Overlap Number of Graphs

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Joint work with
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Representation of Graphs

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Def. finite representation: each $f(v)$ is finite
size of repn: $|f| = \left|\bigcup_{v \in V(G)} f(v)\right|$
inters. number $\theta_1(G)$ = min size of finite inters. repn
overlap number $\varphi(G)$ = min size of finite overlap repn

Overlap vs. Intersection

Thm. (Erdős–Goodman–Pósa [1966]) $\theta_1(G)$ is the min size of a decomposition of G into complete subgraphs.

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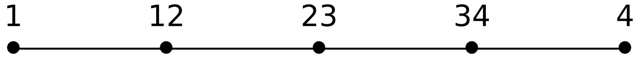

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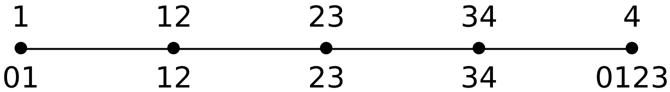

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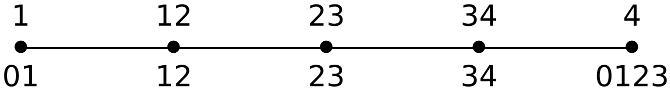
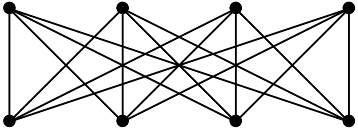


Ex. $\theta_1(K_{r,r}) = r^2$, but $\varphi(K_{r,r}) = 3$.



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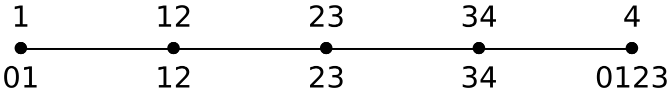

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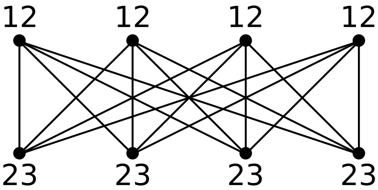

Results (n-vertex graphs)

Rosgen (masters' thesis)
Caterpillars: $\varphi(G) =$ order of the longest path.
Trees: $\varphi(G) \leq n + 1$.
Chordal Graphs: $\varphi(G) \leq 2n$.
Planar Graphs: $\varphi(G) \leq (10/3)n - 6$.
All Graphs: $\varphi(G) \leq \lfloor n^2/4 \rfloor + n$.

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New Results
Trees: linear algorithm, $\varphi(G) =$ order of the skeleton.
Planar Graphs: $\varphi(G) \leq 2n - 5$ for $n \geq 5$, sharp.
All Graphs: $\varphi(G) \leq \frac{n^2}{4} - \frac{n}{2} - 1$ (large even n), sharp.
Edge Bounds: $\varphi(G) \leq |E(G)| - 1$ (sharp for triangle-free connected graphs without star-cutsets).

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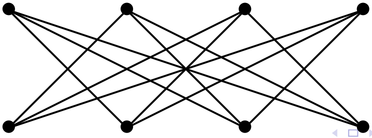

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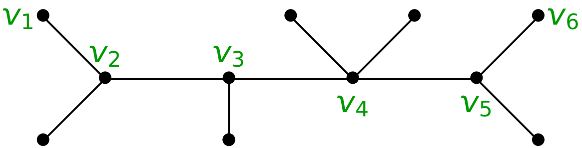
spine $\langle v_1, \ldots, v_l \rangle$: longest path.



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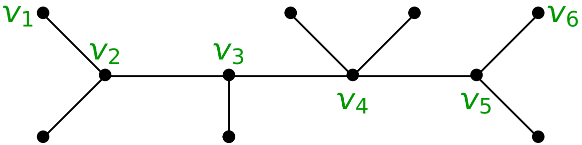

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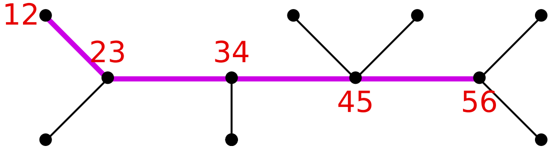

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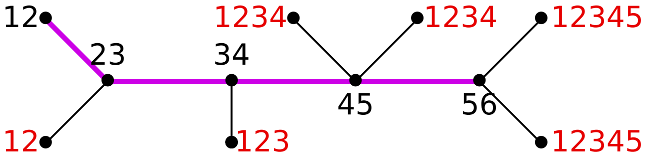

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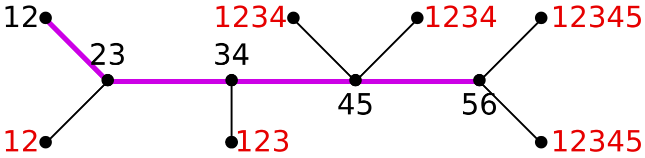

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- $f(v_i)$ is minimal among all sets containing i.

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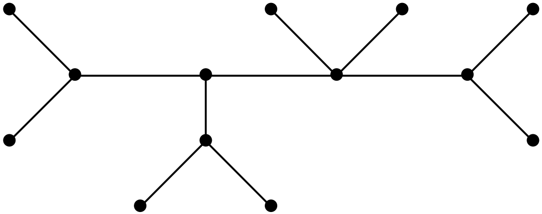
Def. derived tree T': form by deleting every leaf of T.
skeleton of T: keep the derived tree T' plus one leaf neighbor of each leaf of T'.



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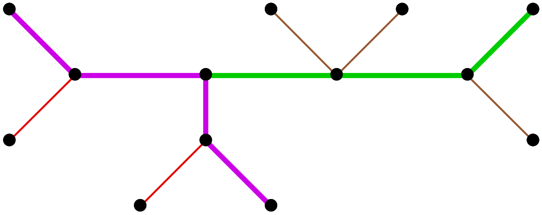

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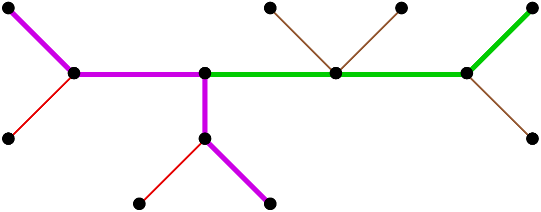

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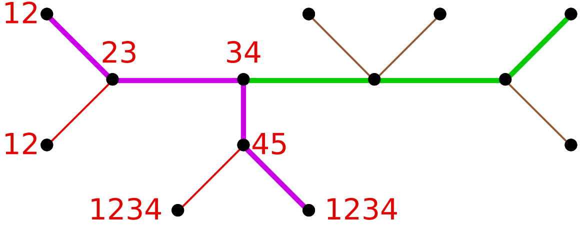

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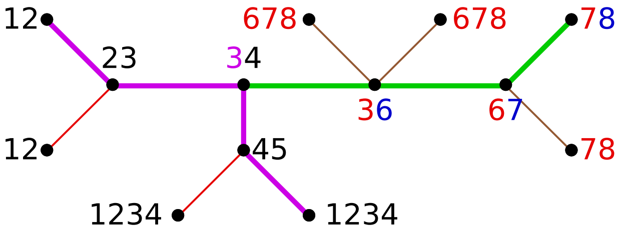

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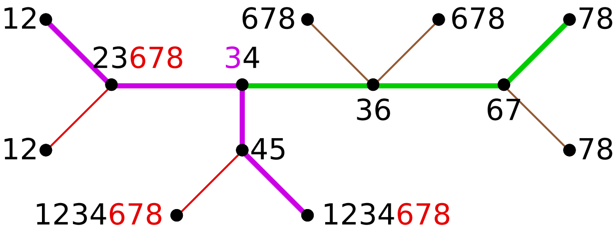

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Sketch of Lower Bound

Suffices to prove that if T is a skeleton, then $\varphi(T) = n$.

Idea Induct on n. Given overlap repn f for a skeleton T, find leaf x [or leaf x and nbr v] whose deletion yields a skeleton T' for which $f - \{a\}$ [or $f - \{a, b\}$] is an overlap repn. This needs $|V(T')|$ elements, so $|f| \geq n$.

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Lem. If f is overlap repn of G, then $f - S$ is an overlap repn of G iff S does not contain $f(u) \cap f(v)$ or $f(u) - f(v)$ or $f(v) - f(u)$ for any edge uv.

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Lem. If f is overlap repn of G, and $N(v)$ is independent and contains no leaves, and $\{a, b\}$ is f-uniform except at v, then $f - \{a\}$ or $f - \{b\}$ is an overlap repn of G.

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Find a doubly-minimal leaf x.

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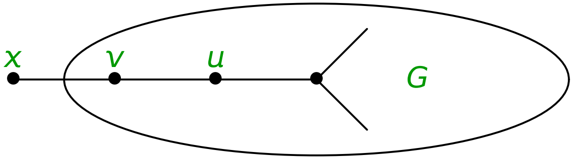

Case 1: $d(u) = 2$. G is a skeleton. Choose $a, b \in f(x)$.
x minimal $\Rightarrow$ $f(x)$ is uniform except at v.
Lemma makes $f - \{a\}$ or $f - \{b\}$ an overlap repn of G.

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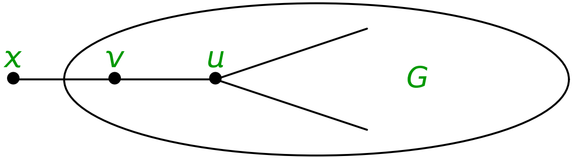

Case 2: $d(u) > 2$. $G - v$ is a skeleton.
Show that $f - \{a, b\}$ restricts to an overlap repn of $G - v$, where $a \in f(x) - f(v)$ and $b \in f(x) \cap f(v)$.

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Cor. $\delta(G) \geq 2 \Rightarrow \varphi(G) \leq \Phi(G) \leq |E(G)|$.

Lem. If $d(v) \leq 2$, then $\Phi(G) \leq \Phi(G - v) + 2$.
Pf. For $d(v) = 2$, let $f(v) = \{a, b\}$, add each to one nbr.

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Lem. Every n-vertex planar graph, with $n \geq 3$, decomposes into at most $2n - 4$ edges and triangles.

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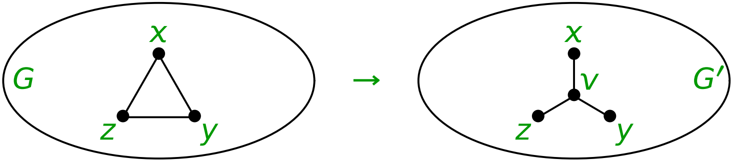
Pf. Induct on # facial triangles. If none, $|E(G)| \leq 2n - 4$. Otherwise, let $[x, y, z]$ be a facial triangle. Form G'.



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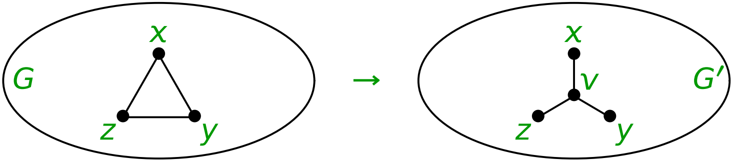

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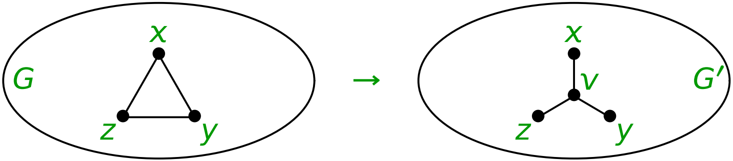

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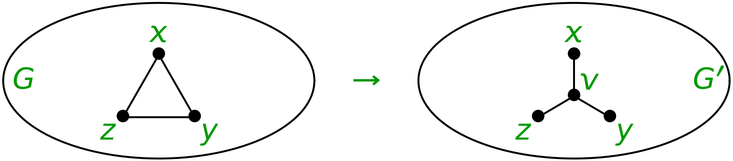
Cor. If G is planar and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-4$.

Upper Bounds for Planar Graphs

Lem. Every n-vertex planar graph, with $n \geq 3$, decomposes into at most $2n-4$ edges and triangles.

Pf. Induct on # facial triangles. If none, $|E(G)| \leq 2n-4$. Otherwise, let $[x, y, z]$ be a facial triangle. Form G'.



G' decomposes into $2n-2$ pieces, using vx, vy, vz. Replace with $[x, y, z]$ to decompose G into $2n-4$. ■

Cor. If G is planar and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-4$.

Cor. If G is planar and not $K_{1,n-1}$, then $\Phi(G) \leq 2n-4$.

The Extremal Result for Planar Graphs

Thm. (much more effort, not yet fully written)
For planar G with n vertices, $\varphi(G) \leq 2n - 5$ (if $n \geq 5$).

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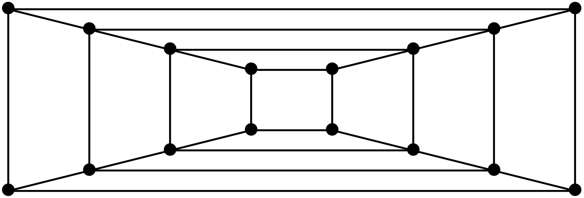

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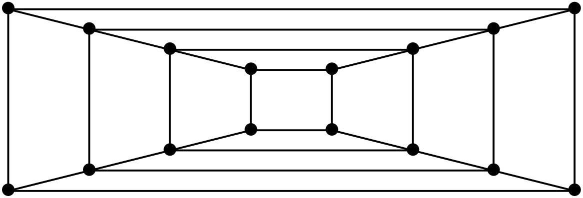

Thm. If G is triangle-free and connected and has no star-cutset, then $\varphi(G) = |E(G)| - 1$.

Def. A star-cutset of a graph G is a separating set S such that $G[S]$ has a spanning star (some vertex of S is adjacent to the rest of S).

Upper Bound Using $|E(G)|$

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Pf. Fix an edge uv. Define f using an element for each edge other than uv.
For $w \notin \{u, v\}$, let $f(w) = \{e \in E(G)\colon\ w \in e\}$.
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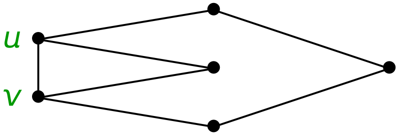
For $w \notin \{u, v\}$, f restricts to a pure overlap repn, since $\delta(G) \geq 2$ prohibits containments. Also, $f(u)$ and $f(v)$ overlap the sets for their nbrs (except in $K_2 \vee \overline{K}_{n-2}$). Finally, $f(u)$ contains the labels for all edges incident to its nonneighbors. ■



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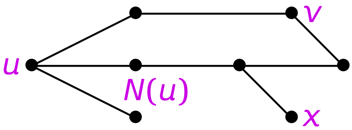
Pf. $uv \notin E(G) \;\Rightarrow\; v \in V(G) - N[u]$.
No star-cutset $\;\Rightarrow\;$ $G - N[u]$ connected.
u non-minimal $\;\Rightarrow\;$ $f(u) \supset f(v)$. Similarly, $f(v) \supset f(u)$. ■



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Case 2: two. This case is similar; one element can be saved. The configuration of the upper bound is forced. ■

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Better still: $\varphi(G) \leq \frac{n^2}{4} - \frac{n}{2} - 1$ for large enough even n. That is, the construction above is extremal (much more work, not yet all written down).

Sketch of the Better bound

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If G has a triangle uvw, use $\Phi(G - \{u, v, w\}) \leq \frac{(n-3)^2}{4}$.
Add labels $f(u) = 12$, $f(v) = 23$, $f(w) = 31$.
For other x, give a label to x and to $N(x) \cap \{u, v, w\}$.
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If G is bipartite with no repeated nbhd, an edge must be deleted from all but one vertex of the larger part.
$\delta(G) \ge 2$, so $\Phi(G) \le |E(G)| \le (k - 1)(n - k) \le \frac{(n-1)^2}{4}$. ■

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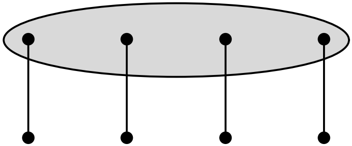

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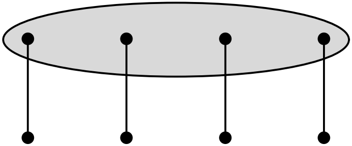

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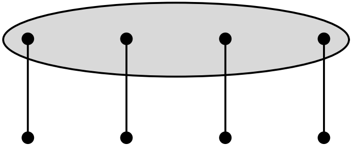

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4) How can one get the paper finished? (7 authors.)