On Pattern Ramsey Numbers of Graphs

Robert E. Jamison
Mathematical Sciences
Clemson University
Clemson, SC 29634-0975
e-mail: rejam@clemson.edu

Douglas B. West*
Department of Mathematics
University of Illinois
Urbana, IL 61801-2975
e-mail:west@math.uiuc.edu

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Abstract

A *color pattern* is a graph whose edges are partitioned into color classes. A family $\mathscr{F}$ of color patterns is a *Ramsey family* if there is some integer N such that every edge-coloring of K_N has a copy of some pattern in $\mathscr{F}$. The smallest such N is the *(pattern) Ramsey number* $R(\mathscr{F})$ of $\mathscr{F}$. The classical Canonical Ramsey Theorem of Erdős and Rado [4] yields an easy characterization of the Ramsey families of color patterns.

In this paper we determine $R(\mathscr{F})$ for all families consisting of equipartitioned stars, and we prove that $5\lfloor \frac{s-1}{2} \rfloor + 1 \le R(\mathscr{F}) \le 3s - \sqrt{3s}$ when $\mathscr{F}$ consists of a monochromatic star of size s and a polychromatic triangle.

1 Introduction

The classical Canonical Ramsey Theorem of Erdős and Rado [4] states that for every positive integer n there is an integer N such that if the edges of the complete graph K_N with vertex set $\{1, \ldots, N\}$ are arbitrarily colored, then there is a complete subgraph with n vertices on which the coloring is of one of four canonical types:

1) *monochromatic* — all edges have the same color;

2) *polychromatic* — all edges receive different colors;

3) *upper lexical* — two edges have the same color if and only if they have the same higher endpoint;

4) *lower lexical* — two edges have the same color if and only if they have the same lower endpoint.

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(See [11] for bounds on the smallest such N.)

The notions of upper and lower lexical are defined with respect to a fixed vertex ordering. They can be combined by loosening them somewhat. Let an edge-coloring of a graph be *lexical* if there is *some* linear ordering of the vertices so that two edges have the same color if and only if they have the same lower endpoint. The classical Canonical Ramsey Theorem considers one ordering and its reverse specified in advance. In the following formally weaker version, the ordering can be obtained afterwards in terms of the coloring.

> For every $p \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that every edge-coloring of K_N contains a monochromatic, a polychromatic, or a lexically colored K_p.

The minimum such N may be smaller for this formulation than in the classical case. The weaker formulation plays a pivotal role for the theory developed here and is itself an example of the kind of theorem we prove.

We define a *color pattern* to be a graph with a partition of its edges into color classes. A color pattern G *contains* a color pattern H if H can be expressed as a subgraph of G in such a way that two edges in H have the same color in H if and only if their images in G have the same color in G. We then say that H is a *subpattern* of G. We use $V(G)$ and $E(G)$ for the vertex set and edge set of a pattern G. The number of vertices is the *order* of G, denoted by $n(G)$. The number of edges is the *size* of G, denoted by $e(G)$.

A family $\mathscr{F}$ of color patterns is a *Ramsey family* if for some integer N every edge-coloring of K_N contains a pattern in $\mathscr{F}$. The smallest such N is the *(pattern) Ramsey number* $R(\mathscr{F})$ of $\mathscr{F}$. The second version of the Canonical Ramsey Theorem above says that the three-pattern family consisting of a monochromatic complete graph K_p^{mono}, a polychromatic complete graph K_p^{poly}, and a lexically colored complete graph K_p^{lex} is a Ramsey family.

The canonical Ramsey theorem immediately implies the following characterization of Ramsey families of color patterns.

Theorem 1.1 *A family of color patterns is a Ramsey family if and only if it contains a monochromatic pattern, a polychromatic pattern, and a lexical pattern.*

Proof. Let $\mathscr{F}$ be a Ramsey family with pattern Ramsey number N. The properties "monochromatic", "polychromatic", and "lexical" are hereditary in the sense that every subpattern of a pattern with one of these properties also has that property. By considering in turn monochromatic, polychromatic, and lexical colorings of K_N, we see that $\mathscr{F}$ must contain a pattern of each type.

Conversely, suppose that $\mathscr{F}$ contains a monochromatic pattern M, a polychromatic pattern P, and a lexical pattern L. Let $p = \max\{n(M), n(P), n(L)\}$. By the Canonical Ramsey Theorem, there exists N such that every edge-coloring of K_N contains a canonically colored K_p. Consider an edge-coloring of K_N. If it contains a monochromatic or polychromatic K_p, then it contains M or P.

More care is needed when K_N contains a lexical K_p. The lexical coloring of K_p is relative to some linear order $\prec_1$ on $V(K_p)$. Similarly, the lexical coloring of L assumes a linear order

$\prec_2$ on $V(L)$. Having L as a subpattern requires an order-preserving injection of $V(L)$ into $V(K_p)$. Since both orders are linear and $p \geq n(L)$, such an injection exists. ■

In [6] it is shown that lexical colorings can be recognized in time that is linear in the number of edges. The same is obviously true for monochromatic and polychromatic patterns, so Ramsey families of color patterns can be recognized in linear time.

There is no requirement that the monochromatic, polychromatic, and lexical patterns in a Ramsey family be different. The only pattern that is both monochromatic and polychromatic is a single edge. Since every color class in a lexical pattern is a star, it follows that monochromatic stars are the only patterns that are both monochromatic and lexical. The essential content of Proposition 2.1 in [7] is that the only patterns that are both polychromatic and lexical are polychromatic forests.

Hence a family consisting of two patterns is Ramsey if and only if it consists of a monochromatic star ($K_{1,s}^{mono}$) and any polychromatic pattern or consists of a polychromatic forest and any monochromatic pattern. The pattern Ramsey numbers of such families are called *constrained Ramsey numbers* in [7]. When $\mathscr{F}$ consists of a monochromatic G and a polychromatic star of size t, the pattern Ramsey number is the minimum N such that every edge-coloring of K_N with fewer than t colors at each vertex has a monochromatic G; studied in [5], this is called the *local Ramsey number* of G. Using local Ramsey numbers, [7] obtained a recursive bound on $R(\mathscr{F})$ when $\mathscr{F}$ consists of a monochromatic G and a polychromatic tree. In particular, when $\mathscr{F}$ consists of monochromatic and polychromatic patterns on a single tree T, [7] obtained an upper bound that is cubic in the size of T ([2] proved a quartic upper bound, and it is conjectured that the truth is quadratic).

Other papers on Ramsey numbers for families consisting of a monochromatic G and a polychromatic H include [3, 9, 10]. These authors studied $R(K_{1,s}^{mono}, P_n^{poly})$, $R(P_m^{mono}, P_n^{poly})$, and $R(K_n^{mono}, P_4^{poly})$ and made remarks about $R(G^{mono}, H^{poly})$ like those above.

In this paper we continue the study of pattern Ramsey numbers for specific families. In Section 3 we study the family consisting of a monochromatic star and a polychromatic triangle. Here the general result is $5\lfloor \frac{s-1}{2} \rfloor + 1 \leq R(K_{1,s}^{mono}, K_3^{poly}) \leq 3s - \sqrt{3s}$. The lower bound is exact for $s \in \{3, 5, 7, 9\}$. Also, the exact value is 7 when $s = 4$. It would be worthwhile to determine the asymptotic behavior of $R(K_{1,s}^{mono}, K_3^{poly})$ for large s. A recent paper obtained the order of growth for a more general problem: [1] determined that $R(K_{1,s}^{mono}, K_t^{poly})$ is bounded between constants times $st^3/\ln t$.

Every Ramsey family has a subset consisting of at most three patterns that guarantees finiteness of the Ramsey number, but additional patterns may reduce the Ramsey number and/or make it easier to compute. In Section 2 we consider families in which each pattern is a star partitioned into color classes of equal size. We determine the pattern Ramsey number exactly for all such families. Let $M_{r,q}$ be the pattern consisting of a star with r color classes, each of size q. We reduce the problem to that of determining $R(\mathscr{F})$ when $\mathscr{F} = \{M_{r_0,q_0}, \ldots, M_{r_l,q_l}\}$, where $1 = r_0 < r_1 < \cdots < r_l$ and $q_0 > q_1 > \cdots > q_l = 1$. We then prove that $R(\mathscr{F}) = 2 + \sum_{i=1}^{l}(r_i - r_{i-1})(q_{i-1} - 1)$.

2 Families of Star Patterns

Consider patterns of the form $M_{r,q}$. By Theorem 1.1, every Ramsey family of star patterns has a monochromatic star $M_{1,q}$ and a polychromatic star $M_{r,1}$. Each of these is also lexical, so the remaining patterns are arbitrary. That is, every family containing a monochromatic star and a polychromatic star is a Ramsey family; this motivates our focus on star patterns.

The pattern Ramsey number of a family $\mathscr{F}$ is determined by its minimal members; when a coloring contains a pattern, it also contains all its subpatterns. Hence we may assume that no pattern in $\mathscr{F}$ contains another. For families of equipartitioned stars, we may therefore assume that $\mathscr{F} = \{M_{r_0,q_0}, \ldots, M_{r_l,q_l}\}$, where $1 = r_0 < r_1 < \cdots < r_l$ and $q_0 > q_1 > \cdots > q_l = 1$.

Theorem 2.1 *If $\mathscr{F} = \{M_{r_0,q_0}, \ldots, M_{r_l,q_l}\}$, with $1 = r_0 < r_1 < \cdots < r_l$ and $q_0 > q_1 > \cdots > q_l = 1$, then $R(\mathscr{F}) = 1 + f(\mathscr{F})$, where $f(\mathscr{F}) = 1 + \sum_{i=1}^{l}(r_i - r_{i-1})(q_{i-1} - 1)$.*

Proof. For the upper bound, consider an edge-coloring of a complete graph that contains no pattern in $\mathscr{F}$, and let x be one of the vertices. For $1 \le i \le l$, at most $r_i - 1$ colors can each appear on at least q_i edges incident to x. The color that appears most often appears on at most $q_0 - 1$ incident edges.

If we list the multiplicities of color usage in nonincreasing order, then we iteratively obtain the most among the first j colors, for all j, if we have $r_i - r_{i-1}$ colors appearing on $q_{i-1} - 1$ incident edges for each i. Summing over i shows that x has at most $f(\mathscr{F}) - 1$ incident edges. Adding 1 for x bounds the number of vertices by $f(\mathscr{F})$, and hence $R(\mathscr{F}) \le 1 + f(\mathscr{F})$.

Let $n = f(\mathscr{F})$. To show that $R(\mathscr{F}) = n + 1$, we construct an edge-coloring of K_n with no pattern in $\mathscr{F}$. We must match the counting bound established above at each vertex. We construct an edge-coloring of K_n such that the color pattern on edges at each vertex consists of $r_i - r_{i-1}$ colors appearing on $q_{i-1} - 1$ incident edges for each i.

We use induction with respect to lexicographic order on the list $(r_0, \ldots, r_l)$. Since $l \ge 1$ is required and the list is increasing, the base case is $\mathscr{F} = \{M_{1,s}, M_{2,1}\}$. We have $n = 1 + (2-1)(s-1) = s$, and at each vertex we should have one color with $s - 1$ edges. The construction is a monochromatic copy of K_s.

For the induction step, we have $l > 1$ or $r_1 > 2$. Let $t = r_l$, $r = r_{l-1}$, and $q = q_{l-1}$. Let $\mathscr{F}'$ be the family obtained by replacing t with $t - 1$ as r_l. Note that if $t = r + 1$, then $M_{t-1,1}$ is contained in $M_{r,q}$, and the family $\mathscr{F}'$ has the same Ramsey number as the family $\mathscr{F}''$ obtained by deleting q and t and setting $q_{l-1} = 1$, reducing the number of minimal patterns in the family. Since $t - r - 1 = 0$, the formula for $f(\mathscr{F}')$ is the same as that for $f(\mathscr{F}'')$ in this case, and the list for $\mathscr{F}''$ is also lexicographically smaller. Hence we really only have one case for the induction step.

Partition $V(K_n)$ into two sets A and B, where $|A| = q - 1$ and $|B| = f(\mathscr{F}')$. By the induction hypothesis, the complete subgraph induced by B has an edge-coloring in which at each vertex the color pattern is exactly what is desired for K_n except that it is missing one color class of size $q - 1$. Add one color class that is the complete graph induced by A together with one vertex x of B. It remains to color the edges of the complete bipartite graph with partite sets A and $B - x$. Partition $B - x$ into sets $X_1, \ldots, X_{t-2}$, altogether having $r_i - r_{i-1}$

sets of size $q_{i-1}-1$ for each i. Complete the construction using $t-2$ more color classes; the jth such class is the complete bipartite graph whose partite sets are A and X_j.

We have now given one additional color class of size $q-1$ to each vertex of B and all the needed color classes to the vertices of A. ■

3 Monochromatic Stars and Polychromatic Triangles

We now consider the family consisting of a monochromatic star of size s and a polychromatic triangle T. Asymptotically, the upper and lower bounds are $3s$ and $2.5s$. Families consisting of a monochromatic pattern and a polychromatic pattern were studied in [7]. Theorem 3.2 improves Theorem 4.1 of [7], which gives $2s-1$ as a lower bound. Theorem 3.1 has no analogue in [7], which gives upper bounds only when the polychromatic pattern is a tree.

Our upper bound follows from a counting argument, and the lower bound arises from a construction involving congruence classes. We discuss the two bounds separately.

We say that a pair of edges is *pure* if the two edges are incident and have the same color.

Theorem 3.1 *$R(\{M_{1,s},T\})\le 3s-\sqrt{3s}$ for $s\ge 2$.*

Proof. Consider an edge-coloring of K_n with no monochromatic star of size s and no polychromatic triangle. Every triangle must have a pure pair of edges, and every pair of incident edges lies in exactly one triangle. Hence there are at least $\binom{n}{3}$ pure pairs of edges.

On the other hand, we can count the pure pairs of edges by grouping them by their shared vertex. When the multiplicities of colors $1,\ldots,k$ at vertex v are $m_1,\ldots,m_k$, the number of pure pairs incident at v is $\sum_{i=1}^{k}\binom{m_i}{2}$. Forbidding $M_{1,s}$ requires $m_i\le s-1$ for each i. The convexity of $\binom{u}{2}$ implies that the number of pure pairs at v is maximized by greedily choosing multiplicities equal to $s-1$ until what remains of $n-1$ is less than $s-1$.

In particular, when $n=3(s-1)-x$, we obtain the most pure pairs at v by having the color multiplicities at v be $s-1$, $s-1$, and $s-2-x$. The resulting inequality is

$$n\left[2\binom{s-1}{2}+\binom{s-2-x}{2}\right]\ge\frac{n}{6}(n-1)(n-2).$$

Cancel n and set $n=3(s-1)-x$. For $x=0$, we can cancel $3s-5$ (when $s\ge 2$) to simplify the inequality to $\frac{s-2}{2}\ge\frac{3s-4}{6}$, which is impossible. Hence x must be positive. Simplifying yields a quadratic inequality for x in terms of s; it is $x^2+3x+(5-3s)\ge 0$. Since we already have $x>0$, we obtain $x\ge\sqrt{3s-2.75}-1.5$.

The resulting inequality for n is $n\le\lfloor 3s-\sqrt{3s-2.75}-1.5\rfloor$. The upper bound on the Ramsey number exceeds this by 1. Accounting for the 2.75 under the square root still yields a uniform upper bound of $R(\{M_{1,s},T\})\le 3s-\sqrt{3s}$ for $s\ge 2$. ■

Theorem 3.2 $R(\{M_{1,s},T\})\ge 5\lfloor\frac{s-1}{2}\rfloor+1$.

Proof. For odd s, let $n=5(s-1)/2$. We prove the bound by constructing an edge-coloring of K_n that avoids $M_{1,s}$ and T. For even s, using the construction for $s-1$ then yields $R(\{M_{1,s},T\})\ge R(\{M_{1,s-1},T\}\ge 5(s-2)/2+1$.

For $s = 3$, decompose K_5 into two monochromatic cycles in colors 1 and 2. Each color has two edges at each vertex, so there is no $M_{1,3}$. Only two colors are used, so there is no T.

For larger odd s, expand each vertex in this edge-coloring of K_5 into a set of size $(s-1)/2$. The edges within a set all receive color 3. The edges between two sets all receive the color that was on the corresponding edge in the edge-coloring of K_5.

Along with the $(s-3)/2$ incident edges of color 3, each vertex is incident to $s-1$ edges of color 1 and $s-1$ edges of color 2 (going to two of the four other sets), so there is no $M_{1,s}$. A copy of T must use all three colors. However, if u, v, w are distinct vertices with uv having color 3, then uw and vw have the same color. ■

The upper and lower bounds are close enough to give exact results for small s.

Proposition 3.3 *If s is odd and $3 \leq s \leq 9$, then $R(\{M_{1,s}, T\}) = 5(s-1)/2 + 1$. Also the value is 7 when $s = 4$.*

Proof. Let $R(s) = R(\{M_{1,s}, T\})$. It is immediate that $R(1) = 2$ and $R(2) = 3$.

For the upper bounds, recall from the proof of Theorem 3.1 that $n < R(s)$ requires at least $(n-1)(n-2)/6$ pure pairs incident to some vertex. When $s = 2k+1$ and $n = 5k+1$, this requires $2\binom{2k}{2} + \binom{k}{2} \geq 5k(5k-1)/6$. For $k > 0$, this simplifies to $k \geq 5$. Hence $R(s) \leq 5(s-1)/2 + 1$ when $s \in \{3, 5, 7, 9\}$. Equality holds for such s by Theorem 3.2. (Similarly, $R(11) \in \{26, 27\}$, and $R(11) = 27$ requires an edge-coloring of K_{26} that avoids T and at every vertex has three colors with multiplicities $(10, 10, 5)$.)

We have $R(4) \geq 7$ by using one color on $K_{3,3}$ and another on its complement.

The argument of Theorem 3.1 yields $R(4) \leq 8$; we show that $R(4) \leq 7$. Let f be an edge-coloring of K_7 avoiding $M_{1,4}$ and T. Each vertex contributes at most 6 pure pairs, using color multiplicities $(3, 3)$. The next best is $(3, 2, 1)$, contributing 4 pure pairs. Let U be the set of vertices of type $(3, 3)$. Since there are 35 triples, 35 pure pairs are needed, so $|U| \geq 4$.

Consider $u, u' \in U$, and let $a = f(uu')$. Each of u and u' has three incident edges not having color a. These edges go to five other vertices, so there exists v with $f(uv) \neq a \neq f(u'v)$. These edges complete a triangle with uu', so $f(uv) = f(u'v)$. Hence all vertices of type $(3, 3)$ have the same two colors a and b on their incident edges.

If some color appears on 11 edges, then that subgraph has degree-sum 22, and some vertex has four incident edges of the same color. Hence three colors must be used among the 21 edges of K_7, and there is an edge xy of a third color c. This implies that $|U| \leq 5$.

If $|U| = 5$, then for all $u \in U$, we have $f(ux) = f(uy) \in \{a, b\}$. Hence u has an odd number of neighbors in U along edges of color a, and U induces a 5-vertex subgraph of color a in which every vertex has odd degree. This is impossible, so $|U| = 4$.

Let z be the third vertex outside U. Since $z \notin U$, some color not in $\{a, b\}$ appears at z. We may assume that $f(xz) \notin \{a, b\}$. Since every edge incident to U has color a or b, for $u \in U$ we have $f(ux) = f(uy)$ and $f(ux) = f(uz)$. Hence each vertex of U has three incident edges of the same color to $\{x, y, z\}$ and three incident edges of the same color within U. This requires all the edges leaving U to have the same color, which makes four edges of that color incident to each of $\{x, y, z\}$.

We have eliminated all possibilities and conclude that $R(4) = 7$. ■

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