Pagenumber of Complete Bipartite Graphs

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ABSTRACT

Given an ordering of the vertices of a graph around a circle, a *page* is a collection of edges forming noncrossing chords. A *book embedding* is a circular permutation of the vertices together with a partition of the edges into pages. The *pagenumber* $t(G)$ (also called *book thickness*) is the minimum number of pages in a book embedding of G. We present a general construction showing $t(K_{m,n}) \leq \lceil (m + 2n)/4 \rceil$, which we conjecture optimal. We prove a result suggesting this is optimal for $m \geq 2n - 3$. For the most difficult case $m = n$, we consider vertex permutations that are *regular*, i.e., place vertices from each partite set into runs of equal size. Book embeddings with such orderings require $\lceil (7n - 2)/9 \rceil$ pages, which is achievable. The general construction uses fewer pages, but with an irregular ordering.

1. INTRODUCTION

Recent work on the efficient layout of networks in VLSI [3] has sparked interest in *book embeddings* of graphs. This name arises by viewing the vertices as placed along a line, called the spine of the book, in some order. The edges are then embedded without crossing in half-planes bounded by the line, which are called *pages* of the book embedding. A central goal in the study of book embeddings is to find the minimum number of pages in any book embedding of a graph G; this is called the *pagenumber* or *book thickness* $t(G)$.

For analysis, an equivalent formulation is often more helpful. The ordering on the spine can be viewed as an ordering of the vertices on a circle, and then

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the pages are collections of noncrossing chords. The early paper of Bernhart and Kainen [1] characterized the graphs with $t(G) = 1$ as the outerplanar graphs and those with $t(G) = 2$ as the subgraphs of planar Hamiltonian graphs. Although graphs with pagenumber 3 may have arbitrarily high genus [1], the maximum pagenumber of planar graphs was a lively subject of investigation (e.g., [2,5]) until Yannakakis [6] determined it to be 4.

Book embeddings have also been studied for special classes of graphs. Games [4] considered several networks important in VLSI. Bernhart and Kainen showed $t(K_n) = \lceil n/2 \rceil$ (the pages are $\lceil n/2 \rceil$ rotations of a simple path), but $t(K_{n,n})$ has proved far more elusive. They gave a construction and elementary argument to show $n/2 \le n - 1$, for $n \ge 4$.

In this paper we begin a systematic study of book embeddings of $K_{m,n}$. Any circular vertex ordering for $K_{m,n}$ has some number of runs of vertices from each partite set; an *r-bucket ordering* is a circular ordering of m X's and n Y's with r runs of each type. Let $t(m,n) = t(K_{m,n})$, and let $t_r(m,n)$ be the minimum number of pages in a book embedding of $K_{m,n}$ using an r-bucket ordering. An r-bucket ordering is *regular* if for each partite set the runs all have the same size.

Given a vertex ordering, a set of edges forming a pairwise crossing set of chords are called a *twist* and must go on separate pages. The lower bound of [1] can be obtained by observing that any vertex ordering of $K_{n,n}$ has a twist of size $n/2$ (any division of the ordering into halves has at least $n/2$ X's in one half and at least $n/2$ Y's in the other half). Intuition suggests that lack of large twists permits efficient embeddings. Every vertex ordering has a twist of size $n/2$, and only the regular 2-bucket ordering of $K_{n,n}$ has no larger twist. We show in Section 5 that optimal embeddings for this ordering have $\lceil (7n - 2)/9 \rceil$ pages. More generally, regular embeddings of $K_{m,n}$ require at least $\min\{n, \lceil (5n + 2m - 2)/9 \rceil\}$ pages.

Surprising, intuition fails to produce the best result here; the regular 2-bucket ordering does not achieve $t(m,n)$. We can embed $K_{n,n}$ in $\lceil 3n/4 \rceil$ pages, and more generally in Section 3 provide a 2-bucket embedding of $K_{m,n}$ in $\lceil (m + 2n)/4 \rceil$ pages. Section 4 contains useful reductions of the 2-bucket problem. Then Section 6 contains a result toward optimality of the construction; $t_2(2n - 3, n) = n$. Bernhart and Kainen used the pigeonhole principle to show that $t(m,n) = n$ for $m > n(n - 1)$; our result suggests that this can be improved to $t(m,n) = n$ when $m > 2n - 4$. The main difficulty in doing so, and indeed in making significant further progress on the computation of $t(m,n)$, is to show that optimality is achieved by a 2-bucket ordering.

2. ELEMENTARY RESULTS

To prove their upper bound of $t(n,n) \le n - 1$ for $n \ge 4$, Bernhardt and Kainen provided an inductive construction, using a 2-bucket ordering, in which they added 4 to n at each step, and thus gave explicit constructions for $n = 4,5,6,7$ as a basis. Applying the following more general lemma with $r = 2$, one need only supply constructions for $n = 4,5$ to get the same result.

Lemma 1. $t_r(m, n) \le t_r(m - r, n - r) + r$.

Proof. Consider an optimal r-bucket embedding of $K_{m-r,n-r}$. Extend each bucket of X's and each bucket of Y's by adding one vertex at the clockwise end of the ordering. We need only add r pages that embed all edges involving at least one of these new vertices. Each of these pages contains a complete matching on the $2r$ new vertices, consisting of "parallel" edges in a configuration that rotates from page to page. In addition, include edges joining each new vertex to each old vertex in the bucket containing its mate. Since there is a new vertex in each bucket, the new pages will contain edges from each new vertex to each old vertex of the other type. Figure 1 illustrates the new pages when $r = 2$. ∎

Next we restate the pigeonhole argument for general m, n.

Lemma 2. $t(kn, n) \ge (k/(k + 1))n$.

Proof. We show every ordering has a twist of the desired size. Partition the vertices into $k + 1$ groups of n consecutive vertices. With kn Y's and n X's, one of these groups must have at least $nk/(k + 1)$ Y's. Since the group size is n, there must be at least $nk/(k + 1)$ X's outside this group. This set of X's yields a twist with the specified set of Y's. ∎

Finally, another simple counting lemma disposes of regular orderings with more than two buckets. For vertices around a circle, let the *distance* between vertices be the number of spaces between vertices that must be crossed to get from one to the other (at most $(m + n)/2$ in a set of $m + n$ vertices), and let the *length* of an edge be the distance between its endpoints.

Lemma 3. At most $\lfloor p/k \rfloor$ edges of length exactly k can appear on the same page in a book embedding of a graph on p vertices. In particular, any regular embedding of $K_{m,n}$ with $r > 2$ buckets requires at least n pages, where $n \le m$.

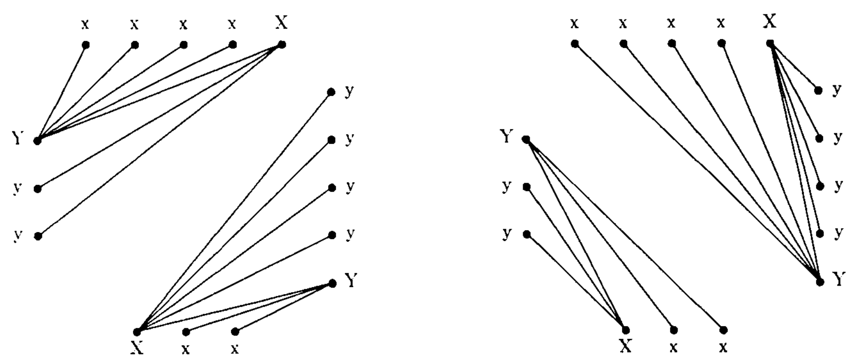


FIGURE 1. Additional pages for inductive construction with $r = 2$.

Proof. Edges of the same length cannot fit "inside" each other on a page, and hence must cut off disjoint arcs of the circumference, possibly meeting at a vertex. The total circumference is p, so there can be at most $\lfloor p/k \rfloor$ of these edges.

Now assume $m \geq n$ and the ordering has r buckets. If r is odd, pair each bucket of size n/r with the diametrically opposite bucket of size m/r, which has vertices of the other type. Choosing a twist of size n/r from each such pair yields a twist of size n for the overall ordering.

If r is even, each Y vertex has an edge to the vertex at distance $k = ((m + n)/2) - (n/r)$ in each direction. Since $m \geq n$, we have $k \geq ((m + n)/3) + n(1/3 - 1/r)$, which exceeds $(m + n)/3$ when $r > 3$. Hence no three of these edges can lie on the same page. Since there are $2n$ edges of length k, we require at least n pages for these edges. ∎

3. THE ENCODING AND THE CONSTRUCTION

There is an encoding that makes book embeddings of complete bipartite graphs somewhat easier to visualize and describe. Label the X vertices with indices $1, \ldots, m$ in clockwise order; these indices will correspond to the rows of an $m \times n$ grid. Label the Y's similarly, corresponding to the columns of the grid. For a complete bipartite graph, location i, j of the grid corresponds uniquely to the edge from X_i to Y_j. Placing this edge on page c is equivalent to placing label c or *color* c at that location of the grid. The usage of a single color in the grid must satisfy restrictions equivalent to forbidding crossings on a page.

For convenience, we assume that X_1 and Y_1 begin runs (buckets), and that X_1 begins the run clockwise following Y_1. This partitions the grid into subgrids such that the i,jth subgrid corresponds to the edges from the ith run of X's to the jth run of Y's. The numbering is illustrated in Figure 2, along with two feasible pages for the 2-bucket case. In the 2-bucket case, we call the subgrids *quadrants,* and we label the runs as $Y_1, \ldots, Y_q$, $X_1, \ldots, X_p$, $Y_{q+1}, \ldots, Y_m$, and $X_{p+1}, \ldots, X_n$. Now we translate the noncrossing condition into the grid encoding. First consider the 2-bucket case; here it is easy to visualize what constitutes a legal page. Each quadrant corresponds to a pair of neighboring runs in the ordering. Edge pairs with endpoints in all four runs belong to diagonally opposite subgrids and never conflict, so we need only consider positions within the same quadrant or neighboring quadrants. Each quadrant has a "free" position corresponding to an edge of length 1; it crosses no other and can be added to any page. These positions are marked in Figure 2. Edges encoded in two neighboring quadrants all have endpoints in a single run; suppose this run is in X. If (x, y) and (x', y') are two such edges on a page and we move clockwise within in the run to reach x' from x, then we must move counterclockwise (or stay fixed) outside the run to reach y' from y. In terms of the encoding, this means that in two neighboring quadrants a single color must occupy a subset of a lattice path between the two free corners. The grid is actually a discrete torus; for the pairs of quadrants involving the upper right quadrant, the lattice path wraps around in columns or rows.

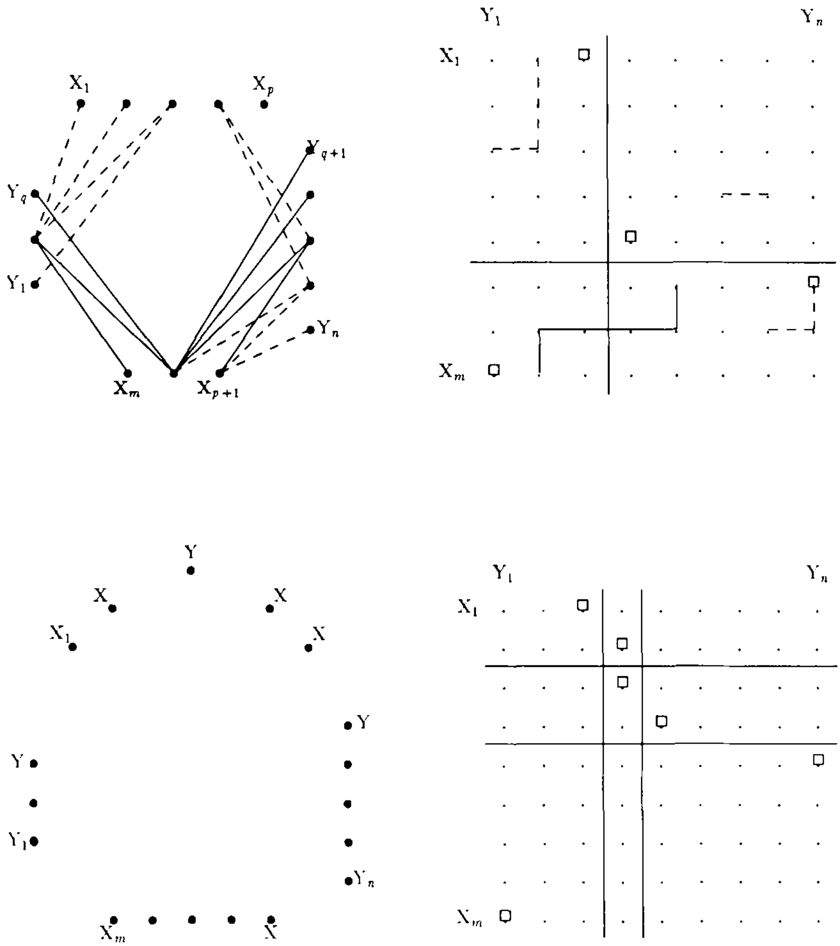


FIGURE 2. Encoding of the edges in 2- or 3-bucket embeddings.

For more buckets, the condition generalizes naturally. Again suppose two edges have endpoints in four distinct runs. For the corresponding subgrids, either all pairs of positions conflict or all pairs are compatible. For example, in the 3-bucket case any pair of edges between disjoint pairs of diametrically opposite runs cross, but any pair of edges between disjoint pairs of consecutive runs are always compatible. This describes the situation when the subgrids containing two positions are not in the same row of subgrids or the same column of subgrids. When they are in the same row [column], the edges have endpoints in the same run S, and the order of the endpoints yields a lattice path condition in the same manner as before. In each row [column] of subgrids, there are two subgrids with a free position, corresponding to the edges between S and the two neighboring runs from the other partite set. The free positions correspond to consecutive vertices in the other partite set and therefore appear in cyclically consecutive columns [rows]. The edges of a single page that are incident to S correspond to a subset of a lattice path between these two free positions. The path wraps around in columns [rows] to avoid stepping directly between the consecutive columns [rows] containing the free positions.

These lattice paths can be described technically in terms of the positions (i,j), but we prefer to avoid that so as to make more geometric arguments. Having specified the noncrossing conditions for the grid coloring, we henceforth often avoid verbiage by using the word *edge* to describe either an edge of $K_{m,n}$ or the corresponding grid position, and we can refer to a legal k-coloring of the grid as a k-page book embedding. In the remainder of the paper, we discuss only 2-bucket vertex orderings. It is convenient, therefore, to refer to a k-page book embedding using a 2-bucket ordering as a *k-page 2-embedding*, and a k-page book embedding using a regular 2-bucket ordering as a *regular k-page 2-embedding*. A k-page 2-embedding of a subgraph of $K_{m,n}$ is a *partial* k-page 2-embedding.

Using the encoding, we can give a simple description of an efficient 2-embedding for general $K_{m,n}$.

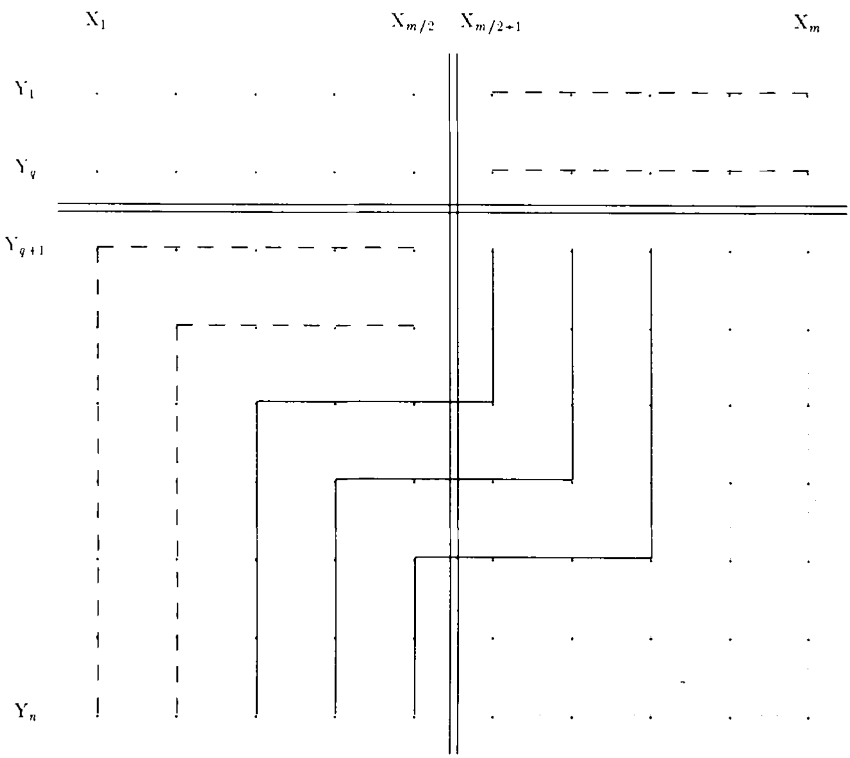
Theorem 1. $t(K_{m,n}) \le t_2(m,n) \le \min\{\lceil (m + 2n)/4 \rceil, n\}$, where $m \ge n$.

Proof. Since we can always use a separate page for each vertex of Y, we need only exhibit a $\lceil (m + 2n)/4 \rceil$-page 2-embedding when $n \le m \le 2n$. Note that $\lceil (m + 2n)/4 \rceil = \lceil (m + 1 + 2n)/4 \rceil$ if m is odd. Hence we can obtain the desired embedding when m is odd by deleting a vertex from the embedding for $K_{m+1,n}$, and we may assume henceforth that m is even. Similarly, when m is even we have $\lceil (m + 2n)/4 \rceil = \lceil (m + 2(n + 1))/4 \rceil$ if n has opposite parity from $m/2$. Hence we may also assume n has the same parity as $m/2$, obtaining the desired embedding by deleting a vertex from the embedding for $K_{m,n+1}$ if not.

We give an explicit 2-embedding. Let $q = (2n - m)/4$. Split X evenly into runs of length $m/2$. Split Y into runs of length q and $n - q$. Since $q \le n/4$, this produces two large lower quadrants and two small upper quadrants, as indicated in Figure 3 for $m, n = 10, 9$, where for aesthetic reasons we have drawn the encoding with the roles of X and Y interchanged. We therefore index rows by j and columns by i in describing this construction.

The edges of a page in a quadrant appear as a path in Figure 3. We use three types of pages (colors). Type 1 colors (solid paths) appear in the two lower quadrants. Type 2 colors (dashed paths) appear in the upper right and lower left quadrants. Type 3 colors (dotted paths) appear in the upper left and lower right quadrants. We use q colors of the two latter types and $m/2 - q$ colors of the first type. The total number of colors is $m/2 + q = (m + 2n)/4$.

For $j = 1, \dots, q$, the positions occupied by the jth Type 3 color in the upper left are $\{(j, 1), \dots, (j, m/2)\}$, and those of the jth Type 2 color in the upper right are $\{(j, m/2 + 1), \dots, (j, m)\}$. In the lower left quadrant, the jth Type 2 color occupies $\{(n,j), \dots, (q + j, j), \dots, (q + j, m/2)\}$. These two "segments" in the grid correspond to edges in $K_{m,n}$ incident to X_j or Y_{q+j}. Next, for $1 \le j \le m/2 - q$ the edges of the jth Type 1 color are all incident to Y_{2q+j}, X_{q+j}, or



$X_{m/2+j}$, and they occupy grid positions $\{(n, q + j), \ldots, (2q + j, q + j), \ldots, (2q + j, m/2 + j), \ldots, (q + 1, m/2 + j)\}$. Finally, the jth Type 3 color occupies grid positions $\{(m/2 + q + j, m/2 + 1), \ldots, (m/2 + q + j, m - q + j), \ldots, (q + 1, m - q + j)\}$, all incident to $Y_{m/2+q+j}$ or X_{m-q+j}.

By construction, the pages comprise disjoint sets of noncrossing edges. The horizontal portions of the colors in the lower quadrants march successively from row $q + 1$ to row $m/2 + 2q$. Since $m/2 + 2q = n$, all edges are colored, which completes the proof. ∎

In particular, $t(K_{n,n}) \leq \lceil 3n/4 \rceil$.

4. REDUCTIONS FOR THE 2-BUCKET PROBLEM

In defining the encoding, we noted that each quadrant has a free position corresponding to an edge of length 1, which we henceforth call HOME. Given any book embedding of the rest of $K_{m,n}$, HOME can be added to any page. We want to identify other positions we may ignore in considering colorings.

For the remainder of the paper, we return to the convention of rows indexed $1 \leq i \leq m$, columns indexed $1 \leq j \leq n$, and vertex ordering $Y_1, \ldots, Y_q$,

$X_1, \ldots, X_p, Y_{q+1}, \ldots, Y_n, X_{p+1}, \ldots, X_m$, as in Figure 2. Think of the X's as being on the top and bottom of the ordering and the Y's on the left and right of the ordering, so the top quadrants have p rows and the left quadrants have q columns. When $m = n$, we assume $p \leq q \leq n/2$.

Let the edge farthest from HOME in a quadrant be called AWAY. The *distance* between two edges (i,j), (i,j') in a quadrant is $|i - i| + |j - j'|$. The distance of an edge from HOME is one less than its length. The edges of a given length in a quadrant form a *diagonal*; they have the same distance from HOME and have a constant value of $|i - j|$. Edges getting the same color belong to a lattice path from HOME to AWAY that increases length by one at each step. Note that the edges of a diagonal form a twist and must get different colors; in fact, any set of positions from two adjacent quadrants that have no pair on any lattice path between the free positions forms a twist.

A diagonal whose size is the number of rows or columns of the quadrant is call a *full diagonal*. The full diagonal farthest from HOME is the *major diagonal;* it always contains a corner of the quadrant. Let M be the length of edges in the major diagonal; let L be the length of AWAY. The edges of length $M + j$ in the quadrant form the *jth superdiagonal*, denoted D_j. The triangular set of grid positions consisting of the major diagonal and all superdiagonals is called the *essential triangle*. This name suggests that if the long edges are properly colored then the rest comes for free, which is justified by Lemma 5. Lemmas 5 and 6 say that we need only find a partial k-page 2-embedding for a particular subgraph of $K_{m,n}$ to have a k-page 2-embedding of $K_{m,n}$. Lemma 7 allows us to make further assumptions about what that embedding looks like. The extensions and recolorings needed to prove these lemmas rest on the following simple remark.

Lemma 4. Consider an edge $e = xy$ in quadrant Q in a 2-bucket ordering of $K_{m,n}$. Let S denote the set of edges in Q that are shorter than e and do not cross e. Let T denote all other edges that do not cross e. Then no edge of S crosses any edge of T.

Proof. This statement is a simpler instance of the reasoning used in Section 3 to describe legal usage of a color in terms of lattice paths. From the vertex ordering, define two segments, A, B of vertices, each with endpoints x, y. No edge of S crosses any edge of T because all endpoints of edges of S lie in one of A, B, and all endpoints of edges of T lie in the other. ∎

Note that $S \cup \{e\}$ and $T \cup \{e\}$ are rectangles in the discrete torus. S is the rectangle whose opposite corners are e and HOME of Q. T is the rectangle whose opposite corners are e of Q and AWAY of the quadrant diagonally opposite Q. $K_{m,n} - S - T - \{e\}$ are the edges crossed by e. Typically, usage of this lemma is to extend partial embeddings. If a legal page contains e and no edge of S, then we can add an edge of S.

Lemma 5. Any partial k-page 2-embedding including the essential triangles extends to a k-page 2-embedding of $K_{m,n}$.

Proof. The edges on the major diagonal of a quadrant have distinct colors; let C be the ordered list of these colors. Complete the coloring of the quadrant by using C in order on each successive diagonal closer to HOME. When the diagonals begin to get shorter, at each step delete a color from one end of C. By Lemma 4, the choices on each successive diagonal yield no crossings. ∎

We call the edges of the essential triangles the *essential edges*. For some vertex orderings, we can ignore additional positions. Consider two adjacent quadrants A, B. If the dimensions are suitable, a color close to HOME in A is forbidden from the essential triangle in B (and hence all B, by Lemma 5), which allows us to use it more freely in A. The application of this lemma to a 2-bucket ordering of $K_{n,n}$, where we may assume $p \leq q \leq n/2$, appears in Figure 4. The dots in Figure 4 designate the only positions that need to be colored.

Lemma 6. Given a 2-bucket ordering, let A, B be two adjacent quadrants, together comprising c columns [rows]. Suppose A, B have a, b rows [columns], and suppose $a < c$. Let $s = a - \max\{0, c - b\}$. Then any partial k-page 2-embedding whose edges in B include $D_0 \cup \cdots \cup D_s$ extends to a partial k-page 2-embedding including all of B.

Proof. We may assume that A and B are vertically adjacent and comprise c columns. Any position in D_s is at least a columns from HOME in B, but all points of the essential triangle in A are less than a columns from AWAY in A. Hence no color in $C = D_s$ appears in the essential triangle of A. This means that the color α in row i of C can be used on row i of D_j in B without penalty

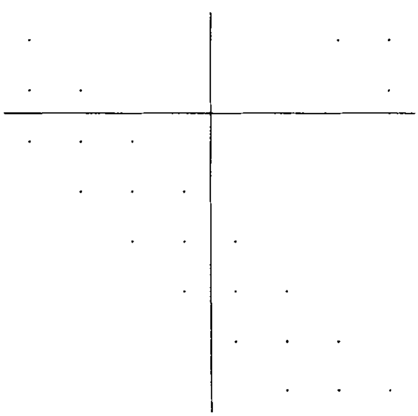


FIGURE 4. Result of Lemma 6 for a 2-bucket ordering of $K_{8,8}$.

for all $j > s$. These positions exert no constraint on usage of α in the quadrant horizontally adjacent to B that is not already exerted by the presence of α in C. This extension of the colors on C completes the coloring of the essential triangle in B, and we apply Lemma 5. ∎

For a 2-embedding of $K_{m,n}$, Lemma 6 reduces us to coloring a continuous band of positions on the discrete torus, with $p + 1$ positions in every row and column. For general $K_{m,n}$, Lemma 6 can be very effective; if $a + b \leq c$, then we need only color the major diagonals in A and B. For regular 2-embeddings of $K_{m,n}$ with $m < 2n$ ($m \geq 2n$ yields a twist of size n), we get two bands with $n + 1 - (m/2)$ positions in each row.

We could consider only partial k-page 2-embeddings of the edges remaining after applying Lemma 6, but it will be convenient to retain the full essential triangles, which explains why we called these the essential edges of $K_{m,n}$. Next, we further restrict partial k-page 2-embeddings of the essential edges to a canonical form. In any book embedding, define the longest edge of a quadrant on a given page to be the *leading edge* of that page in that quadrant. A *staircase embedding* of $K_{m,n}$ is a 2-embedding in which each quadrant has exactly one leading edge of each length $M, \ldots, L$, which are the lengths of edges in the essential triangle. Note that in a staircase embedding, since there is only one leading edge of each length, a page whose leading edge has length $M + j$ must also have edges of length $M, \ldots, M + j - 1$.

Lemma 7. If the essential edges of $K_{m,n}$ have a partial k-page 2-embedding, then $K_{m,n}$ has a k-page staircase embedding.

Proof. Select a quadrant Q. We recolor to obtain the desired embedding iteratively, in decreasing order of length. The edges of D_{r-M} have length r. For the single edge of length L there is nothing to prove; it is the leading edge of that page. Now assume $L > r > M$ and the edges of length exceeding r in Q are embedded in pages $c_1, \ldots, c_{L-r}$ with one leading edge of each length. Since D_{r-M} has $L - r + 1$ edges, there is some color c_{L-r+1} on D_{r-M} that does not appear among $c_1, \ldots, c_{L-r}$ and thus is the leading edge of its page in Q. We need only show that the other edges of D_{r-M} can be given colors $c_1, \ldots, c_{L-r}$. In fact, there is a unique way to do this; the order of $c_1, \ldots, c_r$ on D_{L-M-r} must be the same as their order on $D_{L-M-r+1}$, as indicated in Figure 5 for the two possible orientations of the essential triangle. By Lemma 4, this introduces no crossings, except possibly with the edges yet to be recolored. Since only edges of colors not yet used will be unchanged on later diagonals, any such crossing will be corrected. ∎

5. REGULAR 2-BUCKET EMBEDDINGS

Additional arguments depend on knowing the sizes of the runs, so in this section we consider only regular 2-embeddings of $K_{m,n}$. Lemma 7 allows us to assume

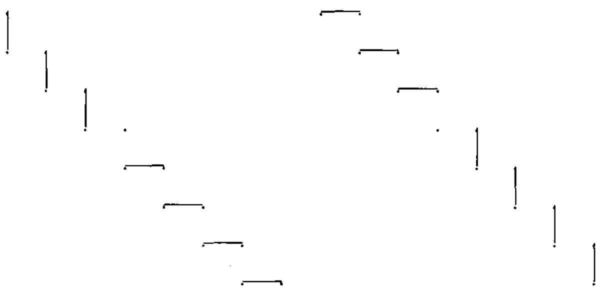


FIGURE 5. Extension of staircase embeddings.

the optimal embedding is a staircase embedding. Since a staircase embedding is completely determined by the choice of leading edges for each page, it is natural to consider what sets of lengths are allowable for the leading edges. In each quadrant of a regular embedding, the length of edges on the major diagonals is $M = m/2$, and the length of AWAY is $(m + n)/2 - 1$. The *length* of a page is the sum of the lengths of its leading edges. The bound on the length of a page depends on the number of quadrants it appears in.

Lemma 8. In a regular staircase embedding of $K_{m,n}$, a page with edges in $4, 3, 2, 1$ quadrants has length at most $m + n$, $m + n - 1$, $m + n - 2$, or $(m + n - 2)/2$, respectively, yielding at most 4, $n - m/2 + 2$, n, or $n/2$ essential edges. A page with edges in two adjacent quadrants has length at most $m + n/2 - 1$, with at most $n/2 + 1$ essential edges. If $m \neq n$, no page appears in four quadrants. Finally, the j longest pages using two opposite quadrants have total length at most $(m + n)j - \frac{1}{2}(j^2 + 2j + \epsilon(j))$ and yield at most $(n + 2)j - \frac{1}{2}(j^2 + 2j + \epsilon(j))$ essential edges, where $\epsilon = 0, 1$ as j is even or odd.

Proof. Consider the circular vertex ordering, and think geometrically. The leading edges of a page must enclose disjoint portions of the ordering. If each of the leading edges starts at the same vertex where the previous one ends, the total length of $m + n$ can be attained. However, this is only possible with edges from all four quadrants. With three, no edge encloses the free edge corresponding to HOME of the unused quadrant. With two quadrants, two such spots are missing, and if the two quadrants are adjacent, then in addition one entire run of the ordering must be subtracted from the length. For one quadrant, the bound is the length of AWAY. To count the essential edges obtained, we must subtract $s(m/2 - 1)$ from the length of a page in s quadrants, because the essential edges begin with the major diagonal, whose edges have length $m/2$. A s-quadrant page thus has length at least $sm/2$, which forbids 4-quadrant pages when $m > n$ and 3-quadrant pages when $m \geq 2n$.

We must be careful about the length of pages in two opposite quadrants. Length $m + n - 2$ requires two edges of length $(m + n - 2)/2$, but there are only four such edges, so there are at most two such pages. In general, each quadrant has exactly one leading edge of each length $m/2, \ldots, (m+n-2)/2$. A page in two opposite quadrants has length $(m+n)/2 - s + (m+n)/2 - t$, where its leading edges are the sth and tth longest leading edges in those quadrants. The total "loss" will be least if the leading edges of these j pages are the $\lceil j/2 \rceil$ longest in one pair of quadrants and the $\lfloor j/2 \rfloor$ longest in the other pair. The resulting count of $(m+n)j - 2\binom{\lceil j/2 \rceil + 1}{2} - 2\binom{\lfloor j/2 \rfloor + 1}{2}$ simplifies to the formula claimed above. ∎

In the four essential triangles there are $n(n+2)/2$ positions, of which $2n$ positions are on major diagonals. The limits on what a page can contribute to these two counts give us a lower bound on the number of pages needed.

Theorem 2. A regular embedding of $K_{m,n}$ requires at least $\min\{n, \lceil (5n + 2m - 2)/9 \rceil\}$ pages.

Proof. Consider a k-page 2-embedding; we may assume it is a staircase embedding. Let a_i be the number of pages appearing in i quadrants. We want to minimize $k = \sum a_i$ subject to two conditions. The requirement from diagonal edges is $\sum i a_i = 2n$, and we need a quadratic inequality $f(a_1, a_2, a_3, a_4) \geq n(n+2)/2$ enforcing the requirement that all essential positions be covered. The function f simply adds up the limits obtained in Lemma 8 for the usage of pages among the essential edges. This is a purely numerical argument and ignores whether the configuration is realizable, but the resulting value of k is a lower bound on the number of pages in a regular embedding of $K_{m,n}$.

There are two simplifications we can make in f. First, we may assume $a_4 = 0$. Lemma 8 already guarantees this if $m \neq n$. Consider $K_{n,n}$. Except for pages in two opposite quadrants, we get at most $n/2 + 2$ essential edges per page, which requires $n(n+2)/(n+4) \geq n - 2$ pages if $a_2 = 0$. Hence we may assume $a_2 > 0$. Choose the smallest value of a_4 in an $\bar{a}$ minimizing k. If $a_4 > 0$, we can add $-1, +2, -1$ to a_2, a_3, a_4 to preserve the diagonal requirement and increase f by at least $-n + n + 4 - 4 = 0$, because any page counted by a_2 contributes at most n essential positions. The resulting $\bar{a}$ is also a solution.

Now consider a_1. If $a_3 = 0$, then the diagonal requirement forces $k \geq n$. Hence we may assume $a_3 > 0$, and if $a_1 > 0$ we add $-1, +2, -1$ to a_1, a_2, a_3 to obtain an $\bar{a}$ that satisfies the diagonal requirement, has the same value of k, and increases f by at least $-n/2 + 2(n/2 + 1) - (n - m/2 + 2) \geq 0$. The last statement rests on a closer look at the contribution from a_2. When the increase from $a_2 = j - 1$ to $a_2 = j$ in the contribution $(n+2)j - \frac{1}{2}(j^2 + 2j + \epsilon(j))$ from using pages in opposite quadrants falls below $n/2 + 1$, we switch to pages in adjacent quadrants, and thereafter count a possible contribution of $n/2 + 1$ to f for each such page. Therefore, the contribution to f from each increase in a_2 is at least $n/2 + 1$.

We are left with a minimization problem in two variables. Letting $y = a_3$ and $z = a_2$, we want to minimize $y + z$ subject to $3y + 2z = 2n$ and $f(y, z) \geq$

$n(n+2)/2$. The equality constraint allows us to reduce to one variable by setting $y = \frac{2}{3}(n - z)$. The bound on pages is now $(2n + z)/3$, so we want to find the smallest value of z satisfying $f(z) \geq n(n+2)/2$. We have $f(z) = (n - m/2 + 2)(2/3)(n - z) + (n + 2)z - \frac{1}{2}(z^2 + 2z + \epsilon(z))$ for $z \leq n/2$. If $f(z) \geq n(n+2)/2$ somewhere in this range, we need not consider the modification needed for $z > n/2$. Treating $\epsilon(z)$ as a constant, we solve the quadratic equation $f(z) = n(n+2)/2$. It is helpful to set $m = \alpha n$ and clear fractions to obtain $3z^2 - 2[n(1 + \alpha) - 1]z + (2\alpha - 1)n^2 - 2n + 3\epsilon = 0$. The solution to this simplifies surprisingly to $z = (n + \alpha n - 1 + \frac{1}{3}\sqrt{[n(\alpha - 2) - 1]^2 - 9\epsilon})$. Ignoring ϵ yields $z = (2m - n - 2)/3$ and $y = (8n - 4m + 4)/9$. Both y and z must be integers; if $\lceil (2m - n - 2)/3 \rceil$ is even, then ϵ may allow z to be one smaller, but then y may be bigger. In all cases we find that the number of colors must be at least $\lceil (5n + 2m - 2)/9 \rceil$. ∎

In fact, satisfying these counting requirements is also sufficient for the existence of regular k-page staircase embeddings. To see this, we define matrices that will summarize the potential usage of pages in quadrants in a staircase embedding. Given a k-page staircase embedding, we want the jth entry of the ith row of the corresponding matrix to be the number of essential edges page i contributes to quadrant j. Lemma 8 has given us some necessary requirements for this matrix. With those in mind, we define a class of matrices.

Definition A. *k-book* for (m, n) (with $m \geq n$) is a k by 4 matrix in which each column consists of $\{1, \ldots, n/2\}$ and $k - n/2$ 0's, and each row $\bar{x}$ satisfies the following:

(1) If $\bar{x}$ has four nonzero terms, they are all 1. (This is allowed only if $m = n$.)
(2) If $\bar{x}$ has three nonzero terms, then $\Sigma x_j \leq n - m/2 + 2$.
(3) If $\bar{x}$ has $\{x_1, x_4\}$ or $\{x_2, x_3\}$ nonzero, then $\Sigma x_j \leq n/2 + 1$.
(4) If $\bar{x}$ has $\{x_1, x_2\}$ or $\{x_3, x_4\}$ nonzero, then $\Sigma x_j \leq n - m/2 + 1$.

Note that there is no explicit restriction when one term or two terms with indices of the same parity are nonzero. However, the fact that each column contains each of $\{1, \ldots, n/2\}$ exactly once constrains the total sum in rows of this type.

Lemma 9. There exists a k-page regular staircase embedding of $K_{m,n}$ if and only if there exists a k-book for (m, n).

Proof. Necessity was mostly shown in Lemma 8. Given a k-page regular staircase embedding, form a k by 4 matrix by recording in row i the number of essential edges in page i in each of the four quadrants, with quadrants indexed clockwise from the upper left of the grid encoding. Since a staircase embedding has for each quadrant one page contributing i essential edges for each $1 \leq i \leq n/2$, the column constraint for k-books holds. As discussed in the proof of Lemma 8, the quadrants in which page i appears restrict its length and thus the number of essential edges it has in such a way that the rows of the matrix satisfy the conditions for a k-book. The distinction between the two types of adjacent quadrants arises from whether the relevant vertices omit a run of length $n/2$ or $m/2$ from the ordering. Note that the column conditions on the k-book enforce the global limitation on pages using opposite quadrants.

Sufficiency is not much harder. By the argument in Lemma 7, a staircase embedding is determined by the placement of the leading edges in each page. No matter where on the diagonals the leading edges are placed, the quadrant can be completed in exactly one way. Lemma 4 guarantees that the resulting assignment of edges to pages yields no crossings if and only if for each page the leading edges have no crossings. Therefore, it suffices to show that for any row $\bar{x}$ of a k-book, there is a way to assign noncrossing edges from the specified diagonals in each quadrant; these diagonals are D_{x_j-1} from quadrant j if $x_j > 0$, otherwise no edge from quadrant j.

If $m = n$ and $\bar{x}$ has four nonzero terms, we can choose any four noncrossing edges of length $m/2$, such as those corresponding to the lower right corner of each quadrant in the grid encoding—positions $(n/2, n/2)$, $(n/2, n)$, (n, n), and $(n, n/2)$. If $\bar{x}$ has two nonzero terms with indices of the same parity, there are no crossings between edges of these quadrants, and we can choose the positions arbitrarily from the specified diagonals. Similarly, the choice is arbitrary when $\bar{x}$ has only one nonzero term.

For pages on three quadrants or two adjacent quadrants, we want edges of lengths $x_j + m/2 - 1$, for the nonzero x_j. By reflection and rotation of the circular vertex ordering, we may assume x_1, x_2, x_3 are nonzero, or x_1, x_2 are nonzero, or x_1, x_4 are nonzero (recall we indexed the quadrants clockwise from the upper left, starting with the quadrant of edges between $Y_1, \ldots, Y_{m/2}$ and $X_1, \ldots, X_{n/2}$).

If x_1, x_2 are nonzero, choose $(m/2 - n/2 + x_1, 1)$ and $(n/2 + 1 - x_2, n)$. The condition $x_1 + x_2 \leq n - m/2 + 1$ is the same as $m/2 - n/2 + x_1 \leq n/2 + 1 - x_2$, so there is no crossing. If x_1, x_4 are nonzero, choose $(m/2, n/2 + 1 - x_1)$ and $(m/2 + 1, x_4)$. The condition $x_1 + x_4 \leq n/2 + 1$ is the same as $n/2 + 1 - x_1 \geq x_4$, so again there is no crossing.

If x_1, x_2, x_3 are nonzero, choose the grid positions $(m/2 - n/2 + x_1, 1)$ and $(m, n + 1 - x_3)$ from quadrants 1 and 3; these are the edges of the desired lengths that exert the least influence on quadrant 2. The distance between the endpoints $X_{m/2-n/2+x_1}$ and Y_{n+1-x_3} of these edges is $n/2 - x_1 + n + 1 - x_3 - n/2$, which is at least $m/2 - 1 + x_2$ because $\Sigma_{x_i} \leq n - m/2 + 2$. Hence there is room for an edge of length $m/2 - 1 + x_2$ in quadrant 2. We can choose any positions from D_{x_2-1} lying on a lattice path from HOME to $(m/2 - n/2 + x_1, n + 1 - x_3)$ in quadrant 2. ∎

For regular embeddings, we have reduced the question of finding k-page embeddings to that of constructing k-books. In principle, the same analysis can be followed for any 2-bucket ordering. Again there is an optimal staircase embedding, and it suffices to specify compatible lengths for the leading edges of the pages. The difficulty is that the corresponding constraints in Lemma 8 and the definition of k-books become more difficult to keep track of. In the regular case, however, we can show that the bound of Theorem 2 for $K_{n,n}$ can be achieved. We presume it is also achievable for $K_{m,n}$, but have not worked out the details of the k-book construction.

Theorem 3. The optimal regular embedding of $K_{n,n}$ has $\lceil(7n - 2)/9\rceil$ pages.

Proof. We may assume n is even; Theorem 2 gives the lower bound. By Lemma 1, we need only present constructions of k-books for four of the nine even congruence classes of n mod 18—the places where $\lceil(7n - 2)/9\rceil = \lceil(7(n - 2) - 2)/9\rceil + 1$. These are $n \equiv 0, 4, 8, 14 \bmod 18$. Given the reductions in the proof of Theorem 2, it is not surprising that we use only pages appearing in three quadrants or in two opposite quadrants, and that we use approximately $n/3$ pages in two opposite quadrants and $4n/9$ pages in three quadrants. More precisely, we use values of y and z as indicated in Table 1, for the four important congruence classes.

With y and z chosen appropriately, Table 2 contains a single construction of a $\lceil(7n - 2)/9\rceil$-book that works in each case. We use 10 types of pages. The last two types are those in two opposite quadrants, about $z/2$ of each. There are about $y/8$ (rounded in various ways) of each type of page in three quadrants. These eight types come in four pairs, Type j and Type j'. Recall that the values x_j in the row for a page are the number of essential edges it receives from the four quadrants. The pages of Type j have a small odd number of essential edges from the jth quadrant (numbered cyclically), and the pages of Type j' have a small even number of essential edges from the jth quadrant. Note that y is even in each case.

To verify this construction, we must show that the pages obey the length limits and that each column contains $1, \ldots, n/2$ once each. The main idea for the latter is that the 3-quadrant page types, when taken in pairs j, j', cover a consecutive segment of $1, \ldots, n/2$. For this to hold in quadrant j, with Type j having small odd x_j and Type j' having small even x_j, the number of Type j pages must exceed the number of Type j' by 0 or 1. We can discuss the four congruence cases at once via a notational convenience. In specifying a formula, a

TABLE 1. Number of Pages in Staircase Embedding for $K_{n,n}$

n	$\left\lceil \frac{7n-2}{9} \right\rceil$	y	z
$18p$	$14p$	$8p$	$6p$
$18p + 4$	$14p + 3$	$8p + 2$	$6p + 1$
$18p + 8$	$14p + 6$	$8p + 4$	$6p + 2$
$18p + 14$	$14p + 11$	$8p + 6$	$6p + 5$

TABLE 2. A $\lceil (7n-2)/9 \rceil$-Book for $K_{n,n}$

Page Type	Index	Leading edge in Quadrant j: $j=1$	$j=2$	$j=3$	$j=4$
1	$1 \le i \le \lceil y/8 \rceil$	$2i-1$	$\frac{y}{2}+\left\lceil \frac{y}{8} \right\rceil+1-i$	0	$\frac{y}{2}+1-i$
1′	$1 \le i \le \lceil y/4 \rceil - \lceil y/8 \rceil$	$2i$	$\frac{y}{2}+\left\lceil \frac{y}{4} \right\rceil+1-i$	0	$\frac{y}{2}-\left\lceil \frac{y}{8} \right\rceil+1-i$
2	$1 \le i \le \lfloor y/4 \rfloor - \lfloor y/8 \rfloor$	$\frac{y}{2}+1-i$	$2i-1$	$\left\lfloor \frac{3y}{4} \right\rfloor-\left\lfloor \frac{y}{8} \right\rfloor+1-i$	0
2′	$1 \le i \le \lfloor y/8 \rfloor$	$\left\lceil \frac{y}{4} \right\rceil-\left\lfloor \frac{y}{8} \right\rfloor+1-i$	$2i$	$\frac{y}{2}+\left\lfloor \frac{y}{4} \right\rfloor+1-i$	0
3	$1 \le i \le \lceil y/8 \rceil$	0	$\frac{y}{2}+1-i$	$2i-1$	$\frac{y}{2}+\left\lceil \frac{y}{8} \right\rceil+1-i$
3′	$1 \le i \le \lceil y/4 \rceil - \lceil y/8 \rceil$	0	$\frac{y}{2}+\left\lceil \frac{y}{8} \right\rceil+1-i$	$2i$	$\frac{y}{2}+\left\lceil \frac{y}{4} \right\rceil+1-i$
4	$1 \le i \le \lfloor y/4 \rfloor - \lfloor y/8 \rfloor$	$\left\lfloor \frac{3y}{4} \right\rfloor-\left\lfloor \frac{y}{8} \right\rfloor+1-i$	0	$\frac{y}{2}+1-i$	$2i-1$
4′	$1 \le i \le \lfloor y/8 \rfloor$	$\frac{y}{2}+\left\lfloor \frac{y}{4} \right\rfloor+1-i$	0	$\left\lceil \frac{y}{4} \right\rceil+\left\lfloor \frac{y}{8} \right\rfloor+1-i$	$2i$
13	$1 \le i \le \lceil z/2 \rceil$	$\frac{y}{2}+\left\lfloor \frac{y}{4} \right\rfloor+i$	0	$\frac{y}{2}+\left\lfloor \frac{y}{4} \right\rfloor+i$	0
24	$1 \le i \le \lfloor z/2 \rfloor$	0	$\frac{y}{2}+\left\lceil \frac{y}{4} \right\rceil+i$	0	$\frac{y}{2}+\left\lceil \frac{y}{4} \right\rceil+i$

string of four digits indicates the values to be used in the four cases $n \equiv 0, 4, 8, 14 \bmod 18$. For example, all the information in Table 1 above is encoded by writing $n/2 = 9p + 0247$, $y = 8p + 0246$, and $z = 6p + 0125$. The computations we need appear in Table 3.

Comparison of entries in Table 3 shows that in each case the number of pages of Type j is the number of pages of Type j' plus 0 or 1, so combining them yields a consecutive sequence of small leading edges. In the other quadrants, they also occupy consecutive segments, as indicated in Table 4.

Examination of the columns of Table 4 shows that the column conditions hold. The fact that Types 13 and 24 with long edges take them precisely up to $n/2$ follows from the last line of Table 3. Finally, consider the length condition. We must compute $\sum x_i$ for each page in three quadrants. This is $y + \lceil y/8 \rceil + 1$ for pages of Type 1 or 3, $y + \lceil y/4 \rceil - \lceil y/8 \rceil + 2$ for pages of Type 1′ or 3′, $y + \lfloor y/4 \rfloor - \lfloor y/8 \rfloor + 1$ for pages of Type 2 or 4, and $y + \lfloor y/8 \rfloor + 2$ for pages of Type 2′ or 4′. In the last column of Table 3, the computation is completed to show that each of these is at most $n/2 + 2$. ∎

6. PAGENUMBER OF $K_{m,n}$ FOR LARGE m

As mentioned earlier, Bernhart and Kainen noted that $t(K_{m,n}) = n$ for $m > n(n-1)$ by using the pigeonhole principle to obtain a large twist. Our construction in Theorem 1 shows that $t(K_{2n-4,n}) \leq n - 1$; note also that Theorem 2 shows that regular embeddings of $K_{2n-2,n}$ require n pages. We conjecture that $t(K_{2n-4,n}) = n - 1$ and $t(K_{2n-3,n}) = n$. Toward this we offer the following:

Theorem 4. $t_2(K_{2n-3,n}) = n$.

Proof. With the ordering $Y_1, \ldots, Y_q, X_1, \ldots, X_p, Y_{q+1}, \ldots, Y_n, X_{p+1}, \ldots, X_m$, we may assume $p = n - 2$. Otherwise we have a run of n X's, yielding a twist of size n. Lemma 6 reduces us to coloring only D_0 and D_1 in each quadrant, as indicated in Figure 6, but the aim of this proof is to show we cannot do even that with $n - 1$ colors. For ease of discussion, index the quadrants clockwise from the upper left, as usual, and let D_0^s, D_1^s be the diagonals of interest in quadrant s.

In the two bottom quadrants, $D_0^4 \cup D_1^3$ and $D_1^4 \cup D_0^3$ each form a twist of size $n - 1$. Hence all $n - 1$ colors must appear in each, so the color in row i is the same in both twists. Let α be the one color that now appears in both lower quadrants. Note that in the upper quadrants α can appear only in the HOME columns. Also, no color can appear in column 1 and column n of the upper quadrants, because every color appears in a column less than n in the lower right or a column greater than 1 in the lower left.

Because α can only appear in columns $q, q + 1$ in the upper quadrants, α cannot appear in $D_1^1 \cup D_1^2$, the diagonals next to the main diagonal there. How-

TABLE 3. Functions of y and z in Various Congruence Classes

$y = 8p + 0246$	$z = 6p + 0125$	$y + \lceil y/8 \rceil + 1 = 9p + 1468$
$y/2 = 4p + 0123$	$n/2 = 9p + 0247$	$y + \lceil y/4 \rceil - \lceil y/8 \rceil + 2 = 9p + 2469$
$\lceil y/4 \rceil = 2p + 0112$	$\lfloor y/4 \rfloor = 2p + 0011$	$y + \lfloor y/4 \rfloor - \lfloor y/8 \rfloor + 1 = 9p + 1368$
$\lceil y/8 \rceil = p + 0111$	$\lfloor y/8 \rfloor = p$	$y + \lfloor y/8 \rfloor + 2 = 9p + 2468$
$\lceil y/4 \rceil - \lceil y/8 \rceil = p + 0001$	$\lfloor y/4 \rfloor - \lfloor y/8 \rfloor = p + 0011$	
$\lfloor z/2 \rfloor = 3p + 0012$	$\lceil z/2 \rceil = 3p + 0113$	
$\lceil y/4 \rceil + \lfloor z/2 \rfloor = 5p + 0124$	$\lfloor y/4 \rfloor + \lceil z/2 \rceil = 5p + 0124$	
$y/2 + \lceil y/4 \rceil + \lfloor z/2 \rfloor = 9p + 0247$	$y/2 + \lfloor y/4 \rfloor + \lceil z/2 \rceil = 9p + 0247$	

TABLE 4. Sequences of Leading Edges in Combined Types

Types	No.	Range of Leading Edges in Quadrant j			
$1 \cup 1'$	$\lceil y/4 \rceil$	$1 \leftrightarrow \left\lceil \frac{y}{4} \right\rceil$	$\frac{y}{2} + 1 \leftrightarrow \left\lceil \frac{y}{4} \right\rceil$	—	$\left\lfloor \frac{y}{4} \right\rfloor + 1 \leftrightarrow \frac{y}{2}$
$2 \cup 2'$	$\lfloor y/4 \rfloor$	$\left\lceil \frac{y}{4} \right\rceil + 1 \leftrightarrow \frac{y}{2}$	$1 \leftrightarrow \left\lfloor \frac{y}{4} \right\rfloor$	$\frac{y}{2} + 1 \leftrightarrow \left\lfloor \frac{y}{4} \right\rfloor$	—
$3 \cup 3'$	$\lceil y/4 \rceil$	—	$\left\lfloor \frac{y}{4} \right\rfloor + 1 \leftrightarrow \frac{y}{2}$	$1 \leftrightarrow \left\lceil \frac{y}{4} \right\rceil$	$\frac{y}{2} + 1 \leftrightarrow \left\lceil \frac{y}{4} \right\rceil$
$4 \cup 4'$	$\lfloor y/4 \rfloor$	$\frac{y}{2} + 1 \leftrightarrow \left\lfloor \frac{y}{4} \right\rfloor$	—	$\left\lceil \frac{y}{4} \right\rceil + 1 \leftrightarrow \frac{y}{2}$	$1 \leftrightarrow \left\lfloor \frac{y}{4} \right\rfloor$
13	$\lceil z/2 \rceil$	$\frac{y}{2} + \left\lfloor \frac{y}{4} \right\rfloor + 1 \leftrightarrow \frac{n}{2}$	—	$\frac{y}{2} + \left\lfloor \frac{y}{4} \right\rfloor + 1 \leftrightarrow \frac{n}{2}$	—
24	$\lceil z/2 \rceil$	—	$\frac{y}{2} + \left\lceil \frac{y}{4} \right\rceil + 1 \leftrightarrow \frac{n}{2}$	—	$\frac{y}{2} + \left\lceil \frac{y}{4} \right\rceil + 1 \leftrightarrow \frac{n}{2}$

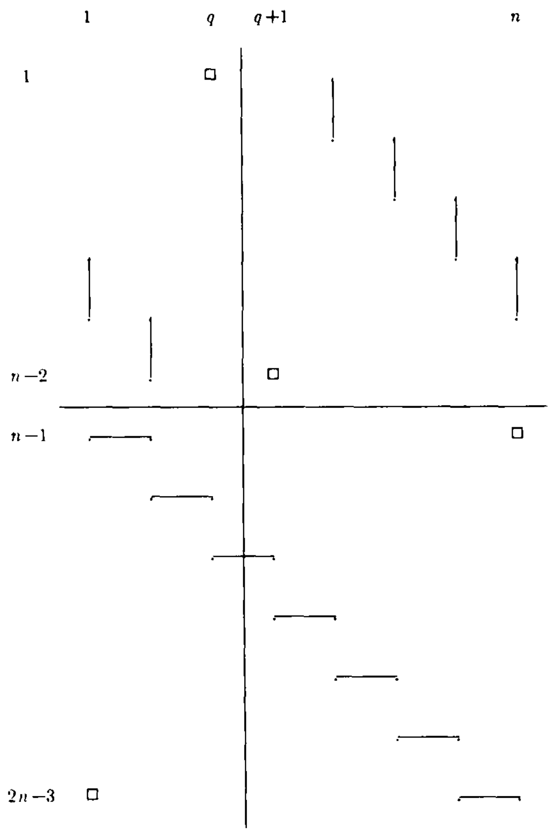


FIGURE 6. Problematic edges for $K_{2n-3,n}$.

ever, $D_1^1 \cup D_1^2$ is a twist of size $n - 2$, so it receives the $n - 2$ other colors. Let β, γ be the colors on D_1^1 in column 1 and D_1^2 in column n. (If $q = 1$, then D_1^1 is empty and we can set $\beta = \alpha$; similarly, if $q = n - 1$, then D_1^2 is empty and we set $\gamma = \alpha$.)

Now consider the colors available for D_0^1 in column 1 and D_0^2 in column n. By considering the twist $D_1^1 \cup D_1^2$, the only colors that can be used are β, γ. Since no color appears in column 1 and column n, we must put β on column 1 in D_0^1 and γ on column n in D_0^2. Looking at successively increasing columns in D_0^1 and decreasing columns in D_0^2, the only color available is the color in the same column of $D_1^1 \cup D_1^2$. Reaching the HOME columns, this leaves only α available to color both $(1, q + 1)$ and $(n - 2, q)$, but it cannot color both. ∎

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Note added in proof. If n is even and $n/3 \leq p < n/z$, then the counting argument of Theorem 2 can be extended to show that at least $\lceil 3n/4 \rceil$ pages are required to embed $K_{n,n}$ when the vertex ordering has buckets of sizes p, $n/2$, $n - p$, and n/z.

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