REPRESENTING DIGRAPHS USING INTERVALS OR CIRCULAR ARCS

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Abstract

Containment and overlap representations of digraphs are studied, with the following results. The interval containment digraphs are the digraphs of Ferrers dimension 2, and the circular-arc containment digraphs are the complements of circular-arc intersection digraphs. A poset is an interval containment poset if and only if its comparability digraph is an interval containment digraph, and a graph is an interval graph if and only if the corresponding symmetric digraph with loops is an interval digraph. In an appropriate model of overlap representation using intervals, the unit right-overlap interval digraphs are precisely the unit interval digraphs, and the adjacency matrices of right-overlap interval digraphs have a simple structural characterization bounding their Ferrers dimension by 3.

Keywords: digraph, intersection representation, containment, overlap, unit interval, Ferrers dimension

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1. INTRODUCTION

The *intersection graph* of a family of sets $F = \{S_v\}$ is the graph with vertices corresponding to the sets such that vertices are adjacent if and only if the corresponding sets intersect. Many classes of intersection graphs in which the sets are restricted in some way have been studied; the best known class is the *interval graphs*; these are the graphs with intersection representations in which F is a family of intervals on a line.

In [17,18,19], we studied an analogous model for representation of digraphs. The *intersection digraph* of a family $F = \{(S_v, T_v)\colon v \in V\}$ is the digraph with vertex set V in which there is an edge from u to v if and only if S_u intersects T_v. A digraph is an *interval digraph* if F is a family of pairs of intervals on a line. Given a graph G, the adjacency matrix of the corresponding "symmetric digraph with loops" $D(G)$ is obtained from the adjacency matrix of G by adding 1's on the diagonal. Many results about interval digraphs in [17,18,19] are generalizations of results about interval graphs, because G is an interval graph if and only $D(G)$ is an interval digraph (see section 2).

Subclasses of interval digraphs have been studied. If the intervals all have the same length, we obtain a *unit interval digraph* [19]. If $T_v \subseteq S_v$ for all v, we obtain an *interval nest digraph*; these were introduced and characterized by Prisner [15]. We can extend the study of intersection graphs of subtrees of a tree to intersection digraphs of pairs of subtrees of a tree. Every digraph is the intersection digraph of a family of pairs of subtrees of a star, so it makes sense to define the *leafage* of a digraph to be the minimum number of leaves in the host tree of a subtree intersection representation. Interval digraphs correspond to leafage 2; leafage is studied in [12].

In this paper, we study generation of digraphs by models related to intersection. This mirrors investigations for undirected graphs. In a *containment representation* for undirected graphs, edges correspond to containment of sets $S_u \subset S_v$ or $S_u \supset S_v$. In an *overlap representation*, edges correspond to pairs of sets that intersect without either set containing the other. Classes of containment graphs have been studied by Golumbic [7,8] and Golumbic and Scheinerman [9]; overlap representations by intervals have been studied by Fournier [5].

We extend these models to representations of digraphs. The *containment digraph* of a family $F = \{(S_v, T_v)\colon v \in V\}$ is the digraph with vertex set V in which there is an edge from u to v if and only if S_u properly contains T_v. It is easy to represent any digraph as a containment digraph; given vertex set $V = \{v_i\}$, let $T_{v_i} = \{i\}$, and let S_{v_i} be the the element $-i$ together with $\{j\colon v_j \in N^+(v_i)\}$, where $N^+(v_i)$ is the set of *successors* (out-neighbors) of v_i. Naturally, we wish to restrict the pairs of sets in F to obtain interesting classes of digraphs. If F is a family of pairs of intervals, then the resulting containment digraph is called an *interval containment digraph.* The characterization of interval containment digraphs uses *Ferrers digraphs*, introduced independently by Guttman [10] and Riguet [16], which are those whose successor sets are linearly ordered by inclusion, which is equivalent to transformability of the adjacency matrix by independent row and column permutations to a 0,1-matrix in which the 1's are clustered in the form of a Ferrers diagram. The Ferrers dimension of a digraph D is the minimum number of Ferrers digraphs whose intersection is D. In section 2 we observe that the digraphs of Ferrers dimension

2, characterized independently by Cogis [2] and Doignon, Ducamp, and Falmagne [3], are precisely the interval containment digraphs.

Just as interval digraphs are closely related to interval graphs, so interval containment digraphs are closely related to interval containment posets. A containment representation of a poset assigns each element $x \in P$ a set S_x such that $x < y$ if and only if $S_x \subset S_y$. It is well known that interval containment posets are precisely the posets of dimension 2 [4,13]. Furthermore, the Ferrers dimension of the comparability digraph of a poset (a digraph that is reflexive, antisymmetric, and transitive) equals the order dimension of the posets [1,3]. Hence it is not surprising that interval containment digraphs are precisely the digraphs of Ferrers dimension 2, which we prove in Section 2. Together, these results imply that a poset P is an interval containment poset if and only if its comparability digraph is an interval containment digraph. Unlike the corresponding result about interval intersection graphs, this does not yet have have a direct proof.

When F is restricted to be a family of arcs on a circle, we obtain the undirected *circular-arc graphs* characterized by Tucker [20] (see [6] for a survey). In [17], we studied the intersection digraphs of families of ordered pairs of arcs on a circle, calling these *circular-arc digraphs.* In section 3 we characterize the circular-arc containment digraphs, proving that D is a circular-arc containment digraph if and only if its complement $\bar{D}$ (the digraph whose adjacency matrix is the difference between the the adjacency matrix of D and the matrix of all 1's) is a circular-arc (intersection) digraph.

For our study of overlap digraphs represented by intervals, we will use a more restrictive model than the direct analogue with undirected graphs. For undirected graphs, it is well known that G has an overlap representation in which F is a family of intervals on a line if and only if G has an intersection representation in which F is a family of chords of a circle. Fournier [5] characterized these graphs in terms of relations, leading us again to digraphs. We define a *right overlap interval digraph (ROI-digraph)* to be one having a representation by a family F of ordered pairs of intervals such that there is an edge from u to v if and only if S_u and T_v overlap (no containment) and $\inf S_u < \inf T_v$. In the last section of this paper, we characterize the adjacency matrices of right-overlap interval digraphs; this is the most difficult result of the paper. We also observe that the digraphs having right-overlap interval representations in which all the intervals have unit length (*unit ROI-digraphs*), are the same as the digraphs having interval intersection representations in which all the intervals have unit length (*unit interval digraphs* or *indifference digraphs*), which were characterized in [19].

Another model for obtaining digraphs from sets was introduced by Harary, Kabell, and McMorris [11]. Let $S_1, \ldots, S_n$ be subsets of a poset P with distinct and well-defined infima. The *intersection acyclic digraph* of $\{S_1, \ldots, S_n\}$ is the digraph with vertices $v_1, \ldots, v_n$ such that v_iv_j is an edge if and only if $S_i \cap S_j \neq \emptyset$ and $\inf S_i < \inf S_j$. Suppose the sets are intervals on a single chain. Since all interval graphs can be represented using distinct integer endpoints for intervals, the interval acyclic digraphs are thus precisely the orientations of interval graphs with respect to the left endpoints. McMorris and Mulder [14] generalize this to subpaths of a rooted tree. For such classes of digraphs, we suggest the name "source" to describe the model, as in "interval source digraphs" or "subpath source digraphs".

2. GRAPHS, DIGRAPHS, AND POSETS

In this section we consider relationships between representations of graphs or posets and representations of digraphs. We write $u \leftrightarrow v$ and $u \to v$ to mean "uv is an edge" (in a graph or digraph) and $u \not\leftrightarrow v$ and $u \not\to v$ to mean "uv is not an edge".

THEOREM 1. A graph G is an interval graph if and only if the corresponding symmetric digraph with loops $D(G)$ is an interval digraph.

Proof: Necessity is trivial. If G is an interval graph, with interval I_v assigned to vertex v, then setting $S_v = T_v = I_v$ yields an intersection representation of the digraph $D(G)$. For sufficiency, suppose $\{(S_v, T_v)\colon\ v \in V(G)\}$ is an interval intersection representation of $D(G)$, where $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$. We claim that setting $I_v = [a_v + c_v, b_v + d_v]$ yields an interval intersection representation of G. The verification depends on the observation that two intervals intersect if and only if each left endpoint is less than or equal to the other right endpoint.

For the desired edges, we want $I_u \cap I_v \neq \emptyset$ if $u \to v$ and $v \to u$ in $D(G)$. This means $c_v \leq b_u$ and $a_v \leq d_u$, which implies $a_v + c_v \leq b_u + d_u$. Similarly we have $c_u \leq b_v$ and $a_u \leq d_v$, which implies $a_u + c_u \leq b_v + d_v$.

The other possibility is $u \not\to v$ and $v \not\to u$ in $D(G)$, but also $u \to u$ and $v \to v$. The non-edges imply $d_v < a_u$ or $b_u < c_v$, and also $d_u < a_v$ or $b_v < c_u$. If we choose the first option in each case, the loops give us $d_v < a_u \leq d_u < a_v \leq d_v$. If we choose the second option in each case, we find $b_u < c_v \leq b_v < c_u \leq b_u$. Hence we must choose first/second or second/first. Summing the resulting inequalities yields $b_v + d_v < a_u + c_u$ or $b_u + d_u < a_v + c_v$. Each of these implies $I_u \cap I_v = \emptyset$, as desired. □

Next we consider interval containment, characterizing the digraphs representable by this model. We use one of the equivalent characterizations of Ferrers digraphs. We prove this for completeness and because we need a slight variant.

LEMMA 1. A digraph is a Ferrers digraph if and only if there exist functions $f, g\colon\ V(G) \to \mathbb{R}$ such that $u \to v$ if and only if $f(u) \leq g(v)$. Furthermore, when such functions exist, they can be chosen so the ranges of f and g are disjoint.

Proof: Sufficiency of the condition is clear, because the successor sets are ordered by inclusion. For necessity, consider an independent permutation of the rows and columns of the adjacency matrix so that the ones appear in the upper left in the positions of a Ferrers diagram. There is a unique walk of length $2n$ from the upper right corner to the lower left corner that separates the 1's from the 0's. Select values for $f(u)$ and $g(v)$ as follows: If the row corresponding to u is the ith row or column crossed in the walk, set $f(u) = i$. If the column corresponding to v is the ith row or column crossed in the walk, set $g(v) = i$. Then $u \to v$ if and only if $f(u) \leq g(v)$.

If any values are shared between f and g, then we can reduce all values of f by an amount smaller than the distance between any two distinct values. □

For interval containment digraphs there is an alternative model, which we call *weak containment representation*, in which there is an edge from u to v if and only if S_u contains T_v, without requiring the containment to be proper. The alternative model is better suited for characterizing interval containment digraphs. Fortunately, the two classes are the same.

LEMMA 2. Every interval containment digraph has an interval containment representation such that no source interval equals any sink interval. This also holds for every interval weak containment digraph. In particular, the classes of interval containment digraphs and interval weak containment digraphs are the same.

Proof: Consider an interval containment representation with the minimum number of identical pairs. If this is nonzero, suppose S_u, T_v are an identical pair with the right endpoint x. By the definition, $u \not\to v$. For all intervals in the representation having right endpoint x, except the sink intervals identical to T_v, move the right endpoint leftward by an amount less than the distance between any pair of distinct endpoints. No proper inclusion relations have changed, and we have reduced the number of equalities between source intervals and sink intervals.

For the statement about interval weak containment digraphs, again take a weak containment representation with the minimum number of identical pairs. However, since this time we want to preserve those edges, we move the right endpoint of the sink intervals identical to S_u leftward and do not change the other intervals sharing that right endpoint.

The final statement follows from the fact that a representation with no source set identical to any sink set is both a containment representation and a weak containment representation. □

THEOREM 2. A digraph is an interval containment digraph if and only if it has Ferrers dimension at most 2.

Proof: For necessity, suppose $\{(S_v, T_v)\colon v \in V(D)\}$ is an interval containment representation of the digraph D, where $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$. By the lemma, we may assume that this is a weak containment representation. Hence we have the chain of inequalities $a_u \le c_v \le d_v \le b_u$ if and only if $u \to v$. We define two Ferrers digraphs whose intersection is D. Define D_1 by $u \to v$ in D_1 if and only if $a_u \le c_v$, and define D_2 by $u \to v$ in D_2 if and only if $d_v \le b_u$ (the converse of a Ferrers digraph is also a Ferrers digraph). Since always $c_v \le d_v$ (from the representation), we have $D = D_1 \cap D_2$.

For sufficiency, suppose $D = D_1 \cap D_2$. Let a, b, c, d be functions representing D_1 and D_2 such that $u \to v$ in D_1 if and only if $a_u \le c_v$, and $u \to v$ in D_2 if and only if $d_v \le b_u$. Reduce every value in $\{a_v\} \cup \{c_v\}$ by the same amount ("translate" the representation) to reach values $\{a'_v \cup \{c'_v\}$ such that $c'_v \le d_v$ for every $v \in V(D)$. Now $uv \in E(D_1 \cap D_2)$ if and only if $a'_v \le c'_v \le d_v \le b_v$, which is the statement that $\{(S_v, T_v)\}$ given by $S_v = [a'_v, b_v]$ and $T_v = [c'_v, d_v]$ is an interval weak containment representation of D. □

COROLLARY 1. A poset P is an interval containment poset if and only if its comparability digraph is an interval containment digraph.

Proof: Let $D(P)$ denote the comparability digraph of P, so that $u \to v$ in $D(P)$ if and only if $u >= v$ in P. Necessity is trivial. If P is an interval containment poset, with interval I_v assigned to element v, then setting $S_v = T_v = I_v$ yields an interval containment representation of the digraph $D(P)$.

For sufficiency, suppose $D(P)$ is an interval containment digraph. By Theorem 2, the Ferrers dimension of $D(P)$ is at most 2. Bouchet [1] proved that the order dimension of a poset equals the Ferrers dimension of its (reflexive, transistive, anti-symmetric) comparability digraph. Hence $\dim P = 2$. This is turn implies that P is an interval containment poset, as observed in [4,13]. □

Since the converse of a Ferrers digraph is also a Ferrers digraph, Theorem 2 also implies that the converse of an interval containment digraph is also an interval containment digraph.

3. CIRCULAR-ARC CONTAINMENT DIGRAPHS

A *circular-arc containment representation* of D assigns pairs (S_v, T_v) of arcs on a circle to the vertices $v \in V(D)$ such that $u \to v$ if and only if S_u properly contains T_v. By complementing all the arcs assigned, we see that the converse of a circular-arc containment digraph is also a circular-arc containment digraph. In [17], we showed that the complement of a digraph with Ferrers dimension 2 is a circular-arc digraph, but the sufficient condition of Ferrers dimension 2 for the complement to be a circular-arc digraph is not necessary. It turns out that the complements of circular-arc digraphs are precisely the circular-arc containment digraphs.

THEOREM 3. A digraph is a circular-arc containment digraph if and only if its complement is a circular-arc digraph.

Proof: Suppose D is a circular-arc containment digraph with circular-arc containment representation $\{(S_v, T_v)\}$, where S_v has endpoints a_v, b_v and T_v has endpoints c_v, d_v in clockwise order. Then $u \to v$ if and only if these endpoints occur in the order a_u, c_v, d_v, b_u in the clockwise sense. By using the argument of Lemma 2, we may forbid shared endpoints between source intervals and sink intervals. Let S'_v be the complement of S_v on the circle, together with its endpoints. If $S_u \supset T_v$, then $S'_u \cap T_v = \emptyset$. If $S_u \not\supset T_v$, then the circular order of endpoints starting with a_u must be a_u, c_v, b_u, d_v or a_u, d_v, b_u, c_v or a_u, b_u, c_v, d_v. In each case, $S'_u \cap T_v \neq \emptyset$. Hence $\{(S'_v, T_v)\}$ is a circular-arc representation for $\bar{D}$.

The other direction follows from the fact that this transformation is reversible; it maps circular-arc containment representations into circular-arc representations and vice versa. □

4. OVERLAP DIGRAPHS

In this section we characterize the adjacency matrices of right-overlap interval digraphs.

DEFINITION . A 0,1-matrix has a *P, R-partition* if its rows and columns can be permuted independently so that its 0's can be labeled P or R such that 1) the positions to the right *and* the positions above any R are also 0's labeled R, and 2) the positions to the left *or* the positions below any P are also 0's labeled P.

Note that this definition applies to non-square matrices (and hence general binary relations) as well as to adjacency matrices of digraphs. As illustrated in the first matrix of Fig. 1, it is easy to see that the R's constitute a Ferrers digraph, and the P's constitute the union of two Ferrers digraphs. Hence any digraph whose adjacency matrix has a P, R-partition is a digraph of Ferrers dimension at most 3.

THEOREM 4. A digraph is a right-overlap interval digraph if and only if its adjacency matrix has a P, R-partition.

Proof: Necessity is relatively easy to show. Let $\{(S_v, T_v)\colon v \in V\}$ be a right-overlap interval representation of a digraph D, where $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$. We have $u \to v$ if and only if $a_u < c_v < b_u < d_v$. If $u \not\to v$, then $c_v \geq b_u$ or $a_u \geq c_v$ or $b_u \geq d_v$. The latter two possibilities are not mutually exclusive. We place the position uv of the matrix in the set R if $c_v \geq b_u$ and in the set P if $a_u \geq c_v$ or $b_u \geq d_v$. Now place the rows of the matrix in increasing order of the values b_u, and place the columns of the matrix in increasing order of the values c_v. If $(u, v) \in R$, then every position to the right of (u, v) and every position above (u, v) is also in R. If $(u, v) \in P$ with $a_u \geq c_v$, then every position to the left of (u, v) is in P. If $(u, v) \in P$ with $b_u \geq d_v$, then every position below (u, v) is in P. See the first matrix of Fig. 1 for an illustration.

For the converse, consider a permutation of the rows and columns of the adjacency matrix M that exhibits a P, R-partition. As observed earlier, $\bar{D}$ is the union of three Ferrers digraphs, which we view as sets of positions in the adjacency matrix: 1) H_1 consisting of the P's in M that have only P's to their left, 2) H_2 consisting of the P's in M that have only P's below them, and 3) H_3 consisting of the R's in M. We want to construct intervals $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$ for all $v \in V$ such that uv is *outside* all of H_1, H_2, H_3 if and only if $a_u < c_v < b_u < d_v$, which makes this a right-overlap interval representation (since we will not have equality between any a's and c's, c's and b's, or b's and d's). The values used for the endpoints will come from a topological ordering of an auxiliary acyclic digraph.

We use H_1, H_2, H_3 to define six partitions of V. Because the successor sets of a Ferrers digraph are ordered by inclusion, we can define a *natural partition* of the rows of the adjacency matrix, with two rows in the same block if and only if the successor sets of the two corresponding vertices are identical. Furthermore, the blocks of the partition received a natural order from the inclusion ordering on the successor sets. The same is true of the predecessor sets and the columns of the adjacency matrix.

The positions of H_3 already occupy those of a Ferrers diagram in the upper right corner of M. For H_1, we can permute the rows to achieve this in the lower left, and for H_2, we can permute the columns to achieve this in the lower left. Hence the natural partitions that H_1 and H_3 induce on the columns have the columns in the same order, and the natural partitions that H_2 and H_3 induce on the rows have the rows in the same order. This is illustrated in Fig. 1, where we have given names to the blocks of the partitions. Because a row or column of zeros can be placed in R, we may assume that A_0, C'_p, B'_0, and D_q are non-empty, although C''_0 and/or B''_r may be empty.

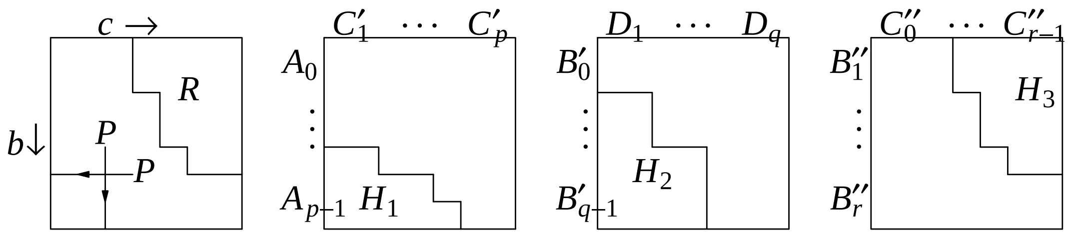

Fig. 1. Decomposition of adjacency matrix of overlap digraph

Let $\mathbf{A} = \{A_i\}$ and $\mathbf{D} = \{D_j\}$. Since the rows of H_2, H_3 and the columns of H_1, H_3 are in the same order, we can define additional partitions $B_0, \ldots, B_s$ and $C_0, \ldots, C_t$ that maintain the order of the rows, where each block B_i is the intersection of one B'_j and one B''_k, and each C_i is the intersection of one C'_j and one C''_l. In other words, the partition $\mathbf{B} = \{B_i\}$ is the common refinement of $\{B'_j\}$ and $\{B''_k\}$ with fewest blocks, indexed by the shared order on the rows, and similarly for $\mathbf{C} = \{C_i\}$. Note that the indexing of the various types of B's agrees with the row order for H_2 and H_3, and the indexing for the C's agrees with the column order for H_1 and H_3.

We construct an auxiliary digraph Q with vertices $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D}$, which we call "nodes" to distinguish them from the vertices of the original digraph. We will assign distinct integers to these nodes via a map f. Each $v \in A_i$ will receive $f(A_i)$ as the value of a_v; similarly b_v, c_v, d_v are set from the values of f on $\mathbf{B}, \mathbf{C}, \mathbf{D}$. We put an edge in Q from one node to another when we want the number assigned to the first node to be less than the number assigned to the second, and then f will be chosen to increase along every edge. Since we want the b-values and c-values to be increasing in rows and columns in accordance with the discussion of the P, R-partition above, we put $B_i \to B_j$ in Q if $i < j$, and similarly $C_i \to C_j$ in Q if $i < j$.

If $u \in A_i$ and $v \in C'_j$ with $i \geq j$, then $uv \notin E(D)$ and we want $a_u > c_v$ to forbid right-overlap, while if $i < j$ we want $a_u < c_v$ to allow right-overlap if the other inequalities are also satisfied. Hence for the pair A_i, C_l with $C_l \subseteq C'_j$, we put $A_i \to C_l$ in Q if $i < j$, but $C_l \to A_i$ if $i \geq j$. This defines a linear ordering on $\mathbf{A} \cup \mathbf{C}$. Similarly, for the pair B_k, D_j with $B_k \subseteq B'_i$, we put $B_k \to D_j$ if $i < j$ and $D_j \to B_k$ if $i \geq j$, which establishes a linear ordering on $\mathbf{B} \cup \mathbf{D}$.

For the interaction between these two orderings, note first that if we want the a's and b's to be the endpoints of real intervals, then we must require $f(A_i) < f(B_k)$ if there is a vertex $v \in A_i \cap B_k$. We represent this by placing an edge from A_i to B_k in Q. Similarly, if

$v \in C_l \cap D_j$, then we add $C_l \to D_j$ to Q. Since we have only added edges from $\{A_i\} \cup \{C_l\}$ to $\{B_k\} \cup \{D_j\}$, this portion Q_1 of Q is still acyclic.

We must still consider the requirements imposed by H_3. Suppose $u \in B_k$ and $v \in C_l$, with $B_k \subseteq B_i''$ and $C_l \subseteq C_j''$. If $i \leq j$, then we have $u \not\to v$ in D, and we want $b_u < c_v$ to enforce this in the representation. On the other hand, if $i > j$, then possibly $u \to v$, and we need to allow this by $c_v < b_u$. Hence if $i \leq j$ we place the edge $B_k \to C_l$ in Q, while if $i > j$ we put $C_l \to B_k$. Let Q_2 consist of these edges together with the edges among $\{B_k\}$ and $\{C_l\}$. The edges of Q_2 impose a linear ordering on $\mathbf{B} \cup \mathbf{C}$.

Our problem now is to show that $Q = Q_1 \cup Q_2$ is acyclic. If it is, consider a numbering $f\colon V(Q) \to \mathbb{Z}$ such that $XY \in E(Q)$ implies $f(X) < f(Y)$. Then using the values of f to determine a_v, b_v, c_v, d_v as described above, we have created intervals $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$ such that $uv \in E(D)$ if and only if $a_u < c_v < b_u < d_v$, and otherwise one of these three inequalities points the other way.

In $Q_1 \cup Q_2$ we have described three linear orderings (paths): on $\mathbf{A} \cup \mathbf{C}$, $\mathbf{B} \cup \mathbf{D}$, and $\mathbf{B} \cup \mathbf{C}$. In each ordering, the indices on a particular type of node appear in increasing order; call this property **I**. If there is a cycle created by adding the edges of Q_2 to Q_1, then it must use an edge of the form $B_k \to C_l$ from Q_2, since all other edges between the two orderings of Q_1 point in the other direction. Choose such an edge so that there is no other edge $B_{k'}C_{l'}$ with $k' \geq k$ and $l' \geq l$. By property **I**, there is no edge from a later $C_{l'}$ to an earlier $B_{k'}$, so the choice of k, l implies the cycle must come back to a $B_{k'}$ with $k' \leq k$ from an A_i later than C_l or come back to a D_j earlier than B_k from a $C_{l'}$ with $l' \geq l$. These two possibilities are symmetric; we need only consider one. Suppose the former. Since every node of $\mathbf{A}$ is adjacent to every node of $\mathbf{C}$ in Q_1 and we have chosen A_i appearing later than C_l in the first linear order, we have $B_k \to C_l \to A_i \to B_{k'}$ in Q with $k' \leq k$. The edge B_kC_l implies that the positions defined by the blocks B_k and C_l are in H_3. The edge C_lA_i implies that the positions defined by the blocks C_l and A_i are in H_1. The edge $A_iB_{k'}$ implies that some row belongs to both A_i and $B_{k'}$. This is impossible, because the rows having positions of H_1 in the original matrix are below those having positions of H_3 in any column. Hence the rows of A_i are below those of B_k, but the rows of $B_{k'}$ are above those of B_k, by the chosen indexing. The contradiction implies that Q is in fact acyclic. □

If the intervals in a right-overlap interval representation all have unit length, then the condition $a_u \geq c_v$ for membership in H_1 is equivalent to the condition $b_u \geq d_v$ for membership in H_2. Hence a unit ROI-digraph has a permutation of rows and columns and a partition of zeros such that every R has only R's above and to its right, and every P has only P's below and to its left. In [20], it was proved that this condition on an adjacency matrix, called a *monotone consecutive arrangement*, characterizes the intersection digraphs of pairs of unit-length intervals. These are called *unit interval digraphs* or *indifference digraphs*. It is easy to show directly that these classes are the same.

THEOREM 5. The class of unit right-overlap interval digraphs is the same as the class of indifference digraphs.

Proof: Given an unit interval intersection representation of D, let $f(u)$ be the midpoint of S_u and $g(v)$ be the midpoint of T_v; this transformation is reversible. Hence D is a unit interval digraph if and only if there exist f, g: $V(D) \to \mathbb{R}$ such that $u \to v$ if and only if $0 \leq |g(v) - f(u)| \leq 1$ (hence the name "indifference" digraph). By a similar association, D is a unit right-overlap interval digraph if and only if there exist f, g: $V(D) \to \mathbb{R}$ such that $u \to v$ if and only if $0 < g(v) - f(u) < 1$.

To show that these conditions are the same, we observe first that the possibility of $|g(v) - f(u)| = 1$ can be ignored in the condition for finite indifference digraphs. If D has such a representation(and has non-edges), let ϵ be the smallest positive value such that some $g(v)$ and $f(u)$ differ by $1 + \epsilon$. If we multiply all values of f and g by a fixed constant between 1 and $1/(1+\epsilon)$, then we have changed no edges and have eliminated all instances where values of f and g differ by 1, without introducing any.

Hence D is a unit interval digraph if and only if there exist f, g: $V(D) \to \mathbb{R}$ such that $u \to v$ if and only if $-1 < g(v) - f(u) < 1$. Given such a representation, let $f'(u) = f(u)/2$ and $g'(v) = (1+g(v))/2$. Then $0 < g'(v) - f'(u) < 1$ if and only if $-1 < g(v) - f(u) < 1$, so f', g' provide a right-overlap interval representation of D. Furthermore, the transformation is reversible, by setting $f = 2f'$ and $g = 2g' - 1$ if f', g' provides a right-overlap interval representation. □

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