Vertex Degrees in Outerplanar Graphs

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Joint work with
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$\alpha_6(n) = \begin{cases} n-4 & n \text{ even} \\ n-5 & n \text{ odd} \end{cases}$ Grünbaum–Motzkin [1963]

$\alpha_k(n) = \left\lfloor \frac{3n-12}{k-3} \right\rfloor$ for $7 \leq k \leq 10$ Griggs–Lin [1995]

$\alpha_{11}(n) = \left\lfloor \frac{3n-18}{8} \right\rfloor$ for $k = 11$ Griggs–Lin [1995]

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Also known: Maximum sum of the degrees of vertices with degree at least k.

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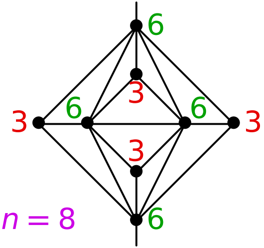

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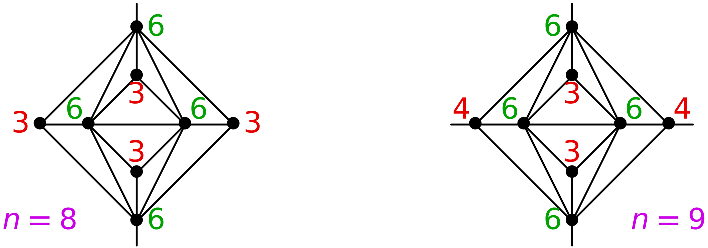

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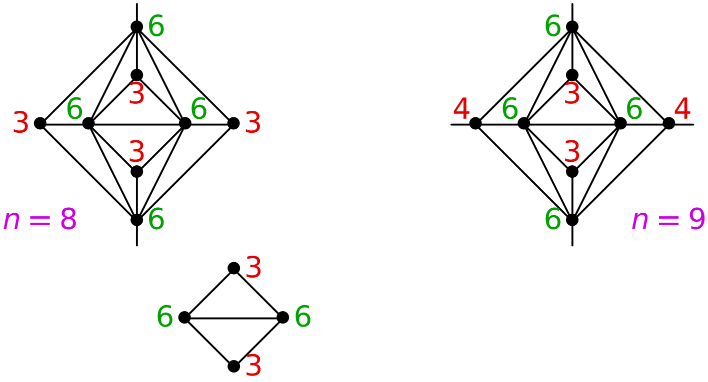

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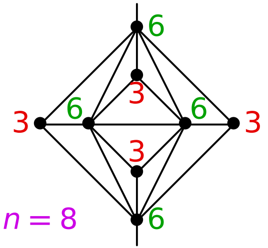

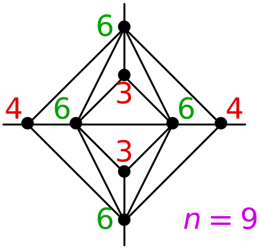

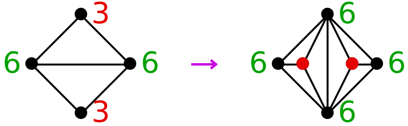

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We also determine:
Maximum sum of the s largest degrees.
Maximum degree-sum for the vertices with degree $\geq k$.

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- $\beta_k(n)$ attained by a maximal outerplanar graph (MOP).

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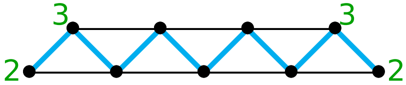

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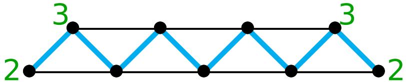

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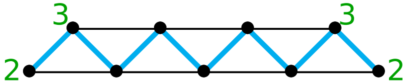

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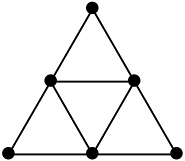

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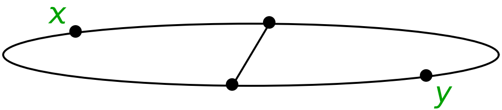

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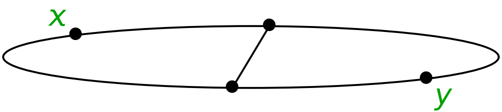

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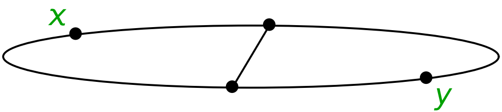

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Pf. Now $n_2 \leq n - \beta - 2$. Put into $(k-3)\beta \leq n + n_2 - 6$. ■

Construction for $k = 5$

Lem. If $n \geq 4$, then $\beta_5(n) \geq \begin{cases} \left\lfloor \frac{2(n-5)}{3} \right\rfloor & \text{if } n \equiv 1 \bmod 6, \\ \left\lfloor \frac{2(n-4)}{3} \right\rfloor & \text{otherwise.} \end{cases}$

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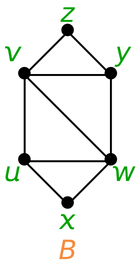

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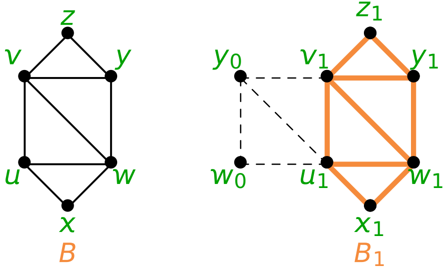

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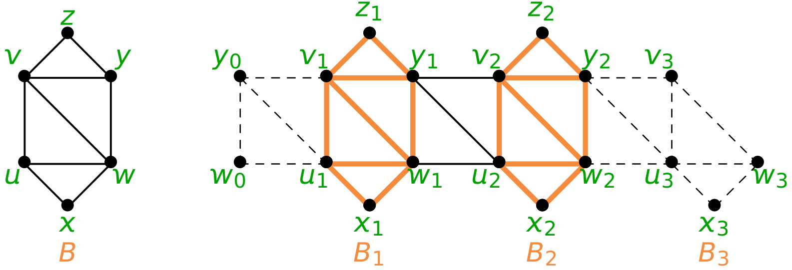

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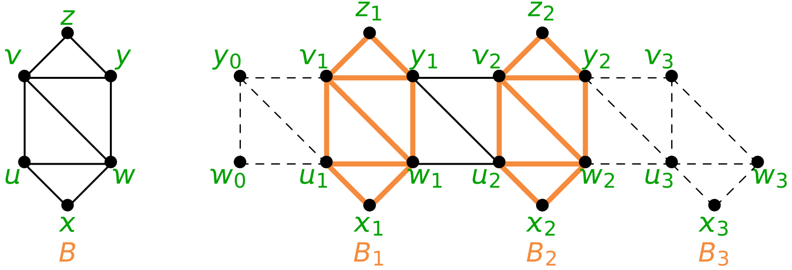

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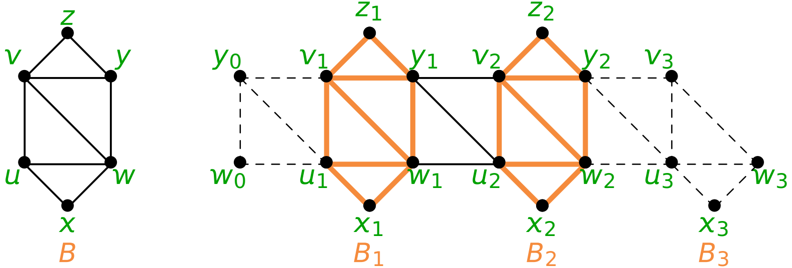

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Now $k = 5$ done except upper bound for $n \equiv 1 \mod 6$.

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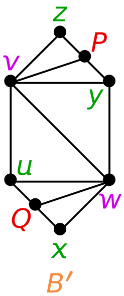

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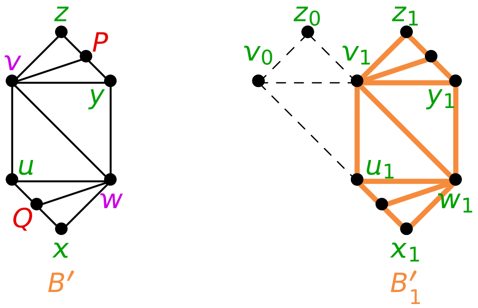

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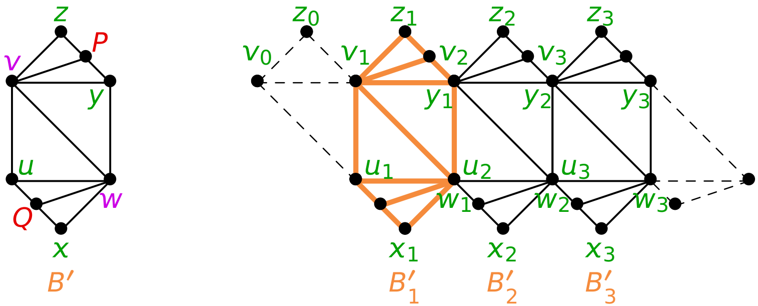

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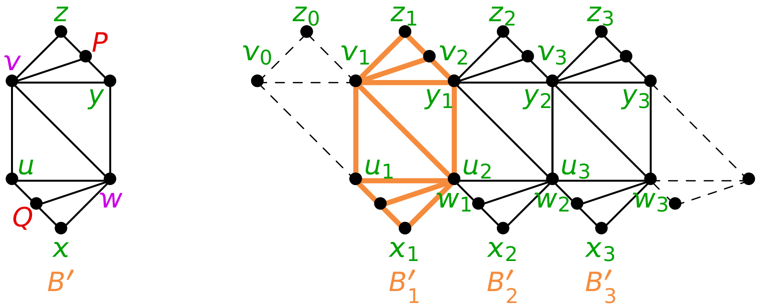

Start with $n \equiv 4 \mod (k-4)$. In the next copy B'_i, add w_i, then one vertex from Q (now $d(u_i) = k$).

Next y_i, then P to z_i (now $d(v_i) = k$),

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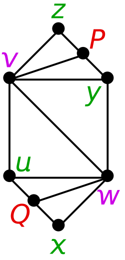
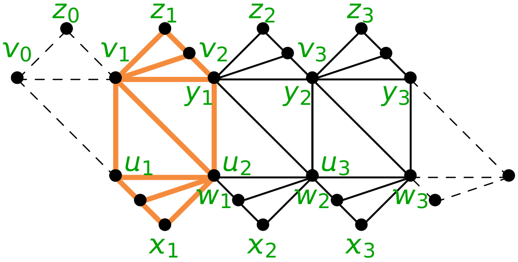


B'



B'_1 B'_2 B'_3

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Pf. Otherwise, G can be altered to get s vertices with larger degree-sum. ■

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Pf. To the s vertices with maximum sum, inducing a MOP, s vertices can be added contributing 2 each to the sum, and if $s < n/2$ the remainder contribute only 1. ■

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Pf. The β vertices with degree $\geq k$ have largest degrees.
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Hence $ks \leq n-6+4s$, which yields $s \leq \frac{n-6}{k-4}$. ∎