Parity Edge-Coloring of Graphs

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(Joint with David Bunde, Kevin Milans, Hehui Wu)
Motivation

Ques. What graphs embed in a $k$-dimensional cube?
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• $k$-coloring the edges by the $k$ coordinates yields natural necessary conditions. In this coloring:
  (1) On every cycle, every color appears even # times.
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Obs. $p(G) \geq \chi'(G),$ and $H \subseteq G \Rightarrow p(H) \leq p(G).$
A Related Parameter

**Def.** Parity walk = walk using each color even \#times.  
Strong parity edge-coloring (spec) = edge-coloring such that every parity walk is closed.  
spec number $\hat{\rho}(G)$ = least \#colors in a spec.
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**Appl.** extends special case of Yuzvinsky’s Thm [1981], which is tight lower bound on $|\{a + b : a \in A, b \in B\}|$ when $A, B \subseteq \mathbb{F}_2^k$ with $|A| = r$ and $|B| = s$. 

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** Conj.** \( \rho(K_n) = 2^{[\lg n]} - 1 \) for all \( n \). (Known for \( n \leq 16 \).)
Embedding Trees in $k$-cubes

**Prop.** A tree $T$ is a subgraph of $Q_k$ $\iff p(T) \leq k$. 
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• Embeddability in hypercubes is NP-complete for trees (Wagner–Corneil [1990]), so computing $p(G)$ is also.
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**Cor.** (Havel-Movárek [1972]) A graph $G$ embeds in $Q_k$ $\iff G$ has a $k$-pec where every cycle is a parity walk.
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**Pf.** Embed a spanning tree $T$ of $G$ in $Q_k$ as done above.
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**Pf.** Embed a spanning tree $T$ of $G$ in $Q_k$ as done above.
Each remaining edge $e$ completes a cycle. When $e = uv$, the color on $e$ is the only color with odd usage on the $u, v$-path in $T$. Hence $f(u) \leftrightarrow f(v)$ in $Q_k$. ■
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Pf. If $T$ is a spanning tree of $G$, then $\rho(G) \geq \rho(T)$. Since $T \subseteq Q_{\rho(T)}$, we have $n(G) = n(T) \leq n(Q_{\rho(T)}) = 2^\rho(T)$. 


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Obs. Always $p(G) \leq p(G - e) + 1$.

Pf. Put optimal pec on $G - e$; add new color on $e$. Each path is okay in $G$ whether it uses $e$ or not.
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Cor. If $n$ is odd, then $\lceil \lg n \rceil \leq p(C_n) \leq \lceil \lg n \rceil + 1$. 
Lower Bound for Odd Cycles

**Lem.** Every pec of $C_n$ is a spec, so $p(C_n) = \hat{p}(C_n)$.

**Pf.** Take a pec of $C_n$. The edges with odd usage in any open walk $W$ form a path $P$ joining the ends of $W$. $P$ has some odd-used color; \[.\therefore W\] is not a parity walk.  \[\blacksquare\]
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**Lem.** If $n$ is odd, then $\hat{p}(C_n) \geq p(P_{2n})$.

**Pf.** Spec of $C_n$ yields pec of $P_{2n}$.

Each path in $P_{2n}$ arises from an open walk in $C_n$ or one trip around the cycle (which is odd length). ■
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**Thm.** If $n$ is odd, then $p(C_n) = \lceil \lg n \rceil + 1$. 
Example Showing $p \neq \hat{p}$

- **Unrolling technique** (like lower bound for odd cycle)

![Graph Image]

$G$

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**Obs.** \[ \hat{p}(G) \geq p(P_{18}) = 5. \]

**Pf.** Copy a spec of $G$ onto $P_{18}$ (path edges doubled).
An $x, y'$-subpath of $P_{18}$ comes from an open walk in $G$.
An $x, x'$-subpath of $P_{18}$ comes from an odd walk in $G$. ■
Complete Graphs, $n = 2^k$

**Def.** canonical coloring of $K_{2^k}$ = edge-coloring $f$ defined by $f(uv) = u + v$, where $V(K_{2^k}) = \mathbb{F}_2^k$.

![Graph Diagram]
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![Diagram of a complete graph with vertices labeled 00, 01, 10, 11 and edges colored in red, purple, and blue.]

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![Graph diagram]

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**Pf.** Canonical coloring uses $n - 1$ colors ($0^k$ not used).

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**Cor.** $\hat{\rho}(K_n) \leq 2^{[\lg n]} - 1 \leq 2n - 3$.

**Conj.** $\rho(K_n) = 2^{[\lg n]} - 1$. (Thm. $\hat{\rho}(K_n) = 2^{[\lg n]} - 1$.)
Just Above the Threshold: $K_2, K_3, K_5, K_9$

- It suffices to prove $p(K_{2^k+1}) = 2^{k+1} - 1$.
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**Prop.** $p(K_5) = 7$. 
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**Thm.** $p(K_9) = 15$. (Longer ad hoc argument.)
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**Def.** Hopf–Stiefel function (Hopf [1940], Stiefel [1940]):

\[ r \circ s = \text{least } n \text{ such that } (x + y)^n \text{ is in the ideal of } F_2[x, y] \text{ generated by } x^r \text{ and } y^s. \]
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Equivalently, \( r \circ s = \text{least } n \text{ such that } \binom{n}{k} \text{ is even for } n - s < k < r. \) (Empty if \( n \geq r + s - 1, \) so \( r \circ s \leq r + s - 1. \))
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**Thm.** (Plagne [2003], Károlyi [2006])
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Yuzvinsky \Rightarrow \text{ canonical coloring of } K_r \text{ needs } \geq 2^{\lfloor \lg r \rfloor} - 1. \]

Our theorem \Rightarrow \text{ same lower bound for any spec.}
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**Thm.** (Yuzvinsky [1981]) If \( A, B \subseteq \mathbb{F}_2^k \) with \( |A| = r \) and \( |B| = s \), then \( |\{a + b : a \in A, b \in B\}| \geq r \circ s. \)

**Thm.** (Plagne [2003], Károlyi [2006])
\[ r \circ s = \min_{j \in \mathbb{N}} 2^j \left( \left\lceil \frac{r}{2^j} \right\rceil + \left\lceil \frac{s}{2^j} \right\rceil - 1 \right). \]

- If \( A = B \), with size \( r \), then \( r \circ r = 2^{\left\lfloor \log_2 r \right\rfloor}. \) (Set \( j = \left\lfloor \log_2 r \right\rfloor. \))

Yuzvinsky ⇒ canonical coloring of \( K_r \) needs \( \geq 2^{\left\lfloor \log_2 r \right\rfloor} - 1. \)
Our theorem ⇒ same lower bound for any spec.

** Conj.** \( \hat{\rho}(K_{r,s}) = r \circ s. \) (Would strengthen Yuzv. & ours.)
Main Steps of the Proof

**Thm.** If $f$ is a spec of $K_n$ with every color class a perfect matching, then $f$ is canonical & $n$ is a 2-power.
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**Lem.** If $G$ (colored by $f$) has a dominating vertex $v$, then $L_f = \text{span}\{\pi(T): T \text{ is a triangle containing } v\}$. 
Specs Consisting of 1-Factors Are Canonical

**Thm.** If \( f \) is a spec of \( K_n \) with every color class a perfect matching, then \( f \) is canonical & \( n \) is a 2-power.

**Pf.** Such a coloring satisfies the 4-constraint: If \( f(uv) = f(xy) \), then \( f(uy) = f(vx) \). (Since every color is at every vertex.)
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Aim: Map $V(K_n)$ to $\mathbb{F}_2^k$ so $f$ is the canonical coloring.
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![Diagram](image)

**Aim:** Map $V(K_n)$ to $\mathbb{F}_2^k$ so $f$ is the canonical coloring.

Every edge is a canonically colored $K_2$. Let $R$ be a largest vertex set on which $f$ restricts to a canonical coloring. If $R \neq V(K_n)$, we obtain a larger such set.
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With $|R| = 2^{j-1}$, we are given a bijection from $R$ to $\mathbb{F}_2^{j-1}$ under which $f$ is the canonical coloring.
Expanding the Canonical Portion

\[ f \text{ canonical on } R \Rightarrow \text{ any color used within } R \text{ pairs up } R. \]
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The 4-constraint copies the coloring from \( R \) to \( U \),
so \( f(uu') = f(\nu\nu') = \nu + \nu' = u + u' \).

Use \( u \) to name the color on \( 0^i u \), so \( f(0^i u) = u = 0^i + u \).

The rest: \( \nu \in R \) & \( w = u + \nu \in U \) \( \Rightarrow \) \( f(\nu 0^i) = f(uw) = \nu \);
4-constraint  \( \Rightarrow \) \( f(\nu w) = f(0^i u) = u = \nu + w \).
Def. Given an edge-coloring $f$ and a walk $W$, the parity vector $\pi(W)$ is the binary vector where bit $i$ is the parity of the usage of color $i$ on $W$.
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**Lem.** If $f$ is an edge-coloring of a connected graph $G$, then $L_f$ is a binary vector space.

**Pf.** When $W$ is a $u, u$-walk and $W'$ is a $v, v$-walk, let $P$ be a $u, v$-path, with $P'$ its reverse. Now $W_1, P, W_2, P'$ is a $u, u$-walk with parity vector $\pi(W) + \pi(W')$.  

\[ \begin{array}{c}
W_1 \\
\text{u} \\
\text{P} \\
\text{v} \\
W_2
\end{array} \]
Parity Space for Spec of $K_n$

**Def.** Let $w(L)$ denote the minimum weight ($\text{wt} = \#1s$) of the nonzero vectors in a binary space $L$.

**Prop.** Edge-coloring $f$ of $K_n$ is a spec $\iff w(L_f) \geq 2$. 
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**Lem.** Given colors $a$ and $b$ in an optimal spec $f$ of $K_n$, some closed $W$ has odd usage for $a$, $b$, and one other.

**Pf.** Merging $a$ and $b$ into one color $a'$ yields non-spec $f'$. $\therefore$ some closed $W$ has odd usage only for $c$ under $f'$.

Since $c = a'$ $\implies$ $\text{wt}(\pi_f(W)) = 1$, we have $c \neq a'$.

$\text{wt}(\pi_f(W)) \geq 2 \implies a$ and $b$ have odd usage in $W$. 

$\blacksquare$
More on Parity Spaces

Lem. If $G$ (colored by $f$) has a dominating vertex $v$, then $L_f = \text{span}\{\pi(T): T \text{ is a triangle containing } v\}$.

Pf. The span is in $L_f$. Conversely, suppose $\pi(W) \in L_f$. 
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Define $H$ by $E(H) = \{\text{edges w. odd usage in } W\}$. 

[Diagram showing $W$ and $H$]
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Define $H$ by $E(H) = \{\text{edges w. odd usage in } W\}$. $H$ is an even subgraph of $G$. Also, $\pi(H) = \pi(W)$. 
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\[\therefore\] it suffices to show that $H$ is the sum (mod 2) of the set of triangles formed by adding $v$ to edges of $H - v$.

Each edge of $H - v$ is in one such triangle.
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Edge $vw$ is in odd $\#$ triangles $\iff d_{H-v}(w)$ is odd $\iff w \in N_H(v)$ (since $d_H(w)$ is even) $\iff vw \in E(H)$. \qed
Enlarging the Clique

**Lem.** If an optimal spec \( f \) of \( K_n \) uses some color \( a \) not on a perfect matching, then \( \hat{\rho}(K_{n+1}) = \hat{\rho}(K_n) \).

**Pf.** Let \( v \) be a vertex missed by \( a \); let \( u \) be a new vertex. We use \( f \) to define \( f' \) on the larger complete graph.
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Pf. Let $v$ be a vertex missed by $a$; let $u$ be a new vertex. We use $f$ to define $f'$ on the larger complete graph.

Let $f'(uv) = a$. For $w \notin \{u, v\}$, let $b = f(vw)$. $3W$ with odd usage of $a, b$, and some $c$. Let $f'(uw) = c$. 

![Diagram](attachment:image.png)
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Let \( f'(uv) = a \). For \( w \notin \{u, v\} \), let \( b = f(vw) \). \( \exists W \) with odd usage of \( a, b, \) and some \( c \). Let \( f'(uw) = c \).

![Diagram](https://example.com/diagram.png)

We show that \( L_{f'} \subseteq L_f \) to get \( w(L_{f'}) \geq 2 \). It suffices that \( \pi(T) \in L_f \) when \( T \) is a triangle in \( K_{n+1} \) containing \( v \), since these vectors span \( L_{f'} \) (by lemma).
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We show that $L_f' \subseteq L_f$ to get $\omega(L_f') \geq 2$. It suffices that $\pi(T) \in L_f$ when $T$ is a triangle in $K_{n+1}$ containing $v$, since these vectors span $L_f'$ (by lemma).

If $u \notin T$, then $\pi(T) \in L_f$ by definition of $L_f$.
If $T = \{u, v, w\}$, then $\pi(T) = \pi(W) \in L_f$. 

\[ \begin{array}{c}
\text{K}_n \\
\text{v} \quad a \\
\text{w} \quad b \\
\text{u} \quad c
\end{array} \]
**Thm.** \( \hat{\rho}(K_n) = 2^{|\lg n|} - 1 \)

**Pf.** Let \( k = \hat{\rho}(K_n) \). Canonical coloring \( \Rightarrow k \leq 2^{|\lg n|} - 1 \).
Thm. \( \hat{\rho}(K_n) = 2^{[\lg n]} - 1 \)

Pf. Let \( k = \hat{\rho}(K_n) \). Canonical coloring \( \Rightarrow k \leq 2^{[\lg n]} - 1 \).

Accumulate additional vertices without increasing \( \hat{\rho} \) until every color class is a perfect matching.

This can’t exceed \( 2^{[\lg n]} \) vertices, since vertex degree then reaches \( 2^{[\lg n]} - 1 \).
Thm. $\hat{\rho}(K_n) = 2^{[\log n]} - 1$

Pf. Let $k = \hat{\rho}(K_n)$. Canonical coloring $\Rightarrow k \leq 2^{[\log n]} - 1$.

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This can’t exceed $2^{[\log n]}$ vertices, since vertex degree then reaches $2^{[\log n]} - 1$.

$\therefore$ It stops with every color class a perfect matching. We showed this occurs only in the canonical coloring.

Hence $\hat{\rho}(K_n) = \hat{\rho}(K_{2^{[\log n]}}) = 2^{[\log n]} - 1$. $lacksquare$
**Thm.** \( \hat{\rho}(K_n) = 2^{\lceil \log n \rceil} - 1 \)

**Pf.** Let \( k = \hat{\rho}(K_n) \). Canonical coloring \( \Rightarrow k \leq 2^{\lceil \log n \rceil} - 1 \).

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This can’t exceed \( 2^{\lceil \log n \rceil} \) vertices, since vertex degree then reaches \( 2^{\lceil \log n \rceil} - 1 \).

\[ \therefore \] It stops with every color class a perfect matching. We showed this occurs only in the canonical coloring.

Hence \( \hat{\rho}(K_n) = \hat{\rho}(K_{2^{\lceil \log n \rceil}}) = 2^{\lceil \log n \rceil} - 1 \). \[ \blacksquare \]

**Cor.** Every optimal spec of a complete graph is obtained by deleting vertices from a canonical coloring.
Other Related Parameters

**Def.** conflict-free coloring = edge-coloring s.t. each path has some color used once; $c(G) = \text{least \# colors.}$

edge-ranking = edge-coloring s.t. each path has the highest color used once; $\chi'_r(G) = \text{least \# colors.}$

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- \( \chi'_r(G) \geq c(G) \geq p(G) \), and the difference can be large.
Indeed, \( \chi'_r(K_n) \in \Theta(n^2) \) [BDJKKMT], but \( p(K_n) \in \Theta(n) \).
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  Indeed, $\chi'_r(K_n) \in \Theta(n^2)$ [BDJKKMT], but $p(K_n) \in \Theta(n)$.

**Def.** nonrepetitive edge-coloring = Thue coloring = edge-coloring with no immediate repetition $c_1, \ldots, c_k, c_1, \ldots, c_k$ on any path; $t(G)$ = least #colors.

Alon–Grytczuk–Hałuszczak–Riordan [2002]

- $p(G) \geq t(G) \geq \chi'(G)$.
Examples Showing $c \neq \hat{c}$

**Ex.** $C_8$: $p(C_8) = \lfloor \lg 8 \rfloor = 3$. Suppose $c(C_8) = 3$. A color used once $\Rightarrow$ parity 4-path in the other two.  
$\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; delete edge of largest class to leave $P_8$ with no color used once.
Examples Showing $c \neq \hat{\rho}$

Ex. $C_8$: $p(C_8) = \lceil \log_2 8 \rceil = 3$. Suppose $c(C_8) = 3$. A color used once $\Rightarrow$ parity 4-path in the other two. $\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; delete edge of largest class to leave $P_8$ with no color used once.

Ex. Let $T_k = \text{broom formed by identifying an end of } P_{2^k-2k+2}$ with a leaf of a $k$-edge star. ($T_5$ below.)
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$T_k$ embeds in $Q_k$, so $p(T_k) = k$. (Induction on $k$.)
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$T_k$ embeds in $Q_k$, so $p(T_k) = k$.  (Induction on $k$.)

For $k \geq 5$, $c(T_k) = k + 1$.  If conflict-free w. $k$ colors, $P_{2^{k-1}+1}$ takes $k$ colors, and $P_{2^{k-2}+1}$ takes $k - 1$.  All $k$ colors are at $x$, use the color missing on $P_{2^{k-2}+1}$.  ■
Open Problems

**Conj. 1** \( p(K_n) = 2^{[\lg n]} - 1 \) for all \( n \).
Known for \( n \leq 16 \); proved \( \hat{p}(K_n) = 2^{[\lg n]} - 1 \) for all \( n \).

**Conj. 2** \( p(K_{n,n}) = \hat{p}(K_{n,n}) = 2^{[\lg n]} \). (\( \hat{p}(K_{r,s}) = r \circ s \)?)

**Conj. 3** \( \hat{p}(G) = p(G) \) for every bipartite graph \( G \).

**Ques. 4** What is \( \max \hat{p}(G) \) (or \( \max c(G) \)) for \( p(G) = k \)?

**Ques. 5** How do \( \hat{p}(K_{k,n}) \) and \( p(K_{k,n}) \) grow with \( k \)?

**Ques. 6** What is \( \max p(T) \) when \( T \) is an \( n \)-vertex tree with maximum degree \( k \)? (That is, what cube contains all \( n \)-vertex trees with maximum degree \( k \)?)

**Ques. 7** When does \( p(G) \) equal \( [\lg n(G)] \)?

**Ques. 8** Is \( p(T) \) NP-hard on trees w. bounded degree?

**Ques. 9** Stability . . . \( \hat{p}(G \Box H) \) . . . Digraphs . . .