Proper Path-Factors and Interval Edge-Coloring of (3,4)-Biregular Bigraphs

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Joint work with
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Necessary condition: must have a proper edge-coloring with $\Delta(G)$ colors (Asratian-Kamalian [1994]).

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Recognizing whether $(3, 6)$-biregular bigraphs have interval 6-colorings is NP-complete.

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Neither result implies the other.

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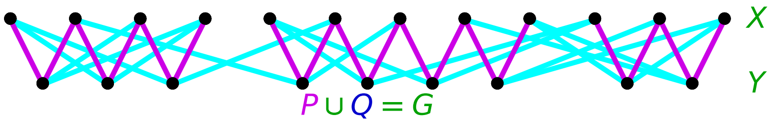

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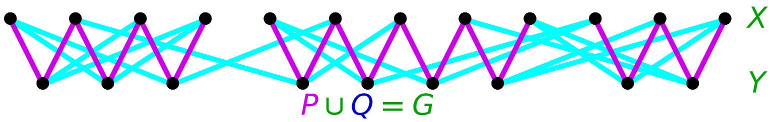

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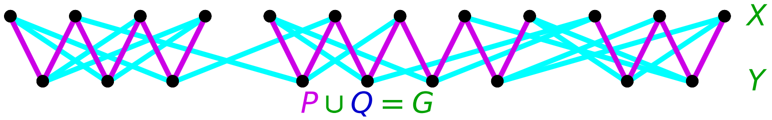


Prop. Every component of Q is an even cycle or is a path with endpoints in X.

Pf. Always $d_Q(y) = 2$ when $y \in Y$. Also $d_Q(x) = 2$ if x is an endpoint of a component of P, while $d_Q(x) = 1$ if $x \in X$ and x is an internal vertex of a component of P. ∎

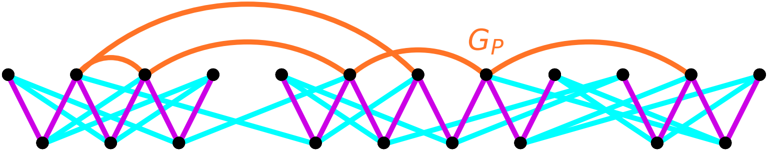
The P-graph of G

Def. Given a proper path-factor P of G, let G_P be the graph with vertex set $\{x \in X\colon d_P(x) = 2\}$, with x_i and x_j adjacent when any condition below holds:

(a) x_i and x_j have degree 2 in one copy of P_7 in P, or

(b) x_i and x_j have degree 2 at distance 4 in one copy of P_9 in P, or

(c) x_i and x_j have degree 1 in one component of Q.



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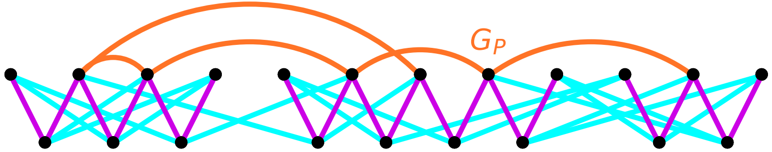

Lem. If P is a proper path-factor, then G_P is bipartite.

Pf. Every vertex of G_P has one incident type (c) edge; some have another of type (a) or (b). Hence $\Delta(G_P) \le 2$ and no odd cycle. ∎

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Color cycles in Q arbitrarily using 3 and 4. A path in Q has endpoints x and x' adjacent in G_P. If $c(x) = A$ and $c(x') = B$, alternate 3 and 4 starting with 3 on the edge at x and ending with 4 on the edge at x'. Now 3 and 4 both appear at every vertex of G with degree 2 in Q.

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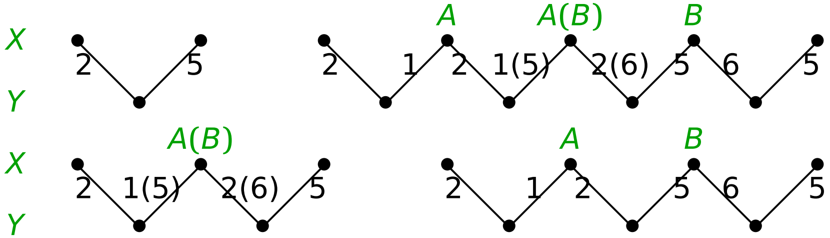
On each path in P, alternate 2 and then 1 from one end, and alternate 5 and then 6 from the other. The choice of which end is which and when to switch from $\{2, 1\}$ to $\{5, 6\}$ uses the colors that c assigns to the internal vertices. They have degree 1 in Q and lie in G_P.

Choosing Colors in P

A component H of P is a copy of P_3, P_5, P_7, or P_9. If $x \in V(H)$ and $c(x) = A$, use 1 and 2 at x; if $c(x) = B$, use 6 and 5. On P_7 and P_9, vertices two steps from the ends are adjacent in G_P and have distinct colors under c.



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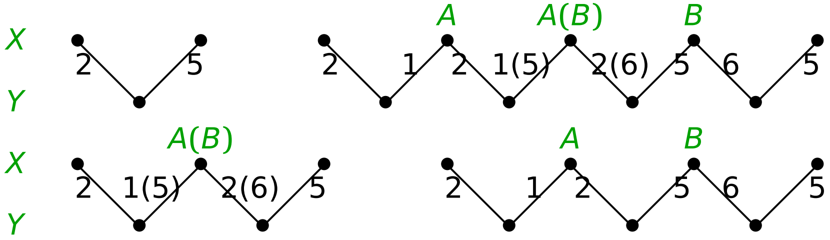

Each Y-vertex gets $\{3, 4\}$ on its edges in Q and gets $\{2, 5\}$ or $\{1, 2\}$ or $\{5, 6\}$ on its edges in P: an interval. Each leaf in P gets $\{3, 4\}$ from Q and 2 or 5 from P. Each non-leaf in P gets 3 from Q and $\{1, 2\}$ from P if $c(x) = A$, but 4 from Q and $\{5, 6\}$ from P if $c(x) = B$. ∎

Constructions

Ex. The containment bigraph of the 3-sets and 2-sets in $\{1, 2, 3, 4, 5, 6\}$ is a $(3, 4)$-biregular bigraph with a simple explicit P_7-factor (with 5-fold cyclic symmetry).

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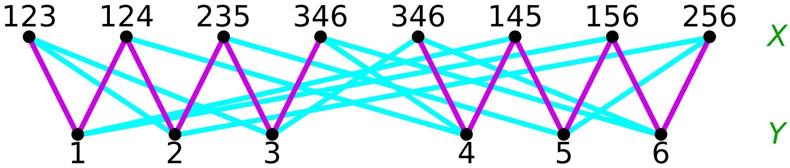

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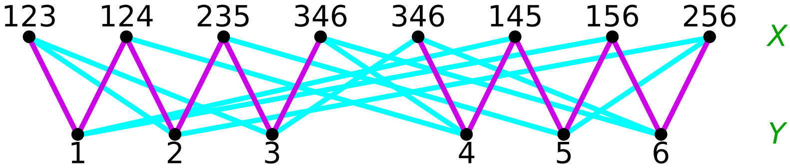

Lem. For $i \in \{1,2\}$, let G_i be 2-edge-connected with a P_7-factor F_i, and $e_i \in E(G_i) - E(F_i)$. Form G from $G_1 + G_2$ by replacing e_1 and e_2 with other edges e_1' and e_2' joining their endpoints. If G_1 has no full 3-regular subgraph, then G is a larger such example.

Sufficient Conditions

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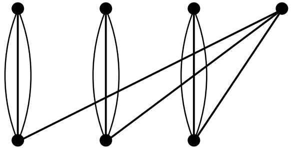

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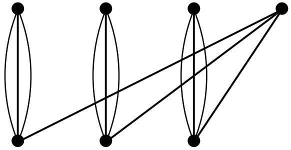

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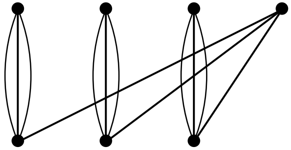

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Conj. Every simple $(3, 4)$-biregular X, Y-bigraph has a proper path-factor.