Extending graph choosability results to paintability

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Joint work with
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slides available on DBW preprint page

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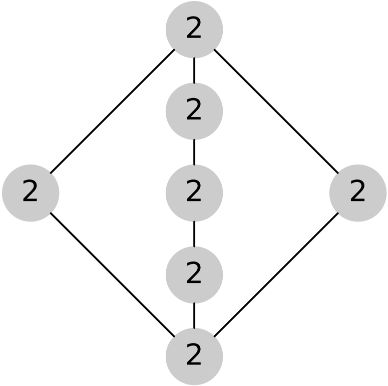

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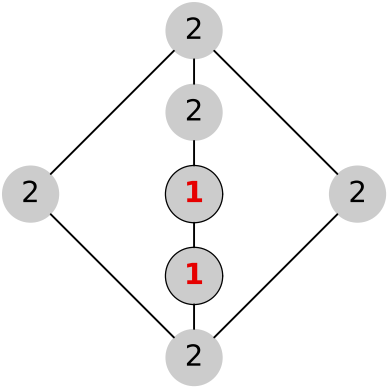

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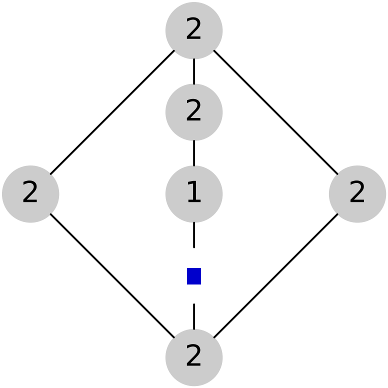

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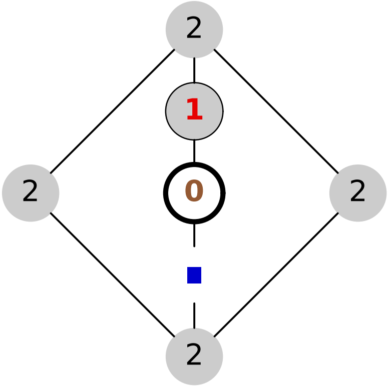

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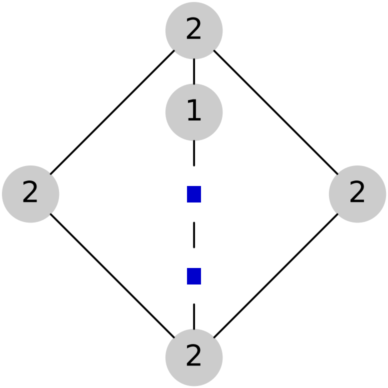

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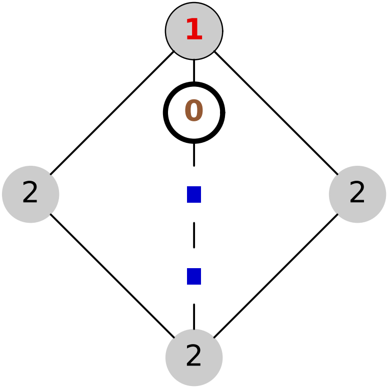

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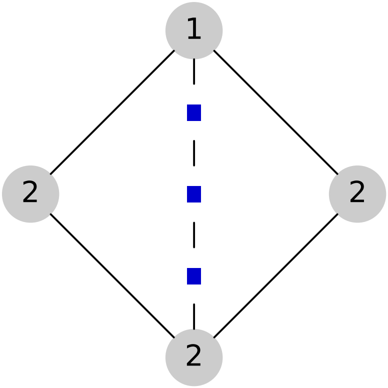

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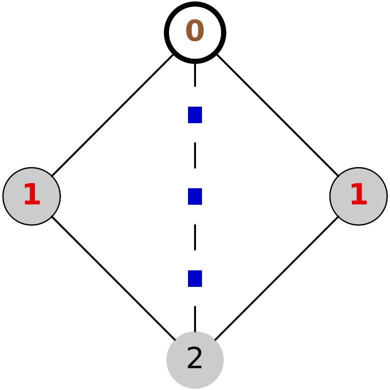

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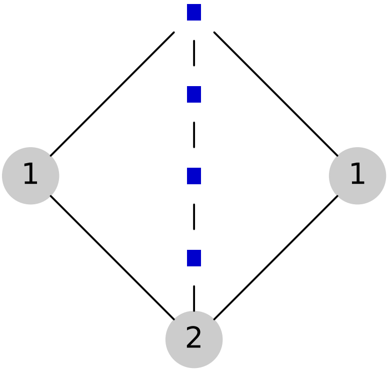

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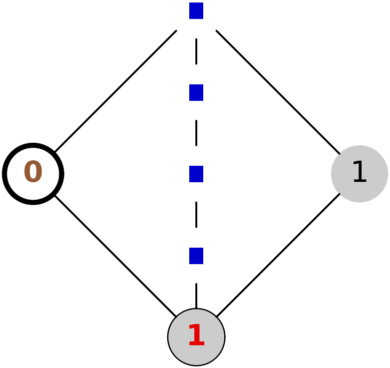

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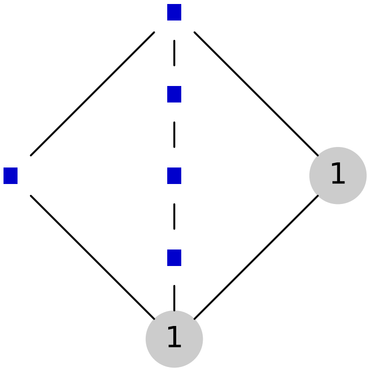

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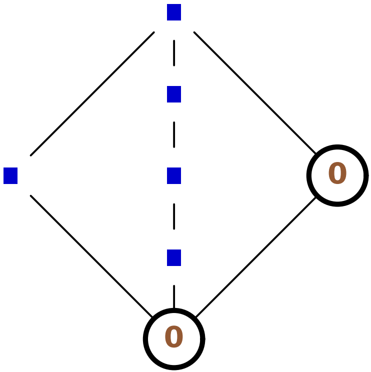

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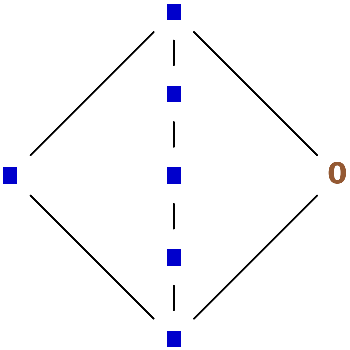

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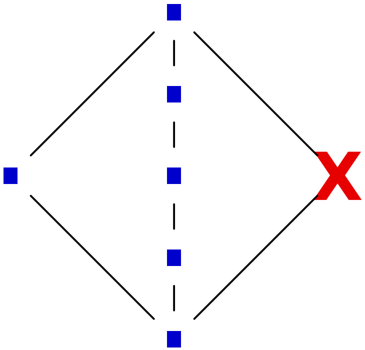

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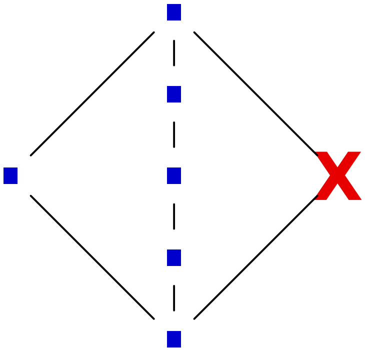

Marker wins on $\Theta_{2,2,4}$ when each vertex has 2 tokens.

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When G is bipartite,

G is $\Delta(G)$-edge-colorable (König [1916])

G is $\Delta(G)$-edge-choosable (Galvin [1995])

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- Kozik-Micek-Zhu [2012]: $|V(G)| \le \chi(G) + \alpha\sqrt{\chi(G)}$ suffices.

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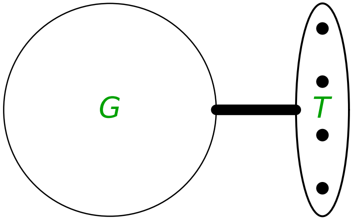
Pf. Idea: Remover uses k-paintability strategy **S** on G, ignoring the added t-set T, until a special round where $M \cap T$ is deleted instead. The extra tokens on $V(G)$ then permit finishing on G, after which we claim each vertex remaining in T still has a token.



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If suffices to show: (1) (∗) occurs after some round r, (2) each $v \in T - M$ then still has more than $\left\lfloor \frac{t}{t-1} k \right\rfloor - r$ tokens. (G has at most $\left\lfloor \frac{t}{t-1} k \right\rfloor - r$ vertices left.)

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(1) When some $v\in T$ is marked before (∗), count rounds with v marked. Each omits some vertex not yet omitted μ times. Hence $\exists$ fewer than $(t-1)\mu$ such rounds. Since $(t-1)\mu\leq k$, no vertex loses all tokens before (∗). ■

Application

Thm. $\chi_p(G) \leq k$ and $|V(G)| \leq \frac{t}{t-1}k \Rightarrow \chi_p(G \oplus \overline{K}_t) \leq k+1$.

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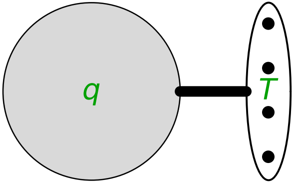
Let $q = \#$singleton parts. If G has one non-singleton part T, then G is chromatic-paintable (vertices of T have $q+1$ tokens; always delete a marked singleton).



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Thm. (Voigt [1998]) G is 3-choice-critical $\iff$ G is

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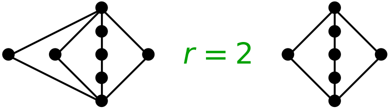

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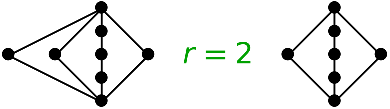

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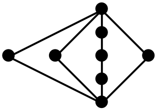
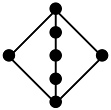


$r = 2$



- Uses charzn of 2-paintable graphs (Zhu [2009]).
- Riasat-Schauz [2012] found the minimal 3-paintable graphs under vertex deletion.

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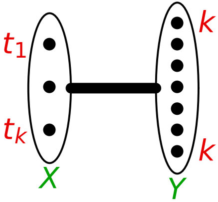

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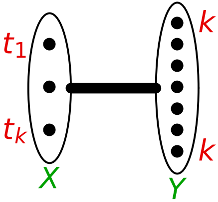

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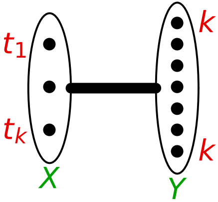

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Ques. What is $\min\{r : K_{k+j,r}$ is not k-paintable$\}$?

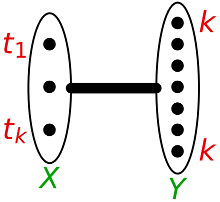
Complete Bipartite Graphs

Thm. (ERT [1979]) $K_{k,r}$ is k-choosable $\iff$ $r < k^k$.

Thm. (CLMPTW) Consider $K_{k,r}$ with parts X of size k and Y of size r. If each vertex of Y has k tokens, then

Remover has a winning strategy $\iff$ $r < \prod_{i=1}^{k} t_i$,

where $t_1, \ldots, t_k$ are the token counts in X.



Cor. $K_{k,r}$ is k-paintable $\iff$ $r < k^k$.

Ques. What is $\min\{r \colon K_{k+j,r}$ is not k-paintable$\}$?

Hard to compute when $j > 0$!

Tools

Prop. (Degeneracy Tool) If $f(v) > d_G(v)$, then G is f-paintable $\iff$ $G - v$ is $f|_{V(G-v)}$-paintable.

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Prop. If Marker can win, then Marker can win with the marked set M never being an independent set (except on the last move).

Pf. If marked set M is independent, then Remover deletes M. Marker must be able to win by skipping M and just playing the rest of the strategy on $G - M$. ■

k-paintability for $K_{k,r}$

Thm. (CLMPTW) Consider $K_{k,r}$ with $|X| = k$ and $|Y| = r$.
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Pf. $r = \prod t_i \Rightarrow K_{k,r}$ *is not f-choosable.*
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Case 2: $q\ge\prod_{i=1}^{k-1} t_i$. Remover deletes $M\cap Y$.
$|Y-M|<\prod t_i-q\le\prod_{i=1}^{k-1} t_i(t_k-1)$; ind. hyp. applies! ■

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Ex. $K_{5,164}$: Move by move, #tokens in X; #verts. in Y.

(4,4,4,4,4);	64	(3,3,2,2,2);	8
(3,3,4,4,4);	36	(2,2,2,2,2);	8
(3,3,3,3,4);	27	(1,1,2,2,2);	2
(3,3,3,2,3);	18	(1,1,1,1,2);	1

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Remover deletes all 164 vertices in Y, but doing so leaves a vertex in X with no tokens, and Marker wins.

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Similarly, $\chi_p(K_{3,3}) > 2$ and $\chi_p(K_{4,19}) > 3$.

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General strategy for Marker on $K_{k+j,r}$:

Present some subset of X (at most $j+1$ vertices) and enough of Y to force Remover to delete $M \cap Y$. Repeat.

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This only gives an upper bound on r where Marker wins.

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Case 2: Remover deletes $M \cap Y$. Marker plays on $(Y - M) \cup X$. The vector on X is now t^S. Since $|Y - M| \geq g(k,t) - g(k-1, t|_{\overline{S}}) = g(k, t^S)$, Marker wins. ■

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Obs. Since all f-paintable graphs are f-choosable, $\chi_{sp}(G) \geq \chi_{sc}(G)$ for every graph G.

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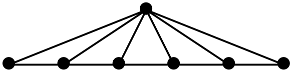

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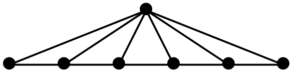

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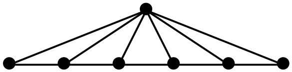

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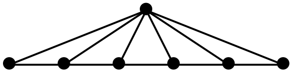

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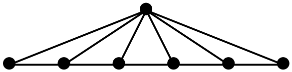

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Obs. The chordal graph $K_2 \oplus \overline{K}_t$ is not sp-greedy (Mahoney–Tomlinson–Wise)

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Ques. What other bounds on choosability are also valid for paintability? (Mahoney has other such results.)