Proper path-factors and interval edge-coloring of $(3,4)$-biregular bigraphs

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Abstract

An *interval coloring* of a graph G is a proper coloring of $E(G)$ by positive integers such that the colors on the edges incident to any vertex are consecutive. A $(3,4)$-*biregular bigraph* is a bipartite graph in which each vertex of one part has degree 3 and each vertex of the other has degree 4; it is unknown whether these all have interval colorings. We prove that G has an interval coloring using 6 colors when G is a $(3,4)$-biregular bigraph having a spanning subgraph whose components are paths with endpoints at 3-valent vertices and lengths in $\{2,4,6,8\}$. We provide several sufficient conditions for the existence of such a subgraph.

Keywords: path factor, interval edge-coloring, biregular bipartite graph
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1 Introduction

An *interval coloring* or *consecutive coloring* of a graph G is a proper coloring of the edges of G by positive integers such that the colors on the edges incident to any vertex are consecutive. The notion was introduced by Asratian and Kamalian [2] (available in English as [3]), motivated by the problem of constructing timetables without "gaps" for teachers and classes. Hansen [9] suggested another scenario: a school wishes to schedule parent-teacher conferences in time slots so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for the people and edges for the required meetings has an interval coloring.

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In the context of edge-colorings, and particularly edge-colorings of bipartite graphs, it is common to consider the general model in which multiple edges are allowed. In this paper, we adopt the convention that "graph" allows multiple edges, and we will explicitly exclude multiple edges when necessary (a *simple graph* is a graph without loops or multiple edges).

All regular bipartite graphs have interval colorings, since they decompose into perfect matchings. Not every graph has an interval coloring, since a graph G with an interval coloring must have a proper $\Delta(G)$-edge-coloring [3]. Furthermore, Sevastjanov [16] proved that determining whether a bipartite graph has an interval coloring is NP-complete. Nevertheless, trees [9, 13], complete bipartite graphs [9, 13], "doubly convex" bipartite graphs [13], grids [7], and simple outerplanar bipartite graphs [4, 8] all have interval colorings. Giaro [6] showed that one can decide in polynomial time whether bipartite graphs with maximum degree 4 have interval 4-colorings.

An (a, b)-*biregular bigraph* is a bipartite graph where the vertices in one part all have degree a and the vertices in the other part all have degree b. Hansen [9] proved that $(2, b)$-biregular bigraphs are interval colorable when b is even. This was extended to all b by Hanson, Loten, and Toft [11] and independently by Kostochka [14]. Kamalian [13] showed that the complete bipartite graph $K_{b,a}$ has an interval coloring using t colors if and only if $a + b - \gcd(a, b) \le t \le a + b - 1$, where gcd denotes the greatest common divisor. Asratian and Casselgren [1] showed that recognizing whether $(3, 6)$-biregular bigraphs have interval 6-colorings is NP-complete.

It is conjectured that all (a, b)-biregular bigraphs have interval colorings (see [12]); the first open case is $(a, b) = (3, 4)$. Hanson and Loten [10] proved that no (a, b)-biregular bigraph has an interval coloring with fewer than $a + b - \gcd(a, b)$ colors; thus $(3, 4)$-biregular bigraphs need at least 6 colors. An X, Y-*bigraph* is a bipartite graph with partite sets X and Y. In our $(3, 4)$-biregular X, Y-bigraphs, the vertices of X will have degree 3. Pyatkin [15] proved that if a $(3, 4)$-biregular bigraph has a 3-regular subgraph covering the vertices of degree 4, then it has an interval 6-coloring.

Here we obtain another sufficient condition for the existence of an interval 6-coloring of a $(3, 4)$-biregular X, Y-bigraph G: If G has a spanning subgraph whose components are paths with endpoints in X and lengths in $\{2, 4, 6, 8\}$ (we call this a *proper path-factor* of G), then G has an interval 6-coloring. A longer proof of this was found earlier by Casselgren [5].

We present infinitely many $(3, 4)$-biregular bigraphs that have proper path-factors but do not satisfy Pyatkin's condition. On the other hand, $(3, 4)$-biregular bigraphs with multiple edges need not have proper path-factors, even if they satisfy Pyatkin's condition. For example, consider the graph formed from three triple-edges by adding a claw; that is, the pairs $x_i y_i$ have multiplicity three for $i \in \{1, 2, 3\}$, and there is an additional vertex x_0 with neighborhood $\{y_1, y_2, y_3\}$. A 3-regular subgraph covers $\{y_1, y_2, y_3\}$, but there is no proper path-factor. Therefore, neither our result nor Pyatkin's result implies the other.

Forbidding multiple edges eliminates some difficulties. We conjecture that every simple $(3, 4)$-biregular bigraph has a proper path-factor and hence also an interval 6-coloring. In Section 3 we obtain several sufficient conditions for the existence of a proper path-factor in such a graph.

2 Interval 6-Colorings from Proper Path-Factors

In general, an $\mathcal{H}$-factor of a graph is a spanning subgraph whose components lie in $\mathcal{H}$. We are interested in a particular family $\mathcal{H}$. Let $d_H(v)$ denote the degree of a vertex v in a graph H.

Definition 1. A *proper path-factor* of a $(3,4)$-biregular X, Y-bigraph G is a spanning subgraph of G whose components are paths with endpoints in X and lengths in $\{2,4,6,8\}$.

Henceforth let G be a $(3,4)$-biregular X, Y-bigraph. Given a proper path-factor P of G, let $Q = G - E(P)$. Note that $d_Q(v) = 2$ if $v \in Y$ or if v is an endpoint of a component of P, and otherwise $d_Q(v) = 1$. Thus every component of Q is an even cycle or is a path with endpoints in X.

Definition 2. Given a proper path-factor P of G, the *P-graph of G*, denoted G_P, is the graph with vertices $\{x \in X : d_P(x) = 2\}$ having x_i and x_j adjacent when any condition below holds:
  (a) x_i and x_j are vertices of degree 2 in one component of P with length 6, or
  (b) x_i and x_j are vertices of degree 2 at distance 4 in one component of P with length 8, or
  (c) x_i and x_j are the endpoints of one component of Q.

Lemma 3. *If P is a proper path-factor of G, then G_P is bipartite.*

Proof. Every vertex of G_P has exactly one incident edge of type (c). Some vertices have one more neighbor, via an edge of type (a) or (b). Thus $\Delta(G_P) \leq 2$. Furthermore, the edges along any path or cycle in G_P alternate type (c) with type (a) or (b). Thus G_P has no odd cycle. □

We say that a color appears "at" a vertex if it appears on an edge incident to that vertex.

Theorem 4. *If G has a proper path-factor, then G has an interval 6-coloring.*

Proof. Let P be a proper path-factor of G. Let c be a proper 2-coloring of G_P using colors A and B on $V(G_P)$. We define a 6-coloring of $E(G)$ that we will show is an interval coloring. Edges of P receive colors from $\{1,2,5,6\}$; edges of Q receive colors from $\{3,4\}$.

First we color $E(Q)$. Properly color cycles arbitrarily using colors 3 and 4. A component of Q that is a path has both endpoints in G_P, and they are adjacent in G_P. Hence $c(x) = A$ for one endpoint x of the path, and $c(x') = B$ for the other endpoint x'. Alternate colors along the path, starting with color 3 on the edge at x and ending with color 4 on the edge at x'. Colors 3 and 4 both now appear at every vertex of G having degree 2 in Q.

The edges of every component of P are colored by alternating 2 and 1 (starting with 2) from one end, and alternating 5 and 6 (starting with 5) from the other end. We use the colors assigned by c to internal vertices of the path to specify which end is which and where to switch from colors $\{2,1\}$ to $\{6,5\}$. See Figure 1.

Let H be a component of P. If $H \cong P_3$, then assign 2 and 5 to the edges arbitrarily. If $H \cong P_5$ with middle vertex x, then assign 2 and 5 to the end edges arbitrarily, but give the middle edges
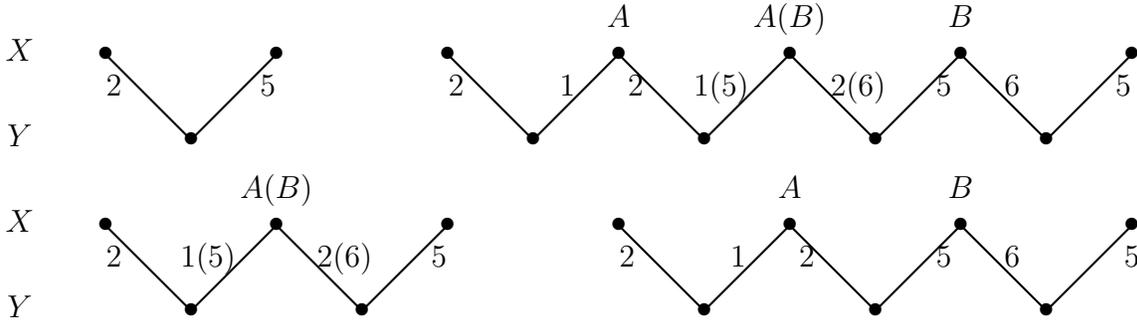
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Figure 1: Coloring the edges of P

colors 1 and 2 if $c(x) = A$, 5 and 6 if $c(x) = B$. If $H \cong P_7$, then the internal vertices are adjacent in G_P and receive distinct colors under c; use $2, 1, 2$ from the end closest to the one colored A and $5, 6, 5$ from the end closest to the one colored B. If $H \cong P_9$, then the internal vertices at distance 4 on the path again are adjacent in G_P, and the three edges from each end are colored in the same way as for P_7. The two central edges are colored like the middle edges of P_5, based on the color under c of the central vertex of the path.

The resulting 6-edge-coloring is an interval coloring. Each vertex of Y has colors 3 and 4 on its incident edges in Q and receives $\{2, 5\}$ or $\{1, 2\}$ or $\{5, 6\}$ on its incident edges in P, forming an interval in each case. Each endpoint of a component of P has colors 3 and 4 from Q and receives color 2 or 5 from P. Each internal vertex x of a component of P receives $\{1, 2, 3\}$ if $c(x) = A$ and $\{4, 5, 6\}$ if $c(x) = B$. $\qquad\qquad\square$

This technique does not extend to arbitrary path and cycle factors. A path in P switches once from $1, 2$-alternation to $5, 6$-alternation and cannot switch back. Thus along any path of P, all internal vertices with color A under c precede those with color B. With longer paths, we have no mechanism for achieving this; the graph G_P can only enforce that vertices receive different colors under c. Introducing more edges into G_P to prevent alternation of A and B along paths destroys 2-colorability of G_P.

3   Constructions and Conditions for Proper Path-Factors

To apply the theorem, we seek proper path-factors of $(3, 4)$-biregular bigraphs. We give some sufficient conditions for their existence and provide some examples related to Pyatkin's condition.

We call a 3-regular subgraph of a $(3, 4)$-biregular bigraph that covers the vertices of degree 4 a *full 3-regular subgraph*. Pyatkin proved that a $(3, 4)$-biregular bigraph with a full 3-regular subgraph has an interval 6-coloring. Our first example satisfies our condition but not Pyatkin's.

Example 1. *The X, Y-bigraph G defined by letting X and Y consist of the 3-sets and 2-sets in $\{1, 2, 3, 4, 5, 6\}$, respectively with adjacency defined by proper containment, has an interval 6-coloring.* By Theorem 4, it suffices to find a proper path-factor. In fact, G has a P_7-factor as shown below.

$$124 \rightarrow 12 \rightarrow 123 \rightarrow 23 \rightarrow 235 \rightarrow 35 \rightarrow 345$$
$$135 \rightarrow 13 \rightarrow 134 \rightarrow 34 \rightarrow 346 \rightarrow 46 \rightarrow 456$$
$$146 \rightarrow 14 \rightarrow 145 \rightarrow 45 \rightarrow 245 \rightarrow 25 \rightarrow 256$$
$$125 \rightarrow 15 \rightarrow 156 \rightarrow 56 \rightarrow 356 \rightarrow 36 \rightarrow 236$$
$$136 \rightarrow 16 \rightarrow 126 \rightarrow 26 \rightarrow 246 \rightarrow 24 \rightarrow 234$$

Here $|X| = 20$ and $|Y| = 15$, with Y corresponding to the edge set of K_6. The neighborhood of a vertex in X corresponds to a triangle in K_6. Hence five vertices can be deleted from G to leave a full 3-regular subgraph if and only if K_6 decomposes into five triangles. It does not, because the vertices of K_6 have odd degree. $\qquad\square$

We next construct infinitely many examples that satisfy our condition but not Pyatkin's, starting with a graph smaller than that of Example 1.

Example 2. The smallest simple $(3, 4)$-biregular bigraph is $K_{3,4}$; it satisfies Pyatkin's condition. The next smallest such graphs have eight vertices of degree 3 and six of degree 4. For example, consider an X, Y-bigraph where $Y = \{1, 2, 3, 4, 5, 6\}$ and the neighborhoods of the vertices in X are eight triples from $\{1, 2, 3, 4, 5, 6\}$, with each element used in four triples. The graph fails Pyatkin's condition if and only if the triple system does not have two disjoint triples.

Case analysis shows that it is not possible to avoid two disjoint triples without a repeated triple. However, it is possible using a repeated triple, as in $\{123, 124, 235, 346, 346, 145, 156, 256\}$. The resulting $(3, 4)$-biregular bigraph has a P_7-factor as shown in bold in Figure 2. $\qquad\square$
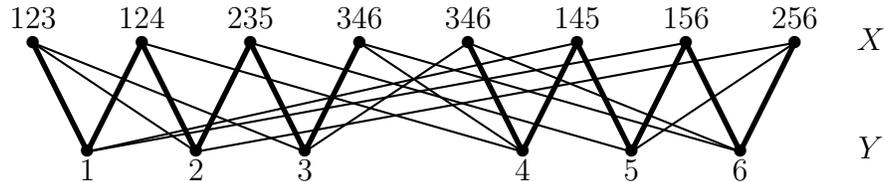

Figure 2: P_7-factor in a bigraph with no full 3-regular subgraph

Using the next lemma, we can generate infinitely many examples that have P_7-factors but have no full 3-regular subgraphs. The number of vertices can be any nontrivial multiple of 7. Here multiple edges are allowed.

Lemma 5. *For $i \in \{1, 2\}$, let G_i be a 2-edge-connected $(3, 4)$-biregular bigraph having a P_7-factor F_i, and choose $e_i \in E(G_i) - E(F_i)$. Let G be the $(3, 4)$-biregular bigraph obtained from the disjoint union of G_1 and G_2 by deleting e_1 and e_2 and replacing them with two other edges e_1' and e_2'*

joining their endpoints. If G_1 has no full 3-regular subgraph, then G is a larger 2-edge-connected $(3, 4)$-biregular bigraph having a P_7-factor but no full 3-regular subgraph.

Proof. Since $e_i \notin E(F_i)$, the subgraph $F_1 \cup F_2$ is a P_7-factor of G. Since each G_i is 2-edge-connected, G is connected. Also, a cycle through e_i in G_i can detour through G_{3-i} using e'_1 and e'_2. Thus G is 2-edge-connected.

A full 3-regular subgraph H of G must have an even number of edges in $\{e'_1, e'_2\}$, since a 3-regular bipartite graph has no cut-edge. If H uses neither e'_1 nor e'_2, then H restricts to full 3-regular subgraphs of G_1 and G_2. If H uses both, then replacing e'_1 and e'_2 with e_1 and e_2 yields full 3-regular subgraphs of G_1 and G_2. Hence H cannot exist. $\qquad\square$

If G_1 and G_2 in Lemma 5 have no multiple edges, then neither does the resulting graph G.

We turn now to sufficient conditions for proper path-factors in $(3, 4)$-biregular bigraphs. First we note an easy lemma.

Lemma 6. *Every $(2, 4)$-biregular bigraph H has a $(1, 2)$-biregular factor with every component isomorphic to P_3. (Indeed, H decomposes into two such factors.)*

Proof. Each component of H is Eulerian and has an even number of edges. Taking the even-indexed edges from an Eulerian circuit in each component takes half the edges from each vertex. Thus it yields a spanning subgraph in which every vertex of one partite set has degree 1 and every vertex of the other has degree 2. Hence each component of the subgraph is isomorphic to P_3. $\qquad\square$

Theorem 7. *A $(3, 4)$-biregular bigraph G has a P_7-factor (and hence an interval 6-coloring) if G has a $(2, 4)$-biregular subgraph covering the set of vertices of degree 3.*

Proof. Let G have bipartition (X, Y), where $|X| = 4k$ and $|Y| = 3k$. Let H be a $(2, 4)$-biregular subgraph of G covering X; we obtain H from G by deleting vertices $u_1, \ldots, u_k$ of Y that have disjoint neighborhoods. Let $\hat{Y} = \{u_1, \ldots, u_k\}$ and $Y' = Y - \hat{Y}$, so H has bipartition (X, Y').

By Lemma 6, H has a spanning subgraph F whose components are copies of P_3 with endpoints in X. Let $T_1, \ldots, T_{2k}$ be these paths. Index X so that $V(T_i) = \{x_{2i-1}, y_i, x_{2i}\}$ (we maintain the flexibility to decide later which end is x_{2i-1} and which is x_{2i}).

Next we obtain from $G - Y'$ a graph H' by merging the two endpoints of each path T_i into a single vertex x'_i (note that T_i is not contained in $G - Y'$, but its endpoints are). Since $G - Y'$ is a $(1, 4)$-biregular $X, \hat{Y}$-bigraph, H' is a $(2, 4)$-biregular $X', \hat{Y}$-bigraph, where $X' = \{x'_1, \ldots, x'_{2k}\}$. Note that multiple edges may arise in H'.

For each of the k vertices of $\hat{Y}$, we construct a path of length 6 in G with endpoints in X. By Lemma 6, H' has a spanning subgraph F' whose components are copies of P_3 with endpoints in X'. For $u \in \hat{Y}$, let x'_i and x'_j be the neighbors of u in F'. Thus in G the vertex u is adjacent to one endpoint of T_i and one endpoint of T_j. We may complete the indexing of X so that

these neighbors of u are $x_{2i} \in V(T_i)$ and $x_{2j-1} \in V(T_j)$. The path we associate with u is then $\langle x_{2j-1}, y_i, x_{2i}, u, x_{2j-1}, y_j, x_{2j} \rangle$, isomorphic to P_7.

We check that these paths are pairwise disjoint. Each uses exactly one vertex of $\hat{Y}$. Since F' has exactly one edge incident to each vertex of X', for each i the vertices of T_i occur in exactly one of the paths. Hence these paths form a P_7-factor, and Theorem 4 applies. $\qquad \square$

We now return to simple $(3, 4)$-biregular bigraphs. Although the examples constructed so far in this section all have P_7-factors, Casselgren [5] found a simple $(3, 4)$-biregular bigraph with no P_7-factor. We conjecture that every simple $(3, 4)$-biregular bigraph has the weaker property of having a proper path-factor. It should also hold that Pyatkin's condition guarantees the existence of a proper path-factor in a simple $(3, 4)$-biregular bigraph, but this also seems difficult. We present a condition that guarantees a proper path-factor when combined with Pyatkin's condition.

Let G be a simple $(3, 4)$-biregular X, Y-bigraph having a full 3-regular subgraph H. Since $|X| = 4k$ and $|Y| = 3k$ for some k, we may let $X' = X \cap V(H)$ and $X_0 = X - X'$, where $X_0 = \{x_1^0, \ldots, x_k^0\}$. Since H is 3-regular, H has a proper 3-edge coloring. Fix such a coloring c, and let H' be the spanning subgraph of H whose edges are those with color 1 or 2 under c. Define an auxiliary digraph F with vertex set Y by putting $y_i y_j \in E(F)$ if H' has a y_i, y_j-path of length 2 via a vertex x such that $c(y_i x) = 1$ and $c(xy_2) = 2$. Note that F is a union of disjoint cycles, since the two color classes in H' are perfect matchings.

Since G is $(3, 4)$-biregular, the neighborhoods of the vertices of X_0 partition Y into triples; let $T_i = N_G(x_i^0) = \{y_i^1, y_i^2, y_i^3\}$. Let $\mathbf{T}$ denote the family $T_1, \ldots, T_k$.

Definition 8. For a family of disjoint triples, we define a *transversal* to be a set S having exactly one element from each triple. For the family $\mathbf{T}$ defined above on the vertex set Y of the digraph F, an *independent transversal* is a transversal S that is an independent set in F. A *spread transversal* is a transversal S such that, for every vertex v of F that does not belong to S, there is a vertex of S among the next three vertices after v along its cycle in F. Let F^* be the 4-regular graph obtained from the underlying undirected graph of F by adding triangles whose vertex sets are the triples of $\mathbf{T}$. A *mixed transversal* is a transversal that restricts on each component of F^* to an independent transversal or a spread transversal (with respect to F).

Note that a spread transversal on a component of F^* intersects each cycle of F having vertices in that component.

Theorem 9. *Let G be a simple $(3, 4)$-biregular X, Y-bigraph having a full 3-regular subgraph H, and let F and $\mathbf{T}$ be the digraph and triple system defined as above. If $\mathbf{T}$ has a mixed transversal, then G has a proper path-factor.*

Proof. Let c be a proper 3-edge-coloring of H, and let M be the perfect matching in color 1. The ith triple in $\mathbf{T}$ is $\{y_i^1, y_i^2, y_i^3\}$; we may let y_i^1 be the vertex of T_i in the mixed transversal Y_1. Let

$Y_2 = \{y_1^2, \dots, y_k^2\}$ and $Y_3 = \{y_1^3, \dots, y_k^3\}$. For $x \in X' = X - X_0$, put $x \in X_j$ if the other endpoint of the edge of M at x lies in Y_j, and write x as x_i^j if that neighboring vertex is y_i^j. Since each vertex of Y has one neighbor via M, we have labeled X so that $X' = \{x_i^j \colon 1 \le i \le k \text{ and } 1 \le j \le 3\}$.

We construct a proper path-factor of G, dealing separately with each component C of F^*. From C we generate paths in G that together cover $V(C)$, the neighbors of $V(C)$ via M, and the vertices of X_0 whose neighborhoods lie in $V(C)$. The construction depends on whether the restriction of $\mathbf{T}$ to C has an independent transversal or a spread transversal. For simplicity of notation, we describe the construction in the case that F^* is connected. In the general case, $V(C)$ is the union of T_i for i in some subset of $\{1, \dots, k\}$, and the construction in Case 1 or Case 2 covers all vertices in $T_i \cup \{x_i^0, x_i^1, x_i^2, x_i^3\}$ for each such index i.

Case 1: Y_1 *is an independent transversal.* We specify k paths of lengths 4, 6, or 8, each containing one vertex of X_0. Consider the paths $\langle x_i^2, y_i^2, x_i^0, y_i^3, x_i^3 \rangle$ for $1 \le i \le k$. These paths are disjoint and cover $V(G) - (X_1 \cup Y_1)$. The $2k$ endpoints of these paths form $X_2 \cup X_3$. Each vertex y_i^1 of Y_1 has one incident edge $y_i^1 x$ having color 2 under c. Since Y_1 is an independent transversal, this neighbor x lies in $X_2 \cup X_3$, not in X_1. Extend the original path of length 4 ending at x by adding xy_i^1 and $y_i^1 x_i^1$. Altogether there are k such extensions to absorb $Y_1 \cup X_1$. Each of the original paths extends by zero or two edges at each end, so we have the factor using paths of the desired lengths.

Case 2: Y_1 *is a spread transversal.* Again each path contains one vertex of X_0, but now we may also use length 2. For each cycle in F, delete the incoming edge to each vertex of Y_1. Since Y_1 is a spread transversal, what remains of F consists of disjoint paths $P_1, \dots, P_k$, starting at $y_1^1, \dots, y_k^1$, respectively, each with length at most 3. By the definition of F, each edge in P_i expands to a path of length 2 in G having edges of colors 1 and 2 under c, yielding paths of even length (at most 6) ending in $Y_2 \cup Y_3$. When such a path ends at y, the neighbor of y in M is not covered by any of these paths. Thus extending each path P_i by adding $x_i^0 y_i^1$ at the beginning and the edge yx of M at the end yields k disjoint paths of lengths in $\{2, 4, 6, 8\}$ that cover $V(G)$. □

This method for finding proper factors is robust, since any proper 3-edge-coloring of H and any indexing of its colors can be used. Care is needed, since there exist digraphs F and triple systems $\mathbf{T}$ where no mixed transversal exists, as shown in our final example.

Example 3. First we construct F_1 with no independent transversal. Let k_1 be a multiple of 6, and let F_1 consist of $k_1/2$ cycles of length 4 and $k_1/3$ cycles of length 3. Name the 4-cycles as $[y_{2i-1}^1, y_{2i}^1, y_{2i-1}^2, y_{2i}^2]$ for $1 \le i \le k_1/2$. Name the 3-cycles as $[y_{6i-3}^3, y_{6i-1}^3, y_{6i+1}^3]$ and $[y_{6i-4}^3, y_{6i-2}^3, y_{6i}^3]$ for $1 \le i \le k_1/6$ (with $y_{k_1+1} = y_1$). An independent partial transversal has at most one vertex in each cycle, and hence the largest independent partial transversal has at most $k_1/2 + k_1/3$ elements.

Next we construct F_2 with no spread transversal; for clarity, we use vertices z_i^j instead of y_i^j. Let k_2 be a multiple of 2, and let F_2 consist of $3k_2/2$ cycles of length 2. Name the 2-cycles as $[z_i^1, z_{i+1}^2]$ for $1 \le i \le k_2$ (where $z_{k+1} = z_1$) and $[z_{2i-1}^3, z_{2i}^3]$ for $1 \le i \le k_2/2$. A transversal has only k_2 elements, but a spread transversal must have an element in each of the $3k_2/2$ cycles.

Both F_1^* and F_2^* are connected graphs. To construct an example with no mixed transversal, we start with disjoint copies of F_1 and F_2. For a small example where the argument holds, take $k_1 = 12$ and $k_2 = 4$. Change the name of y_1^1 in F_1 to z_1^1, and vice versa for z_1^1 in F_2. This modifies two triples to create a new graph F such that F^* is connected; hence a mixed transversal must be an independent transversal or a spread transversal. In $V(F_2)$ there are three full triples plus $\{y_1^1, z_1^2, z_1^3\}$. Hence a transversal can hit at most five of the size 2-cycles in F_2 and thus cannot be spread. On the other hand, at most 10 of the 11 triples entirely contained in F_1 can contribute to an independent transversal, since each 3-cycle and 4-cycle contributes at most once for these triples. We conclude that there is no mixed transversal. $\qquad \square$

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