'(v)$. Otherwise, put 2^{2t} pebbles on v. This defines a distribution D.

When $r'(v) \geq 2^{2t} - 2^{t+1}$, the 2^{2t} vertices on v can reach all vertices at distance at most $2t$ from v. When $r'(v) < 2^{2t} - 2^{t+1}$, the 2^{t+1} pebbles on v can reach vertices at distance $t+1$ from v, including the closest ones in $R'(v)$. The rest of $T(v)$ can then be reached by pebbling along paths through $R'(v)$. Hence D is solvable.

It remains to bound $|D|$. We claim first that for $v \in S$, at least $r'(v)\frac{t}{t-1}$ vertices lie in $T(v)$ that are not within distance t of v. Let p_0 be the number of leaves of $T(v)$. Say that a vertex x of $R'(v)$ has *height* i if a longest path in $T(v)$ from v through x uses i edges beyond x (ending at a leaf). For $1 \leq i \leq t-1$, let p_i be the number of vertices in $R'(v)$ with height i, and consider the leaves to have height 0. For $i \geq 1$, the vertices with height i have distinct children with height $i-1$, so $p_0 \geq p_1 \geq \cdots \geq p_{t-1}$. Also, $r'(v) = \sum_{i=1}^{t-1} p_i$, and $r'(v) + p_0$ is the number of vertices in $T(v)$ that are not within distance t of v. We have $\frac{r'(v)+p_0}{r'(v)} = 1 + \frac{p_0}{r'(v)} \geq 1 + \frac{p_0}{(t-1)p_0} = \frac{t}{t-1}$.

Now we count the pebbles and the vertices in each tree $T(v)$. When $r'(v) < 2^{2t} - 2^{t+1}$, we use $2^{t+1} + r'(v)$ pebbles with $T(v)$ having at least $c_k(t) + r'(v)\frac{t}{t-1}$ vertices. When $r'(v) \geq 2^{2t} - 2^{t+1}$, we use 2^{2t} pebbles with $T(v)$ having at least $c_k(t) + c'(t)$ vertices.

Let $S' = \{v \in S\colon\ r'(v) < 2^{2t} - 2^{t+1}\}$, and let $s = |S|$. Let $r = \sum_{v \in S'}(2^{2t} - 2^{t+1} - r'(v))$. We have $n \geq s[c_k(t) + c'(t)] - r\frac{t}{t-1}$, and we used $2^{2t}s - r$ pebbles. Thus

$$\Pi_{OPT}(G) \leq \frac{2^{2t}s - r}{s[c_k(t) + c'(t)] - r\frac{t}{t-1}} n \leq \frac{2^{2t}}{c_k(t) + c'(t)} n,$$

where the last inequality uses that $\frac{sa-rb}{sc-rd} \leq \frac{a}{c}$ when $ad \leq bc$. Thus we need $2^{2t}\frac{t}{t-1} \leq c_k(t) + c'(t)$, which simplifies to $c_k(t) \geq 2^{t+1}\frac{t}{t-1}$. This inequality holds for $k \geq 3$ and $t \geq 2$ except when $(k,t) \in \{(3,2),(3,3)\}$. $\square$

For $k = 5$, with $c'(t) \geq 4^t\frac{t}{t-1}$ and $c_5(t) = 1 + (4^t - 1)(5/3)$, the upper bound on $\Pi_{OPT}(G)/n$ in Theorem 6.2 tends to $3/8$ as $t \to \infty$. For $k = 2$, always $\Pi_{OPT}(C_n) = \lceil 2n/3 \rceil$. Thus it is natural to ask whether our observation for $k \geq 6$ also holds for $3 \leq k \leq 5$.

Question 6.3. *For $k \in \{3,4,5\}$, does there exist $f_k(t)$ such that $\lim_{t\to\infty} f_k(t) = 0$ and graphs in $\mathbf{G}_{n,k}$ with girth at least $2t+1$ satisfy $\Pi_{OPT}(G)/|V(G)| \leq f_k(t)$?*

We have not constructed graphs to show that the bound in Theorem 6.2 is sharp, and we do not believe that it is sharp. We discuss one more family to show that if an n-vertex graph G has girth 4 and minimum degree 3, then $\Pi_{OPT}(G)$ can be as large as $n/2$. This improves the construction in Proposition 5.4 for $k = 3$ by showing that forbidding triangles does not reduce the number of pebbles that may be needed. Before defining the candidate graphs, we prove results needed to establish the lower bounds. We will need to characterize the optimal 2-solvable distributions on paths; this uses the next lemma.

We previously defined "m-fold r-solvable" to mean that vertex r is m-reachable. For convenience, we henceforth use 2-*solvable distribution* to mean a distribution in which every vertex is 2-reachable. An *optimal* 2-solvable distribution is a 2-solvable distribution having the minimum number of pebbles (by analogy with "optimal distribution").

Lemma 6.4. *In an optimal 2-solvable distribution, each leaf has at most two pebbles.*

Proof. Given a leaf v with neighbor u, let D be a 2-solvable distribution with $D(u) = j$ and $D(v) = k \geq 3$. Obtain D' from D by setting $D'(v) = 1$ and $D'(u) = j + k - 2$; leave other values unchanged.

Since $|D'| < |D|$, it suffices to show that D' is 2-solvable. Since $D'(u) \geq D(u) + \lfloor D(v)/2 \rfloor$ and D is 2-solvable, vertices outside $\{u, v\}$ are 2-reachable under D'.

Now consider $\{u, v\}$. If $j + k \geq 4$, then $D'(u) \geq 2$, which makes u 2-reachable and provides a second pebble for v. Otherwise, $(j, k) = (0, 3)$; now v can send only one pebble to u under D, so the 2-solvability of D requires that another pebble can be moved to join the pebble that D' has on u; they then together provide a second pebble for v. □

A longer case analysis ensures an optimal 2-solvable distribution with at most one pebble on each leaf, but we will not need this.

Theorem 6.5. *Every 2-solvable distribution on P_n has at least $n + 1$ pebbles. Furthermore, the 2-solvable distributions with $n+1$ pebbles consist of "prime segments" separated by single unoccupied vertices, where a* prime segment *is a path with either (1) two pebbles on one vertex and one pebble on all other vertices, or (2) three consecutive vertices having $0, 4, 0$ pebbles, respectively, and one pebble on all other vertices.*

Proof. We use induction on n; when $n \leq 2$ the optimal 2-solvable distributions have $n + 1$ pebbles and are prime segments, as claimed. Consider $n \geq 3$. A distribution with two pebbles on one vertex and one on all others is 2-solvable, so $n + 1$ pebbles suffice.

In a 2-solvable distribution, every vertex is 2-reachable, so every vertex has a 2-reachable neighbor, and hence the result of a smoothing move is also 2-solvable, by Lemma 5.5. By Lemma 4.2, only finitely many smoothing moves can occur, so we obtain a smooth 2-solvable distribution D. If every vertex is occupied, then 2-solvability requires some vertex to have two pebbles, and hence such distributions with $n+1$ pebbles form a single prime segment.

Now consider the case where D is smooth and leaves a vertex v unoccupied. Since v is 2-reachable, Lemma 4.4 implies that v is not an endpoint and that moving two pebbles to v requires one to arrive from each side. Since two pebbles cannot arrive at v from the same side, pebbles on one side of v cannot be used for 2-reachability of any vertex on the other side. Hence $P_n - v$ consists of two subpaths, and D is 2-solvable if and only if the distributions inherited from D by the components of $P_n - v$ are 2-solvable. With these paths having j and $n-1-j$ vertices, the induction hypothesis requires $j+1+n-j$ pebbles in D, and it also completes the decomposition into prime segments after the split at v. Since D is smooth, these segments all have type (1): $(2,1,1,0,2,1)$ is an example.

Now let D be any 2-solvable distribution of size $n+1$ on P_n, not necessarily smooth. By Lemma 6.4, D has at most two pebbles on each leaf. Since smoothing preserves 2-solvability and size, smoothing an optimal 2-solvable distribution cannot put three pebbles on a leaf. Hence all optimal 2-solvable distributions arise by "unsmoothing" smooth ones, which consist of prime segments of type (1) separated by single unoccupied vertices.

An "unsmoothing" move changes consecutive pebble values (i,j,k) to $(i-1,j+2,k-1)$, where $i,j,k \geq 1$. Since $i,j,k \geq 1$, unsmoothing cannot occupy an unoccupied vertex, so the three positions must be within a single prime segment of the original smooth distribution. If the first unsmoothing move changes $(2,1,1)$ to $(1,3,0)$, then the unoccupied vertex is no longer 2-reachable. If it changes $(1,1,1)$ to $(0,3,0)$, then the unoccupied vertex on the side farther from the vertex with two pebbles is not 2-reachable.

Hence the only way to produce a 2-solvable distribution by one unsmoothing move on a prime segment of type 1 is to change the consecutive triple $(1,2,1)$ to $(0,4,0)$. This changes a prime segment of type (1) to a prime segment of type (2). No further unsmoothing move is possible within such a segment, by the reasoning for $(1,1,1)$ above. Prime segments of type (2) are 2-solvable, so this completes the description of optimal 2-solvable distributions. □

For our final result we need the *cartesian product* $G \,\square\, H$ of graphs G and H, the graph with vertex set $V(G) \times V(H)$ such that (u,v) is adjacent to (u',v') if and only if (1) $u=u'$ and $vv' \in E(H)$ or (2) $v=v'$ and $uu' \in E(G)$, where $E(F)$ denotes the edge set of F.

In particular, $C_m \,\square\, K_2$ and $P_m \,\square\, K_2$ are circular and linear "ladders"; two copies of the cycle or path, with corresponding vertices from the two copies adjacent. We call the m copies of K_2 the *rungs* of the graph. In $C_m \,\square\, K_2$, exchanging the matching joining two rungs for the other possible matching joining them yields a graph isomorphic to the graph formed from a $2m$-cycle by adding chords joining opposite vertices (those at distance m along the cycle). This graph has been called the "Möbius ladder", so we denote it by M_m.

The graphs $C_m \,\square\, K_2$ and M_m are special cases of the construction in Theorem 5.7 with $m = r$ and $s = 2$, but the arguments there were not valid when $s = 2$. We take C_2 to mean P_2, so that $C_m \,\square\, K_2 = P_m \,\square\, K_2 = C_4$ when $m = 2$.

Theorem 6.6. *$\Pi_{OPT}(P_m \,\square\, K_2) = \Pi_{OPT}(C_m \,\square\, K_2) = \Pi_{OPT}(M_m) = m$ for $m \geq 2$, except that $\Pi_{OPT}(P_2 \,\square\, K_2) = \Pi_{OPT}(C_2 \,\square\, K_2) = 3$ and $\Pi_{OPT}(P_5 \,\square\, K_2) = 6$.*

Proof. *Upper bounds.* Since both $C_m \,\square\, K_2$ and M_m contain $P_m \,\square\, K_2$, when $m \notin \{2, 5\}$ it suffices to prove the upper bound for $P_m \,\square\, K_2$. Observe that three pebbles on one rung (with neither vertex unoccupied) can reach all vertices on the two neighboring rungs. Similarly, four pebbles on two adjacent rungs (two each on nonadjacent vertices) can reach all vertices on the two neighboring rungs. Thus it suffices to express m as a sum of 3s and 4s, which can be done unless $m \in \{2, 5\}$.

For $m = 5$, six pebbles suffice, since P_m has a 2-solvable distribution with $m + 1$ pebbles. We need smaller distributions for M_5 and $C_5 \,\square\, K_2$, which we describe for general m. It suffices to put a 2-solvable distribution of size m on a subgraph induced by a dominating set. For $C_m \,\square\, K_2$, assign $2, 0, 2, 0$ pebbles to four consecutive vertices of an m-cycle and 1 to each remaining vertex of that cycle. For M_m, view it as a $2m$-cycle with chords and put one pebble each on $m - 1$ consecutive vertices of the cycle, plus a second pebble on one of them.

For $m = 2$, actually $M_2 = K_4$ and two pebbles suffice, but $C_2 \,\square\, K_2$ and $P_2 \,\square\, K_2$ degenerate to 4-cycles and need a third pebble, which suffices.

Lower bounds. The proof of $\Pi_{OPT}(P_5 \,\square\, K_2) \geq 6$ is a case analysis that we omit. For other cases, since $P_m \,\square\, K_2 \subseteq C_m \,\square\, K_2$ and since the argument for $C_m \,\square\, K_2$ (when $m \geq 3$) is valid also for M_m, it suffices to prove $\Pi_{OPT}(C_m \,\square\, K_2) \geq m$. We use induction on m. Since $C_m \,\square\, K_2 = P_m \,\square\, K_2$ for $m \in \{1, 2\}$, we can take $\{1, 2\}$ as the basis. Now consider $m \geq 3$.

Let D be a solvable distribution on $C_m \,\square\, K_2$ with $|D| \leq m$; we show that equality holds. If some pebbling sequence from D results in a rung having two pebbles, we say that the rung is 2-reachable under D. If at least $m - 1$ rungs are 2-reachable, then collapsing each rung to a vertex yields a distribution D' on C_m under which $m - 1$ vertices are 2-reachable, by the Collapsing Lemma. Lemma 5.6 then yields $|D| = |D'| \geq m$.

Now suppose that at least two rungs R and R' are not 2-reachable under D. The pebbles that arrive in pebbling sequences to reach the two vertices of R arrive from the same direction; otherwise, since no pebble can ever emerge from R', the two pebbling sequences could be performed independently and R would be 2-reachable.

Since pebbles can only reach R from one side, and no pebble can traverse an edge joining R to the rung on the other side (because R is not 2-reachable), D remains solvable on the graph obtained by deleting the edges from R to that other rung. If there are two nonadjacent rungs that are not 2-solvable, then doing this for those two rungs splits D into solvable distributions on $P_i \square K_2$ and $P_{m-i} \square K_2$, for some i with $1 \leq i \leq m-1$. The induction hypothesis applies to each subgraph, and we obtain $|D| \geq m$.

In the remaining case, any two rungs R and R' that are not 2-reachable are consecutive. In the special case $m = 3$, we may have all three rungs not 2-reachable, but then each rung has at most one vertex and D is not solvable. Otherwise, we have exactly two such rungs R and R', consecutive. A rung that is not 2-reachable is unoccupied, because if there is one pebble on it, then the sequence to reach the other vertex requires bringing another pebble to the rung. Furthermore, the pebbling sequences that move two pebbles to other rungs cannot use vertices in R or R', since those rungs are not 2-reachable.

Therefore, deleting R and R' and collapsing the remaining rungs yields a 2-solvable distribution D' on P_{m-2}, by the Collapsing Lemma. If $|D'| \geq m$, we have the desired result. Otherwise, D' is a 2-solvable distribution of size $m-1$ on P_{m-2}. We use the description of all such distributions obtained in Theorem 6.5.

Let S be the rung other than R' that neighbors R. In the collapsed path, S is an endpoint. The prime segment of D' ending at S has i vertices at S, for some $i \in \{0, 1, 2\}$. If $i = 2$, then each other rung in the segment has one pebble and S cannot receive another pebble. If $i = 1$, then S can receive one additional pebble from the neighboring rung S', which exists. If $i = 0$, then S can receive two pebbles from S'. Thus S can never acquire a third pebble.

Furthermore, if S is able to have two pebbles, then the vertices of S are not both 2-reachable in D. This is clear if $i \geq 1$, since at least one pebble starts at a fixed vertex of S, and it is not possible to get two pebbles to the other vertex of S. If $i = 0$ and S can acquire two pebbles under D', then S' starts with four pebbles and acquires no others. From any distribution of four pebbles on S', the vertices of S are not both 2-reachable.

Since the vertices of S are not both 2-reachable under D (and neither is 3-reachable), it follows that the vertices of R are not both reachable. Hence $C_m \square K_2$ has no solvable distribution of size $m-1$. □

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