PARSIMONIOUS 2-MULTIGRAPHS

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Abstract

A 2-multigraph is a loopless multigraph with maximum multiplicity 2; pairs of vertices induce 0, 1, or 2 edges. A 2-multigraph is *parsimonious* if it has the minimum number of single edges (multiplicity 1) among all 2-multigraphs with the same degree sequence. In every parsimonious 2-multigraph, the subgraph of single edges consists of isolated stars and possibly one component that is a triangle. We prove the conjecture of Brualdi and Michael that for any fixed degree sequence, either every parsimonious 2-multigraph with those vertex degrees has a triangle of single edges, or no such parsimonious 2-multigraph has a triangle of single edges.

1 Introduction

A p-multigraph is a loopless multigraph with maximum multiplicity p. We can view a p-multigraph as a weighted complete graph where edges have weight from 0 to p. Chungphaisan [3] gave necessary and sufficient conditions for realizing a given degree sequence by a p-multigraph, extending the well known theorem of Erdös and Gallai [4] regarding simple graphs. Here we consider realizations using the minimum number of distinct edges, i.e. edges with non-zero multiplicity. Such realizations are called *parsimonious.*

The problem of finding a parsimonious realization of a degree sequence $d_1, d_2, \ldots, d_n$ can be reformulated as an integer program. For

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each edge e in K_n, define $0,1$-variables $e_1, \ldots, e_p$, and impose the constraint $\sum_{i=1}^{p} e_i \leq 1$. The constraint ensures that either all the associated variables are 0, or exactly one variable, e_m, equals 1, which will indicate that edge e has weight m. For each vertex v_j, we require $\sum_{e \in F} \sum_{i=1}^{p} i e_i = d_j$, where F is the set of edges incident to v_j. The answer is obtained by minimizing $\sum_{e \in E(K_n)} \sum_{i=1}^{p} e_i$ subject to these constraints. The problem of recognizing parsimonious realizations of an input degree sequence may be NP-hard, so as a first step we investigate the structure of parsimonious 2-multigraphs.

For 2-multigraphs, we refer to an edge with weight (or multiplicity) 1 as a "single edge" and an edge with weight 2 as a "double edge". In addition "missing edges" are those with weight or (multiplicity) 0. The choice of weight 0, 1, or 2 for the pair $\{u, v\}$ is denoted by $u \not\leftrightarrow v$, $u \leftrightarrow v$, or $u \Leftrightarrow v$, respectively. In the illustrations, these are indicated by a dashed line, a solid line, and a double line, respectively. Note that minimizing the total number of edges in a 2-multigraph is equivalent to minimizing the number of single edges. Given a 2-multigraph G, let $\overline{G}$ denote the relative complement of G with respect to a complete 2-multigraph. That is, an edge with weight w in G has weight $2-w$ in $\overline{G}$. When we speak of "improving" G, we mean reducing the number of single edges without changing the vertex degrees.

It is not hard to prove that the subgraph formed by the single edges in a parsimonious 2-multigraph consists of disjoint stars and possibly one component that is a triangle. Brualdi and Michael [2] proved several invariant properties for parsimonious 2-multigraphs, and they conjectured [2,5] that for a given degree sequence, either all parsimonious 2-multigraphs have a triangle of single edges, or none do. The main result of this paper is a proof of this conjecture.

2 Stars and A Triangle

In this section our goal is to prove that the subgraph of single edges in any parsimonious 2-multigraph consists of isolated stars and possibly one triangle. These results were known to Brualdi and Michael, but since they are short and necessary lemmas for the main result, we include them for completeness. The first theorem permits the simplification of later arguments by invocation of symmetry.

Theorem 1 *A 2-multigraph G is parsimonious if and only if $\overline{G}$ is parsimonious.*

Proof: The important observation is that G and $\overline{G}$ have the same number of single edges. Suppose G is a parsimonious 2-multigraph with t single edges. If $\overline{G}$ is not parsimonious, then the degree sequence of $\overline{G}$ can be realized by a 2-multigraph H with fewer than t single edges. But then $\overline{H}$ realizes the degree sequence of G with fewer than t single edges, contradicting the assumption that G is parsimonious. □

Lemma 2 *If G is a parsimonious 2-multigraph, then G contains no 4-vertex path of single edges.*

Proof: Suppose $a \leftrightarrow b \leftrightarrow c \leftrightarrow d$ is such a path. For each possible weight for $\{a, b\}$, we can improve G, contradicting the assumption that G is parsimonious. If $a \not\leftrightarrow d$, then put $a \not\leftrightarrow b \Leftrightarrow c \not\leftrightarrow d \leftrightarrow a$. If $a \leftrightarrow d$, then put $a \Leftrightarrow b \not\leftrightarrow c \Leftrightarrow d \not\leftrightarrow a$. If $a \Leftrightarrow d$, then put $a \Leftrightarrow b \not\leftrightarrow c \Leftrightarrow d \leftrightarrow a$. These possibilities are illustrated in Fig. 1. Alternatively, for the third case we could invoke Theorem 1 and the first case.

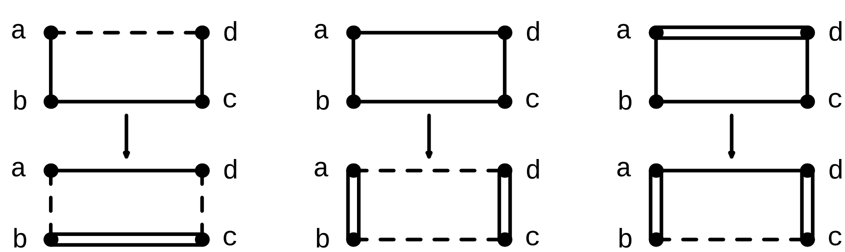

Fig. 1.

□

This implies that the single edges in a parsimonious 2-multigraph consist of isolated stars or triangles. The next lemma will imply that a parsimonious 2-multigraph contains at most one triangle of single edges.

Lemma 3 *If G is a parsimonious 2-multigraph containing a triangle of single edges and e is any other single edge in G, then one vertex of e has a double edge to each vertex of the triangle, and the other vertex of e has no edges to the triangle.*

Proof: Let $T = \{a, b, c\}$ induce a triangle of single edges in G, and let uv be any other single edge in G. First note that lemma 2 forces $\{u, v\}$ to be disjoint from T and to have no single edges to T. Next suppose u and v both have double edges to some vertex of T, say c. Then we can improve G by putting $a \not\leftrightarrow b \Leftrightarrow c \leftrightarrow v \Leftrightarrow u \leftrightarrow c \Leftrightarrow a$, as indicated on the left in Fig. 2. Hence u, v cannot both have double edges to a vertex of T, and by Theorem 1 they also cannot both have missing edges to a vertex of T, so each vertex of T has a double edge to one of u, v and a missing edge to the other. We need only show that u and v cannot have double edges to different vertices of the triangle. If $b \Leftrightarrow u$ and $c \Leftrightarrow v$, for example, then we must also have $b \not\leftrightarrow v$, since u and v cannot both have double edges to b. But now we improve G by putting $v \leftrightarrow b \leftrightarrow u \Leftrightarrow v \not\leftrightarrow c \Leftrightarrow a \not\leftrightarrow b \Leftrightarrow c$, as indicated on the right in Fig. 2.

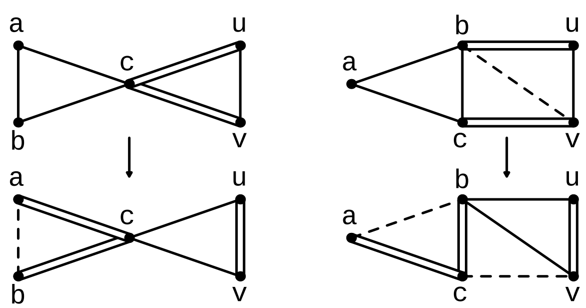

Fig. 2.

□

Corollary 4 *If G is parsimonious, then G contains at most one triangle of single edges.*

Proof: Let uv be an edge of one such triangle, and let xy be an edge of a second. Using the previous lemma, one of $\{u, v\}$ must have double edges to all the vertices of the second triangle. But on the other hand, the lemma implies that only one of $\{x, y\}$ has double edges to any vertices of the first triangle. □

3 Always a Triangle or Never a Triangle

The facts established thus far give a foundation for the proof of the conjecture.

Conjecture 5 *(Brualdi-Michael [2]). If a parsimonious* 2*-multigraph contains a triangle of single edges, then any parsimonious* 2*-multigraph with the same degree sequence contains a triangle of single edges.*

In the remainder of this section, we prove the following theorem, which by Lemma 2 establishes the conjecture, since cycles of length at least four contain 4-vertex paths.

Theorem 6 *If a parsimonious* 2*-multigraph contains a triangle of single edges, then any* 2*-multigraph with the same degree sequence contains a cycle of single edges.*

Note that this theorem applies to *all* 2-multigraphs with this degree sequence, not only the parsimonious ones.

We need to discuss the contribution to vertex degrees from the edges within a set of vertices or the edges between two sets of vertices. If F is a set of d double edges and s single edges, then we write $h(F) = 4d + 2s$. For $S \subset V(G)$, we write $h(S) = h(F)$, where F is the set of edges induced by S. For disjoint $S, T \subset V(G)$, we write $h(S, T) = h([S, T])/2$, where $[S, T]$ is the set of edges with one endpoint in S and one endpoint in T. Note that the sum of the vertex degrees in S is $h(S) + h(S, \overline{S})$. When $S = \{v\}$, we write $h(v, T)$ for $h(S, T)$.

For the remainder of this section, let G be a parsimonious 2-multigraph containing a triangle of single edges on the vertices of $T = \{a, b, c\}$. We will use T to partition the vertices into sets of vertices that have "similar" neighborhoods. Let U be the set of vertices outside T that are incident to single edges in G. By Lemma 3, $h(v, T) \in \{0, 6\}$ for any $v \in U$. Let $K = \{v \in U : h(v, T) = 6\}$ and $I = \{v \in U : h(v, T) = 0\}$; note that $K \cap I = \emptyset$. These definitions establish the following structure: Any vertex in K has a single edge to some vertex in I and a double edge to each vertex of T. Any vertex in I has a single edge to some vertex in K and no edges to T. Moreover, the only single edges in G are those within T and those with one vertex in K and the other in I.

Theorem 7 *The vertices of K form a clique of double edges, and the vertices of I are an independent set.*

Proof: In $\overline{G}$ the sets K and I are interchanged, so by Theorem 1 it suffices to show that K is a clique of double edges; Lemma 3 prevents single edges. Choose $x, x' \in K$, and suppose $x \not\leftrightarrow x'$. By construction, we may choose $y, y' \in I$ such that $y \leftrightarrow x$ and $y' \leftrightarrow x'$, where we may have $y = y'$ or $y \neq y'$. By the definition of I, we have $y \not\leftrightarrow c$ and $y' \not\leftrightarrow c$. In either case we improve G as indicated in Fig.3, the only difference in the reassignment being the fate of c with $\{y, y'\}$. We put $a \Leftrightarrow b \not\leftrightarrow c \not\leftrightarrow a$ and $y \not\leftrightarrow x \leftrightarrow x' \not\leftrightarrow y'$. If $y = y'$ we put $c \Leftrightarrow y$, but if $y \neq y'$ we put $c \leftrightarrow y$ and $c \leftrightarrow y'$.

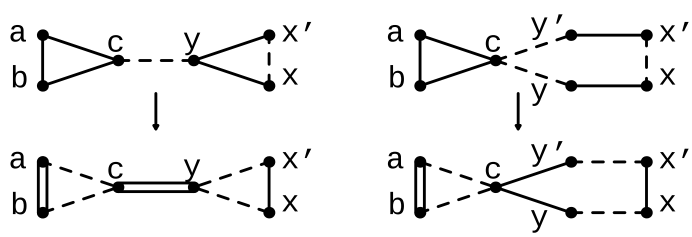

Fig. 3.

□

We will partition the vertices not incident to single edges according to their "reachability" from K and I by alternating trails of particular types. In this context, an *alternating trail* is a sequence of vertices such that successive pairs of vertices alternate between missing edges and double edges of G. A vertex $v \in V(G) \setminus (K \cup I)$ is *reachable* if v is the end of an alternating trail that starts from K along a missing edge or from I along a double edge. We say that v is "reachable on a double edge" or "reachable on a missing edge" if the last pair of some alternating trail reaching it is a double edge or missing edge, respectively.

Lemma 8 *Vertices of T are not reachable.*

Proof: Suppose $c \in T$ is reachable. By Theorem 1, we may assume that c is reached by an alternating trail P from $x \in K$. If we replace missing edges by double edges and double edges by missing edges on P, then vertex degrees remain unchanged, except that $d(x)$ increases by two and $d(c)$ increases or decreases by two according to whether c was reached on a missing edge or double edge, respectively. In either case we can improve G, as indicated in Fig. 4. Choose $y \in I$ such that $y \leftrightarrow x$. By the definition of K and I, we have $y \not\leftrightarrow c$ and $x \Leftrightarrow c$. In addition to switching missing and double edges along the x, c-trail, make yx a missing edge and cx, cy single edges. Finally, if c was reached on a double edge put $a \Leftrightarrow c \Leftrightarrow b \not\leftrightarrow a$, and if c was reached on a missing edge put $a \not\leftrightarrow c \not\leftrightarrow b \Leftrightarrow a$.

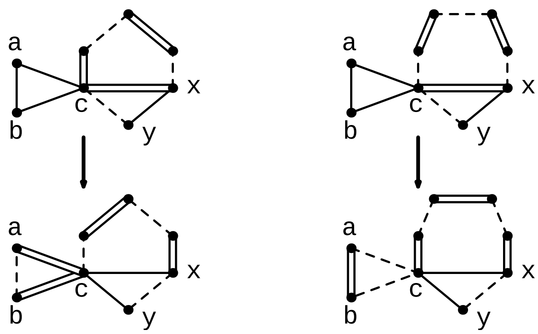

Fig. 4.

□

We now define a partition of the vertex set. Let N be the set of vertices that are not reachable; note that $T \subseteq N$. Let K' be K together with the vertices reachable on a double edge. Let I' be I together with the vertices reachable on a missing edge.

Corollary 9 *The sets K', I', N form a partition of $V(G)$.*

Proof: Since T is not reachable, every vertex in K' has a double edge to every vertex of T, and every vertex in I' has no edge to T. Hence K', I' are disjoint, and $\{K', I', N\}$ partitions $V(G)$. □

Lemma 10 *If $x \in K'$, $y \in I'$, and $z \in N$, then $x \Leftrightarrow z$ and $y \not\leftrightarrow z$. Furthermore, K' induces a clique of double edges and I' is an independent set.*

Proof: In $\overline{G}$ the sets K' and I' are interchanged, so by Theorem 1 it suffices to prove $z \Leftrightarrow x \Leftrightarrow x'$, where $x, x' \in K'$ and $z \in N$. By the construction of the partition, neither pair can be a single edge. If $x \not\leftrightarrow z$, then z is reachable, contradicting the definition of N. If $x \not\leftrightarrow x'$, then x' is also reachable on a missing edge, contradicting Corollary 9. □

Finally, we are ready to prove our main theorem.

Theorem 11 *If a parsimonious 2-multigraph G contains a triangle of single edges, then every 2-multigraph with the same vertex degrees contains a cycle of single edges.*

Proof: Consider the partition N, K', I' obtained above for $V(G)$. By Lemma 10, we have $h(K') = 4\binom{|K'|}{2}$, $h(I') = 0$, $h(K', N) = 2|K'||N|$, and $h(I', N) = 0$. Let $p = h(K')$, $q = h(K', N)$, $s = h(I', K')$, and $t = h(N)$. The totals of the vertex degrees of vertices in I', K', N are s, $s + p + q$, and $t + q$, respectively. Let H be another (labeled) 2-multigraph with the same vertex degrees as G. Let h' denote the function h applied to the graph H.

We claim that $h'(N) = t$, $h'(I', N) = 0$, and $h'(K', N) = q$. To see that this claim completes the proof, let G', H' be the subgraphs of G, H induced by N. The claimed values of h' imply that G' and H' have the same number of edges and the same vertex degrees. The only single edges induced by N in G are the edges of the triangle T; hence G' has an odd number of edges but no vertex of odd degree. We conclude that H' also has an odd number of edges and hence at least one single edge. Since it also has no vertex of odd degree, its single edges must form a non-empty collection of disjoint cycles.

To prove the claim for h', note that the total degree in H of the vertices in the sets I', K', N is the same as in G. Applying this for I' yields $h'(I', K') \leq s$. Since K' induces a double clique in G, we have

$h'(K') \leq p$. Similarly, the existence of double edges in G between all of K' and N forces $h'(K', N) \leq q$. Since the total degree of vertices in K' remains at $s + p + q$, equality holds in these three relations. We have $h'(I', N) = 0$ because the edges to K' have absorbed all the edges involving I'. Together with $h'(K', N) = q$ and the constancy of total degree in N, this implies $h'(N) = t$. □

4 Algorithmic Aspects

We return now to the problem of constructing a parsimonious multigraph from a given degree sequence. The results in this paper give a modest start to a possible algorithm. If a degree sequence is realizable by a 2-multigraph, techniques in [1], provide an efficient means of obtaining such a representation. We can then apply the previous reductions until the graph no longer contains any 4-vertex path of single edges and at most one triangle of single edges remains. The reductions can be done efficiently since the number of single edges decreases by at least two for each reduction, and finding each reduction takes low-order polynomial time. If these reductions remove all triangles, then the results of this paper don't help any further; we are not guaranteed to have a parsimonious graph. If, however a triangle remains, we can continue making reductions until the triangle is destroyed or until the sets K', I', N have been found. If these sets are found and a triangle of single edges remains in N, then the entire graph has been determined except for the edges between the two sets K' and I' and the edges within N. However, the only single edges in N are those of the triangle, so the edges of N will not affect whether the graph is parsimonious. This reduces the problem in this case to finding a parsimonious bipartite 2-multigraph, giving the degree sequence of the subgraph induced by $I' \cup K'$. Although we do not yet know how to produce such a graph, we are hopeful that the restriction to bipartite graphs will simplify the problem.

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