Proper Path-Factors and Interval Edge-Coloring of (3,4)-Biregular Bigraphs

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Joint work with
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The Problem

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Necessary condition: must have a proper edge-coloring with $\Delta(G)$ colors (Asratian-Kamalian [1994]).
More Specific Problem

**Def.** An \((a, b)\)-biregular \(X, Y\)-bigraph is a bipartite graph with degree \(a\) at vertices of \(X\) and degree \(b\) at vertices of \(Y\).
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Recognizing whether \((3, 6)\)-biregular bigraphs have
interval 6-colorings is NP-complete.
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**Thm. Pyatkin [2004]:** If a \((3, 4)\)-biregular \(X, Y\)-bigraph has a 3-regular subgraph covering \(Y\), then it has an interval \(6\)-coloring.
Open Problem: Does every (3, 4)-biregular bigraph have an interval coloring?

**Thm.** Pyatkin [2004]: If a (3, 4)-biregular $X, Y$-bigraph has a 3-regular subgraph covering $Y$, then it has an interval 6-coloring.

**Def.** A *proper path-factor* of a (3, 4)-biregular $X, Y$-bigraph is a spanning subgraph whose components are paths with ends in $X$ and lengths in \{2, 4, 6, 8\}. 
Still More Specific

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Our main result: If a \((3, 4)\)-biregular bigraph has a proper path-factor, then it has an interval 6-coloring.
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Neither result implies the other.
Proper Path Factors

Henceforth let $G$ be a $(3, 4)$-biregular $X, Y$-bigraph. Given a proper path-factor $P$ of $G$, let $Q = G - E(P)$. 
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Prop.  Every component of $Q$ is an even cycle or is a path with endpoints in $X$.

Pf.  Always $d_Q(y) = 2$ when $y \in Y$. Also $d_Q(x) = 2$ if $x$ is an endpoint of a component of $P$, while $d_Q(x) = 1$ if $x \in X$ and $x$ is an internal vertex of a component of $P$.  ■
The $P$-graph of $G$

**Def.** Given a proper path-factor $P$ of $G$, let $G_P$ be the graph with vertex set $\{x \in X: d_P(x) = 2\}$, with $x_i$ and $x_j$ adjacent when any condition below holds:

(a) $x_i$ and $x_j$ have degree 2 in one copy of $P_7$ in $P$, or
(b) $x_i$ and $x_j$ have degree 2 at distance 4 in one copy of $P_9$ in $P$, or
(c) $x_i$ and $x_j$ have degree 1 in one component of $Q$. 
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**Lem.** If $P$ is a proper path-factor, then $G_P$ is bipartite.

**Pf.** Every vertex of $G_P$ has one incident type (c) edge; some have another of type (a) or (b). Hence $\Delta(G_P) \leq 2$ and no odd cycle.
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**Pf.** Let $c$ be a proper 2-coloring of $G_P$, using $A$ and $B$. We will use colors $\{1, 2, 5, 6\}$ on $P$ and $\{3, 4\}$ on $Q$. 
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Color cycles in $Q$ arbitrarily using $3$ and $4$. A path in $Q$ has endpoints $x$ and $x'$ adjacent in $G_P$. If $c(x) = A$ and $c(x') = B$, alternate $3$ and $4$ starting with $3$ on the edge at $x$ and ending with $4$ on the edge at $x'$. Now $3$ and $4$ both appear at every vertex of $G$ with degree $2$ in $Q$. 
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Color cycles in $Q$ arbitrarily using 3 and 4. A path in $Q$ has endpoints $x$ and $x'$ adjacent in $G_P$. If $c(x) = A$ and $c(x') = B$, alternate 3 and 4 starting with 3 on the edge at $x$ and ending with 4 on the edge at $x'$. Now 3 and 4 both appear at every vertex of $G$ with degree 2 in $Q$.

On each path in $P$, alternate 2 and then 1 from one end, and alternate 5 and then 6 from the other. The choice of which end is which and when to switch from $\{2, 1\}$ to $\{5, 6\}$ uses the colors that $c$ assigns to the internal vertices. They have degree 1 in $Q$ and lie in $G_P$. 
Choosing Colors in \( P \)

A component \( H \) of \( P \) is a copy of \( P_3, P_5, P_7, \) or \( P_9 \). If \( x \in V(H) \) and \( c(x) = A \), use 1 and 2 at \( x \); if \( c(x) = B \), use 6 and 5. On \( P_7 \) and \( P_9 \), vertices two steps from the ends are adjacent in \( G_P \) and have distinct colors under \( c \).
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Each $Y$-vertex gets $\{3, 4\}$ on its edges in $Q$ and gets $\{2, 5\}$ or $\{1, 2\}$ or $\{5, 6\}$ on its edges in $P$: an interval.

Each leaf in $P$ gets $\{3, 4\}$ from $Q$ and 2 or 5 from $P$.

Each non-leaf in $P$ gets 3 from $Q$ and $\{1, 2\}$ from $P$ if $c(x) = A$, but 4 from $Q$ and $\{5, 6\}$ from $P$ if $c(x) = B$. 

Ex. The containment bigraph of the 3-sets and 2-sets in \{1, 2, 3, 4, 5, 6\} is a (3, 4)-biregular bigraph with a simple explicit $P_7$-factor (with 5-fold cyclic symmetry).
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**Ex.** \(K_{3,4}\) has a \(P_7\)-factor and satisfies Pyatkin’s condition (a “full” 3-regular subgraph). The graph below has a \(P_7\)-factor but has no full 3-regular subgraph.
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**Ex.** \( K_{3,4} \) has a \( P_7 \)-factor and satisfies Pyatkin’s condition (a “full” 3-regular subgraph). The graph below has a \( P_7 \)-factor but has no full 3-regular subgraph.

**Lem.** For \( i \in \{1, 2\} \), let \( G_i \) be 2-edge-connected with a \( P_7 \)-factor \( F_i \), and \( e_i \in E(G_i) - E(F_i) \). Form \( G \) from \( G_1 + G_2 \) by replacing \( e_1 \) and \( e_2 \) with other edges \( e'_1 \) and \( e'_2 \) joining their endpoints. If \( G_1 \) has no full 3-regular subgraph, then \( G \) is a larger such example.
Sufficient Conditions

**Thm.** A \((3, 4)\)-biregular \(X, Y\)-bigraph \(G\) has a \(P_7\)-factor if \(G\) has a \((2, 4)\)-biregular subgraph covering \(X\).
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**Thm.** A $(3, 4)$-biregular $X, Y$-bigraph $G$ has a $P_7$-factor if $G$ has a $(2, 4)$-biregular subgraph covering $X$.

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![Diagram](image)

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**Conj.** Every simple $(3, 4)$-biregular $X, Y$-bigraph has a proper path-factor.