Parity Edge-Coloring of Graphs

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(Joint with David Bunde, Kevin Milans, Hehui Wu)

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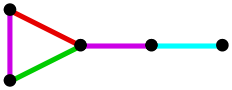

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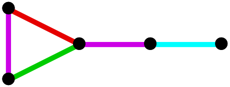

Obs. $p(G) \geq \chi'(G)$, and $H \subseteq G \Rightarrow p(H) \leq p(G)$.

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Conj. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n. (Known for $n \leq 16$.)

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- Embeddability in hypercubes is NP-complete for trees (Wagner–Corneil [1990]), so computing $p(G)$ is also.

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Each remaining edge e completes a cycle. When $e = uv$, the color on e is the only color with odd usage on the u, v-path in T. Hence $f(u) \leftrightarrow f(v)$ in Q_k. ■

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Obs. Always $p(G) \leq p(G - e) + 1$.

Pf. Put optimal pec on $G - e$; add new color on e.
Each path is okay in G whether it uses e or not. ■

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Cor. If n is odd, then $\lceil \lg n \rceil \leq p(C_n) \leq \lceil \lg n \rceil + 1$.

Lower Bound for Odd Cycles

Lem. Every pec of C_n is a spec, so $p(C_n) = \hat{p}(C_n)$.

Pf. Take a pec of C_n. The edges with odd usage in any open walk W form a path P joining the ends of W.
P has some odd-used color; $\therefore$ W is not a parity walk. ∎

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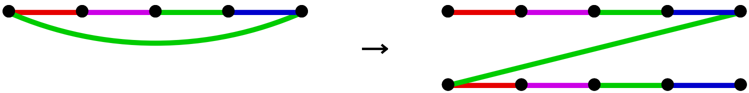
Pf. Spec of C_n yields pec of P_{2n}.



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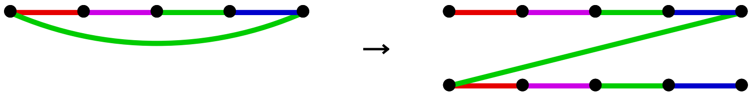

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Thm. If n is odd, then $p(C_n) = \lceil \lg n \rceil + 1$.

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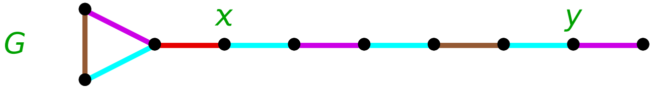
- Unrolling technique (like lower bound for odd cycle)



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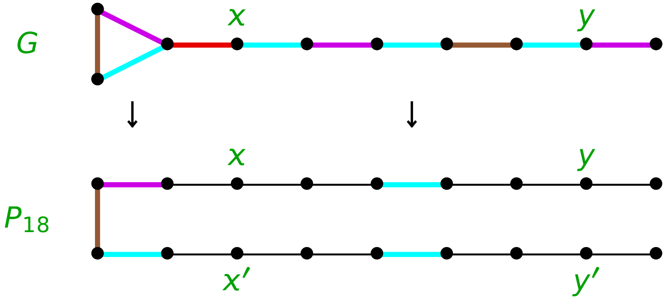

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Obs. $\hat{p}(G) \geq p(P_{18}) = 5$.

Pf. Copy a spec of G onto P_{18} (path edges doubled).
An x, y'-subpath of P_{18} comes from an open walk in G.
An x, x'-subpath of P_{18} comes from an odd walk in G. ∎

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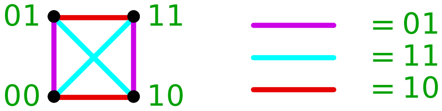

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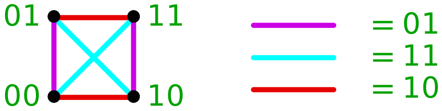

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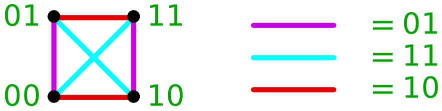

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Pf. Canonical coloring uses $n - 1$ colors (0^k not used).

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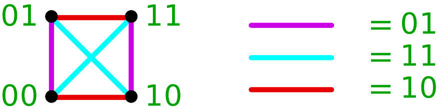

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Cor. $\hat{p}(K_n) \le 2^{\lceil \lg n \rceil} - 1 \le 2n - 3$.

Conj. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$. (**Thm.** $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$.)

Just Above the Threshold: K_2, K_3, K_5, K_9

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$k = 0$: $p(K_2) = 1$; $k = 1$: $p(K_3) = 3$

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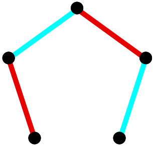

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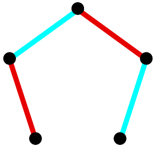

Thm. $p(K_9) = 15$. (Longer ad hoc argument.)

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Def. Hopf–Stiefel function (Hopf [1940], Stiefel [1940]):

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Thm. (Plagne [2003], Károlyi [2006])
$r \circ s = \min_{j \in \mathbf{N}} 2^j \left(\left\lceil \frac{r}{2^j} \right\rceil + \left\lceil \frac{s}{2^j} \right\rceil - 1 \right)$.

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- If $A = B$, with size r, then $r \circ r = 2^{\lceil \lg r \rceil}$. (Set $j = \lceil \lg r \rceil$.)

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Thm. (Yuzvinsky [1981]) If $A, B \subseteq \mathbf{F}_2^k$ with $|A| = r$ and $|B| = s$, then $|\{a + b \colon a \in A, b \in B\}| \geq r \circ s$.

Thm. (Plagne [2003], Károlyi [2006])
$r \circ s = \min_{j \in \mathbf{N}} 2^j(\lceil \frac{r}{2^j} \rceil + \lceil \frac{s}{2^j} \rceil - 1)$.

- If $A = B$, with size r, then $r \circ r = 2^{\lceil \lg r \rceil}$. (Set $j = \lceil \lg r \rceil$.)

Yuzvinsky ⇒ canonical coloring of K_r needs $\geq 2^{\lceil \lg r \rceil} - 1$.
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Yuzvinsky's Theorem

Def. Hopf–Stiefel function (Hopf [1940], Stiefel [1940]):
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Conj. $\hat{p}(K_{r,s}) = r \circ s$. (Would strengthen Yuzv. & ours.)

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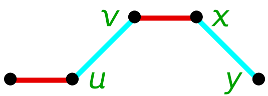

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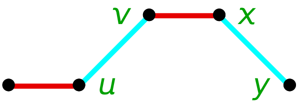

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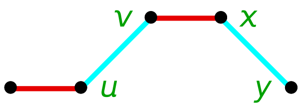

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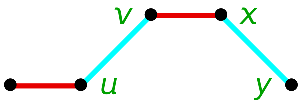

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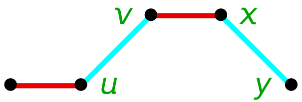

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With $|R| = 2^{j-1}$, we are given a bijection from R to $\mathbf{F}_2^{j-1}$ under which f is the canonical coloring.

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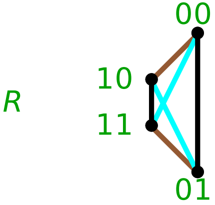
f canonical on R $\Rightarrow$ any color used within R pairs up R.



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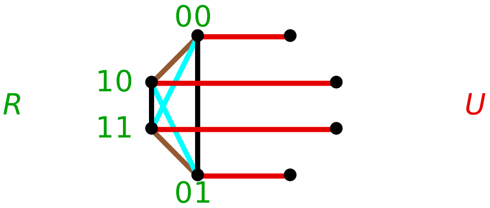

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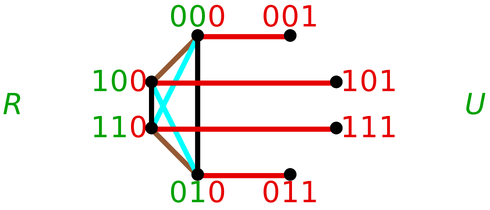

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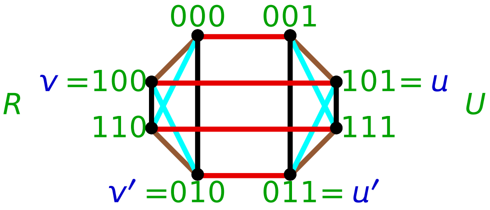

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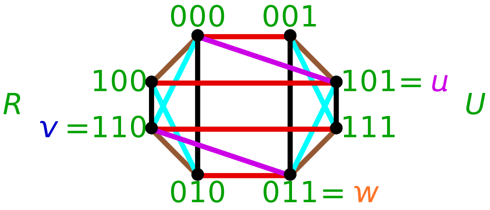

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Use u to name the color on $0^j u$, so $f(0^j u) = u = 0^j + u$.
The rest: $v \in R$ & $w = u + v \in U \Rightarrow f(v0^j) = f(uw) = v$;
4-constraint $\Rightarrow$ $f(vw) = f(0^j u) = u = v + w$. ∎

Algebraic Aspects of Specs

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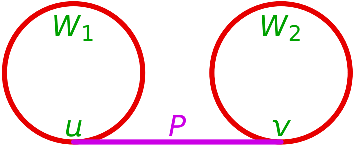
Pf. When W is a u,u-walk and W' is a v,v-walk, let P be a u,v-path, with P' its reverse. Now W_1, P, W_2, P' is a u,u-walk with parity vector $\pi(W) + \pi(W')$. ■



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Pf. Merging a and b into one color a' yields non-spec f'.
$\therefore$ some closed W has odd usage only for c under f'.
Since $c = a'$ $\Rightarrow$ wt$(\pi_f(W)) = 1$, we have $c \neq a'$.
wt$(\pi_f(W)) \geq 2$ $\Rightarrow$ a and b have odd usage in W. ■

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Lem. If G (colored by f) has a dominating vertex v, then $L_f = \mathrm{span}\{\pi(T)\colon T \text{ is a triangle containing } v\}$.

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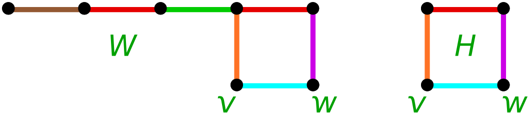

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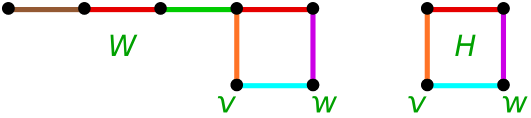

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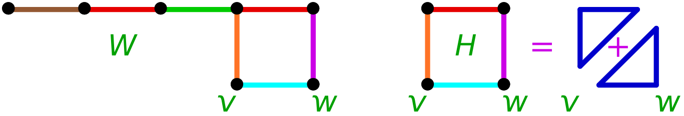

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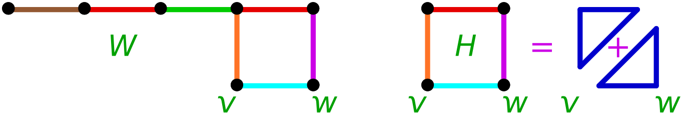

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Edge vw is in odd # triangles $\Leftrightarrow$ $d_{H-v}(w)$ is odd $\Leftrightarrow$ $w \in N_H(v)$ (since $d_H(w)$ is even) $\Leftrightarrow$ $vw \in E(H)$. ∎

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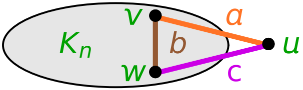

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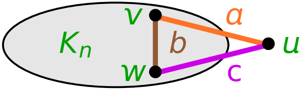

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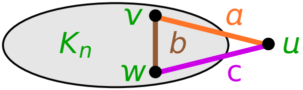

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If $u \notin T$, then $\pi(T) \in L_f$ by definition of L_f.
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Cor. Every optimal spec of a complete graph is obtained by deleting vertices from a canonical coloring.

Other Related Parameters

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Def. nonrepetitive edge-coloring = Thue coloring = edge-coloring with no immediate repetition $c_1, \ldots, c_k, c_1, \ldots, c_k$ on any path; $t(G)$=least #colors.
Alon–Grytczuk–Hałuszczak–Riordan [2002]

- $p(G) \geq t(G) \geq \chi'(G)$.

Examples Showing $c \neq \hat{p}$

Ex. C_8: $p(C_8) = \lceil \lg 8 \rceil = 3$. Suppose $c(C_8) = 3$.
A color used once $\Rightarrow$ parity 4-path in the other two.
$\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; delete edge of largest class to leave P_8 with no color used once.

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Ex. C_8: $p(C_8) = \lceil \lg 8 \rceil = 3$. Suppose $c(C_8) = 3$.
A color used once $\Rightarrow$ parity 4-path in the other two.
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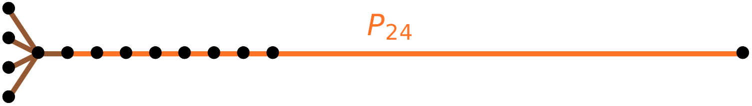
Ex. Let T_k = broom formed by identifying an end of P_{2^k-2k+2} with a leaf of a k-edge star. (T_5 below.)



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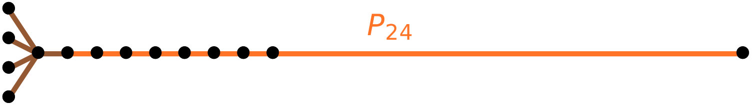

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For $k \geq 5$, $c(T_k) = k+1$. If conflict-free w. k colors, $P_{2^{k-1}+1}$ takes k colors, and $P_{2^{k-2}+1}$ takes $k-1$.
All k colors are at x, use the color missing on $P_{2^{k-2}+1}$. ■

Open Problems

Conj. 1 $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.
Known for $n \le 16$; proved $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.

Conj. 2 $p(K_{n,n}) = \hat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$. ($\hat{p}(K_{r,s}) = r \circ s$?)

Conj. 3 $\hat{p}(G) = p(G)$ for every bipartite graph G.

Ques. 4 What is $\max \hat{p}(G)$ (or $\max c(G)$) for $p(G) = k$?

Ques. 5 How do $\hat{p}(K_{k,n})$ and $p(K_{k,n})$ grow with k?

Ques. 6 What is $\max p(T)$ when T is an n-vertex tree with maximum degree k? (That is, what cube contains all n-vertex trees with maximum degree k?)

Ques. 7 When does $p(G)$ equal $\lceil \lg n(G) \rceil$?

Ques. 8 Is $p(T)$ NP-hard on trees w. bounded degree?

Ques. 9 Stability . . . $\hat{p}(G \square H)$ . . . Digraphs . . .