On-line Choice Number or "Paintability" of Graphs

Douglas B. West

Zhejiang Normal University, Jinhua, China
and
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page

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The lists may be identical, so always $\chi_\ell(G) \geq \chi(G)$.

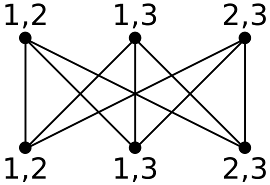
$\chi_\ell(G)$ vs. $\chi(G)$

Ex. $\chi_\ell(G)$ is unbounded on bipartite graphs.

Consider $K_{r,r}$ for $r = \binom{2k-1}{k}$. We give a bad assignment of lists of size k.

Use each k-set from $[2k-1]$ as a list in each part.

$k = 2$



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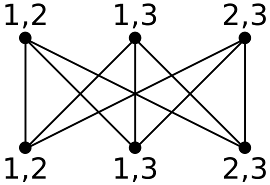

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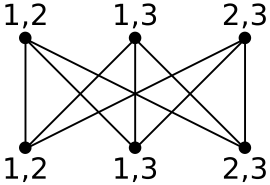

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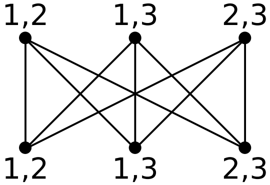

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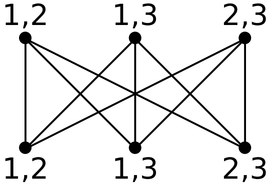

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Let's make it harder: Maybe the lists are not known.

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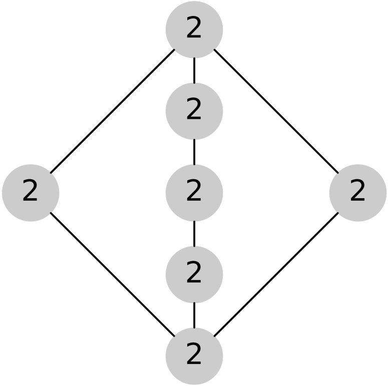

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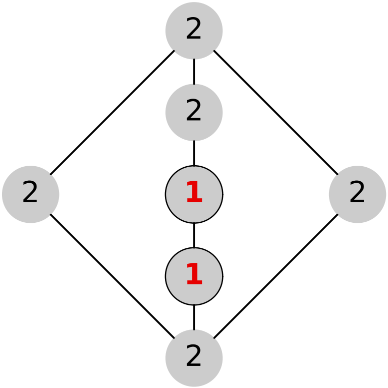

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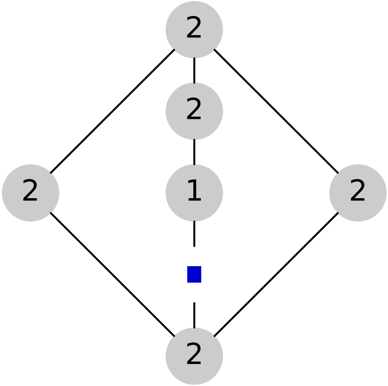

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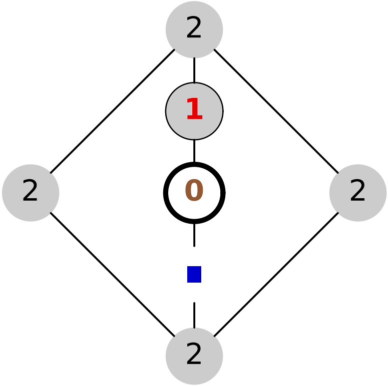

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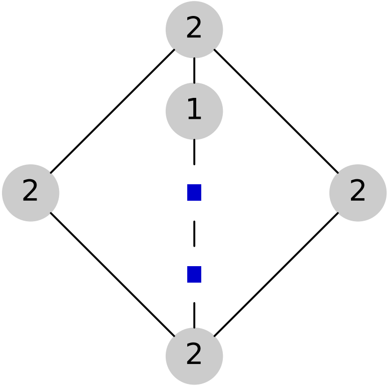

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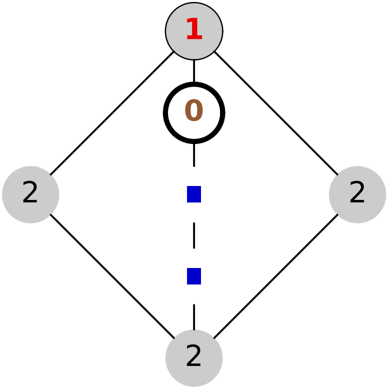

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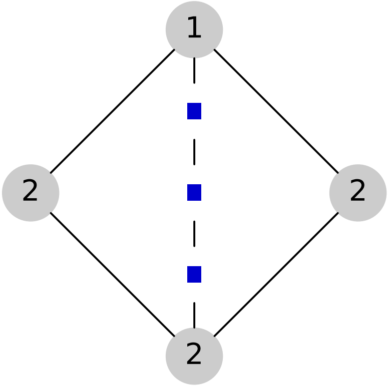

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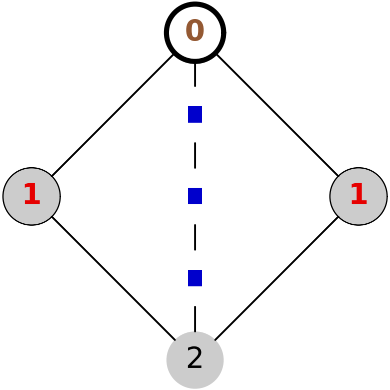

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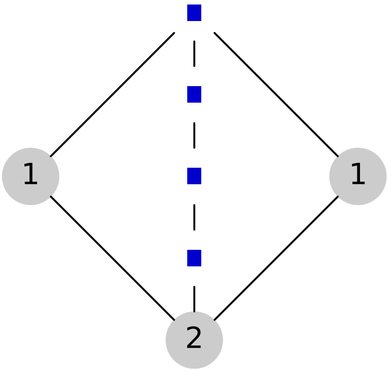

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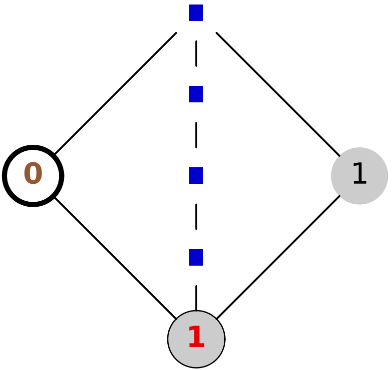

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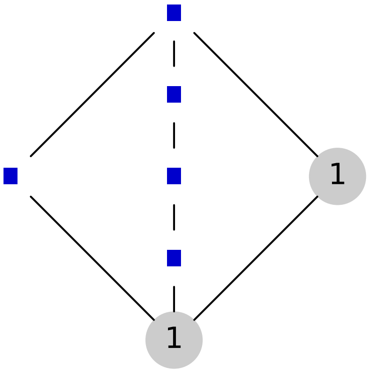

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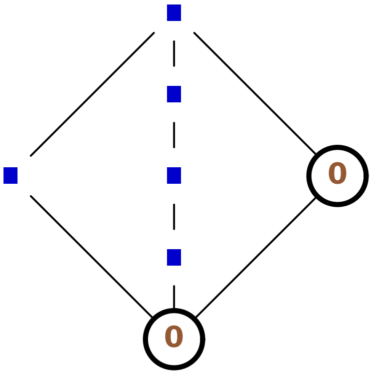

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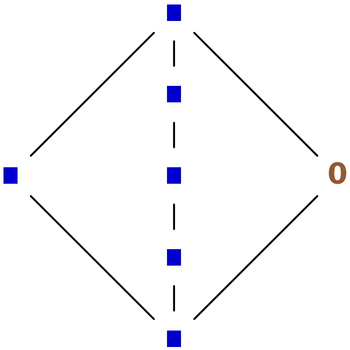

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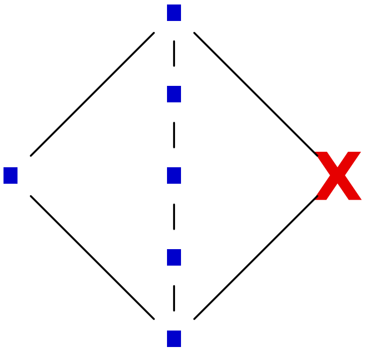

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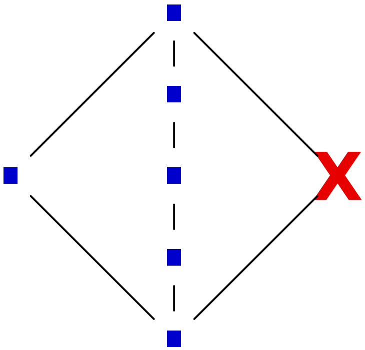

Lister wins on $\Theta_{2,2,4}$ when each vertex has 2 tokens, so $\chi_p(\Theta_{2,2,4}) > 2$.

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Prop. (Erdős–Rubin–Taylor [1979]) $\chi_\ell(\Theta_{2,2,2r}) = 2$.

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Lem. If D is an orientation of G with outdegrees α, then $\text{diff}(D) = \left| |E(\alpha)| - |O(\alpha)| \right| = \left| |E(\alpha^S)| - |O(\alpha^S)| \right|$ for all $S \subseteq V(G)$.

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One then must show that deleting this set R and lowering the number of tokens by 1 on $S - R$ produces a smaller graph (add possibly new orientation) on which the hypothesis $\mathrm{diff}(D) \neq 0$ is again true.

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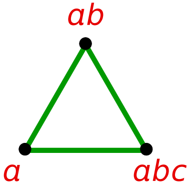

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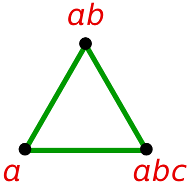

Induction step: Outer cycle $[v_1, \ldots, v_p]$ in order, with $|L(v_1)| = 2$ and $|L(v_p)| = 1$.

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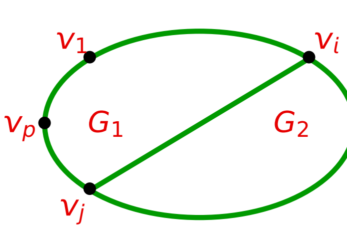

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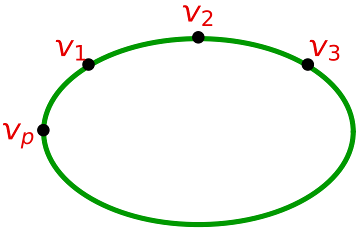

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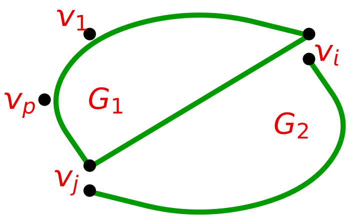

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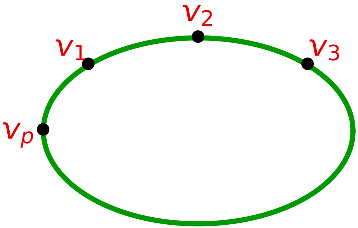

Case 1: The Merge Lemma: If G_i is f_i-paintable, for $i \in \{1,2\}$, $G = G_1 \cup G_2$, $T = V(G_1) \cap V(G_2)$, and $f_2|_T \equiv 1$, then G is f-paintable.

Pf. Painter follows $\mathbf{S_1}$ on G_1 to generate response R_1. Obtain R_2 by using $\mathbf{S_2}$ to respond to $(M - M_1) \cup (T \cap R_1)$ in G_2. Let $R = R_1 \cup R_2$. In effect, tokens for vertices of $M - R_1$ in T are removed in G_1, not G_2.

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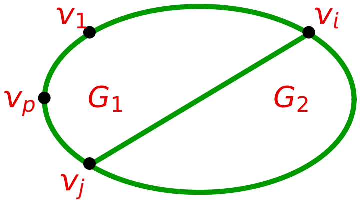

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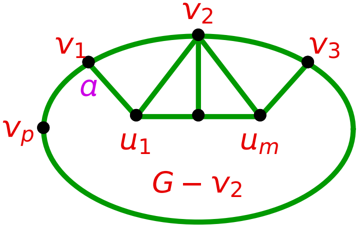

Case 2: Painter uses winning strategy $\mathbf{S}'$ on G', where $G' = G - v_2$, except as follows. When $v_2 \in M$, respond via $\mathbf{S}'$ to $M - U - \{v_2\}$ if $v_1 \notin M$, to $M - \{v_2\}$ if $v_1 \in M$. Remove v_2 when marked if v_1 and v_3 are not being removed; it and U can tolerate waiting twice; the remaining three tokens on U are enough for those neighbors of v_2 to survive in the game on $G - v_2$.

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Total wt never increases, so a vertex is marked and doubled at most $\lg 2r$ times. $\therefore$ $2 + \lg r$ tokens suffice. ■

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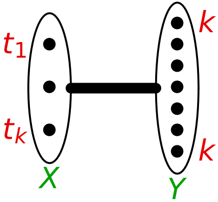

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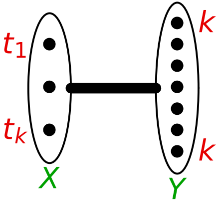

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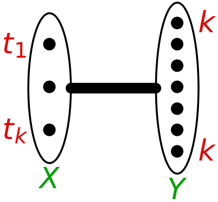

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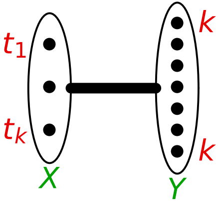
where $t_1, \ldots, t_k$ are the token counts in X.



Cor. $K_{k,r}$ is k-paintable $\iff$ $r < k^k$.

Ques. What is $\min\{r\colon K_{k+j,r}$ is not k-paintable$\}$?

Hard to compute when $j > 0$!

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Pf. If marked set M is independent, then Painter deletes M. Lister must be able to win by skipping M and just playing the rest of the strategy on $G - M$. ■

k-paintability for $K_{k,r}$

Thm. (CLMPTW) Consider $K_{k,r}$ with $|X| = k$ and $|Y| = r$.
If $f(y) = k$ for $y \in Y$ and $f(x_i) = t_i$ for $x_i \in X$, then

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Pf. $r = \prod t_i \Rightarrow K_{k,r}$ *is not f-choosable.*
Let $L(x_i) = U_i$ with $|U_i| = t_i$ and pairwise disjoint.
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Case 2: $q \geq \prod_{i=1}^{k-1} t_i$. Painter deletes $M \cap Y$.
$|Y - M| < \prod t_i - q \leq \prod_{i=1}^{k-1} t_i(t_k - 1)$; ind. hyp. applies! ■

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Ex. $K_{5,164}$: Move by move, #tokens in X; #verts. in Y.

(4,4,4,4,4);	64	(3,3,2,2,2);	8
(3,3,4,4,4);	36	(2,2,2,2,2);	8
(3,3,3,3,4);	27	(1,1,2,2,2);	2
(3,3,3,2,3);	18	(1,1,1,1,2);	1

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Painter deletes all 164 vertices in Y, but doing so leaves a vertex in X with no tokens, and Lister wins.

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Similarly, $\chi_p(K_{3,3}) > 2$ and $\chi_p(K_{4,19}) > 3$.

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General strategy for Lister on $K_{k+j,r}$:

Present some subset of X (at most $j+1$ vertices) and enough of Y to force Painter to delete $M \cap Y$. Repeat.

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This only gives an upper bound on r where Lister wins.

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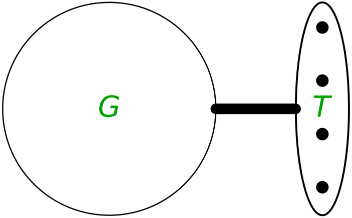
Pf. Idea: Painter uses k-paintability strategy **S** on G, ignoring the added t-set T, until a special round where $M \cap T$ is deleted instead, using up the extra token on $V(G)$. Painter then continues **S** to finish G, after which we each vertex remaining in T still has a token.



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(1) When some $v \in T$ is marked before (∗), count rounds with v marked. Each omits some vertex not yet omitted μ times. Hence $\exists$ fewer than $(t-1)\mu$ such rounds. Since $(t-1)\mu \leq k$, no vertex loses all tokens before (∗). ■

Application

Thm. $\chi_p(G) \le k$ and $|V(G)| \le \frac{t}{t-1}k \Rightarrow \chi_p(G \mathbin{\oplus} \overline{K}_t) \le k+1$.

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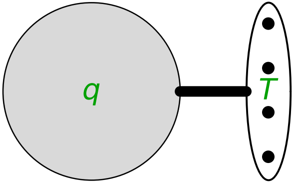
Pf. It suffices to study complete multipartite graphs.

Let $q = $ #singleton parts. If G has one non-singleton part T, then G is chromatic-paintable (vertices of T have $q+1$ tokens; always delete a marked singleton).



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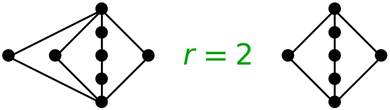

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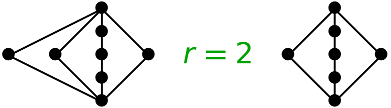

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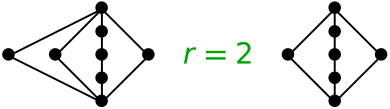

- Uses charzn of 2-paintable graphs (Zhu [2009]).
- Riasat-Schauz [2012] found the minimal 3-paintable graphs under vertex deletion.

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Obs. Since all f-paintable graphs are f-choosable, $\chi_{sp}(G) \geq \chi_{sc}(G)$ for every graph G.

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Painter wins, proving the upper bound. ■

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Mahoney–Tomlinson–Wise [2012+] did $\chi_{sp}(\Theta_{l_1,\ldots,l_r})$ for $l_1 = 1$.

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Obs. sc-greedy $\Rightarrow$ sp-greedy.

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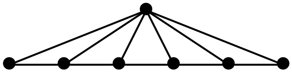

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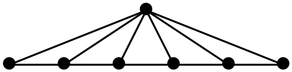

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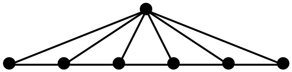

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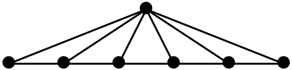

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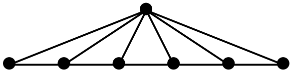

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Obs. The chordal graph $K_2 \oplus \overline{K}_t$ is not sp-greedy (Mahoney–Tomlinson–Wise)

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Ques. What other bounds on choosability are also valid for paintability? (Mahoney has other such results.)