

Vertex Degrees in Outerplanar Graphs

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Abstract

For an outerplanar graph on n vertices, we determine the maximum number of vertices of degree at least k . For $k = 4$ (and $n \geq 7$) the answer is $n - 4$. For $k = 5$ (and $n \geq 4$), the answer is $\lfloor \frac{2(n-4)}{3} \rfloor$ (except one less when $n \equiv 1 \pmod{6}$). For $k \geq 6$ (and $n \geq k + 2$), the answer is $\lfloor \frac{n-6}{k-4} \rfloor$. We also determine the maximum sum of the degrees of s vertices in an n -vertex outerplanar graph and the maximum sum of the degrees of the vertices with degree at least k .

1 Introduction

For $n > k > 0$, Erdős and Griggs [1] asked for the minimum, over n -vertex planar graphs, of the number of vertices with degree less than k . For $k \leq 6$, the optimal values follow from results of Grünbaum and Motzkin [4]. West and Will [5] determined the optimal values for $k \geq 12$, obtained the best lower bounds for $7 \leq k \leq 11$, and provided constructions achieving those bounds for infinitely many n when $7 \leq k \leq 10$. Griggs and Lin [2] independently found the same lower bounds for $7 \leq k \leq 10$ and gave constructions achieving the lower bounds when $7 \leq k \leq 11$ for *all* sufficiently large n .

We study the analogous question for outerplanar graphs, expressed in terms of large-degree vertices. Let $\beta_k(n)$ be the maximum, over n -vertex outerplanar graphs, of the number of vertices having degree at least k . For $k \leq 2$, the problem is trivial; $\beta_k(n) = n$, achieved by a cycle (or by any maximal outerplanar graph).

When $k \in \{3, 4\}$, the square of a path shows that $\beta_3(n) \geq n - 2$ and $\beta_4(n) \geq n - 4$. Since every outerplanar graph with $n \geq 2$ has at least two vertices of degree at most 2, $\beta_3(n) = n - 2$. We will prove $\beta_4(n) = n - 4$ when $n \geq 7$ (Theorem 2.6). For $k = 5$ and

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$n \geq 4$, we prove $\beta_k(n) = \lfloor 2(n-4)/3 \rfloor$, except one less when $n \equiv 1 \pmod{6}$ (Theorem 2.5). For $k \geq 6$ and $n \geq k+2$, we prove $\beta_k(n) = \lfloor (n-6)/(k-4) \rfloor$ (Theorem 3.4).

We close this introduction with a general upper bound that is optimal for $k = 5$ when $n \not\equiv 1 \pmod{6}$. In Section 2 we improve the upper bound by 1 when $n \equiv 1 \pmod{6}$ and provide the general construction that meets the bound; these ideas also give the upper bound for $k = 4$. In Section 3 we solve the problem for $k \geq 6$. The bounds in [5] were obtained by first solving a related problem, which here corresponds to maximizing the sum of the degrees of the vertices with degree at least k . We use this approach in Section 3 to prove the upper bound on $\beta_k(n)$ when $k \geq 6$.

Adding edges does not decrease the number of vertices with degree at least k , so an n -vertex outerplanar graph with $\beta_k(n)$ vertices of degree at least k must be a maximal outerplanar graph, which we abbreviate to *MOP*. For a MOP with n vertices, let β be the number of vertices having degree at least k , and let n_2 be the number of vertices having degree 2. A MOP with n vertices has $2n - 3$ edges, so summing the vertex degrees yields

$$2n_2 + 3(n - n_2 - \beta) + k\beta \leq 4n - 6. \quad (1)$$

This inequality simplifies to $(k-3)\beta \leq n + n_2 - 6$. Using $n_2 \leq n - \beta$ then yields $\beta_k(n) \leq \lfloor 2(n-3)/(k-3) \rfloor$. To improve the bound, we need a structural lemma.

Lemma 1.1. *Let G be an n -vertex MOP with external cycle C . If $n \geq 4$, then G has two vertices with degree in $\{3, 4\}$ that are not consecutive along C .*

Proof. We use induction on n . Note that G contains $n - 3$ chords of C . If every chord lies in a triangle with two external edges, then $n \leq 6$ and $\Delta(G) \leq 4$, and the two neighbors of a vertex of degree 2 are the desired vertices. This case includes the MOPs for $n \in \{4, 5\}$.

Otherwise, a chord xy not in a triangle with two external edges splits G into two MOPs with at least four vertices, each with x and y consecutive along its external cycle. By the induction hypothesis, each has a vertex with degree 3 or 4 outside $\{x, y\}$. In G , those two vertices retain their degrees, and they are separated along C by x and y . \square

Corollary 1.2. *If $k \geq 5$ and $n \geq 4$, then $\beta_k(n) \leq \lfloor 2(n-4)/(k-2) \rfloor$.*

Proof. Lemma 1.1 yields $n - n_2 - \beta \geq 2$, and hence $n_2 \leq n - \beta - 2$. Substituting this improved inequality into the inequality $(k-3)\beta \leq n + n_2 - 6$ that follows from (1) yields $\beta_k(n) \leq 2(n-4)/(k-2)$. \square

Corollary 1.2 gives us a target to aim for in the construction for $k = 5$. For $k \geq 6$ we will need further improvement of the upper bound. The argument of Corollary 1.2 is not valid for $k = 4$, since vertices of degree 4 are counted by β . The upper bound $\beta_4(n) \leq n - 4$ (for $n \geq 7$) will come as a byproduct of ideas in the next section.

2 The solution for $k = 5$

We begin with the construction. Let $\langle v_1, \dots, v_k \rangle$ and $[v_1, \dots, v_k]$ denote a path and a cycle with vertices v_1, \dots, v_k in order, respectively. Let B be the graph formed from the cycle $[v, u, x, w, y, z]$ by adding the path $\langle u, w, v, y \rangle$. (see Fig. 1). The reason for naming the vertices in this way is that we will create copies of B in a large graph by adding the vertices in the order u, v, w, x, y, z .

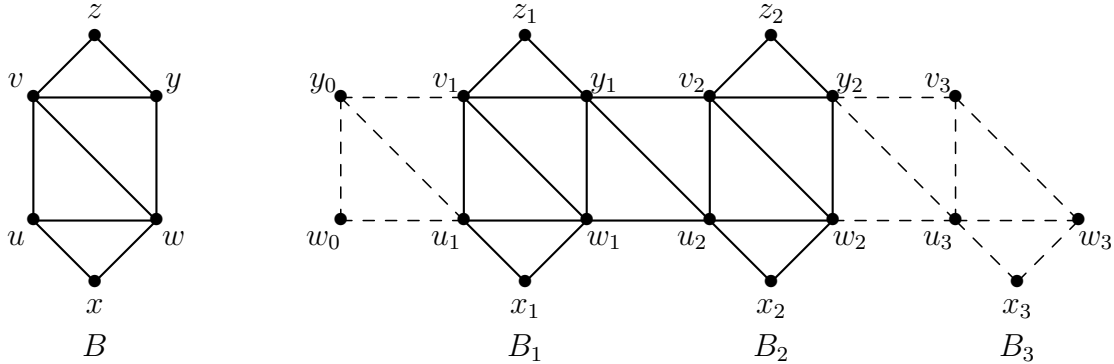


Figure 1: The graph B and its use in constructing F .

To facilitate discussion, define a j -*vertex* to be a vertex of degree j , and define a j^+ -*vertex* to be a vertex of degree at least j .

Lemma 2.1. *If $n \geq 4$, then $\beta_5(n) \geq \begin{cases} \lfloor 2(n-5)/3 \rfloor & \text{if } n \equiv 1 \pmod{6}, \\ \lfloor 2(n-4)/3 \rfloor & \text{otherwise.} \end{cases}$*

Proof. Begin at $n = 2$ with one edge having endpoints w_0 and y_0 . Add vertices one by one, always adding a vertex adjacent to two earlier neighboring external vertices; the result is always a MOP. For $6q - 3 \leq n \leq 6q + 2$, add the vertices $u_q, v_q, w_q, x_q, y_q, z_q$ in order. The added vertex is the center of a 3-vertex path; the paths added are successively $\langle w_{q-1}, u_q, y_{q-1} \rangle$, $\langle y_{q-1}, v_q, u_q \rangle$, $\langle u_q, w_q, v_q \rangle$, $\langle u_q, x_q, w_q \rangle$, $\langle v_q, y_q, w_q \rangle$, and $\langle v_q, z_q, y_q \rangle$. After z_q is added, the subgraph induced by $\{u_q, v_q, w_q, x_q, y_q, z_q\}$ is isomorphic to B ; call it B_q (see Fig. 1).

When $n = 6$, the addition of x_1 augments u_1 to degree 5 (the first such vertex). The second occurs at v_1 when z_1 is added to reach eight vertices. This agrees with the claimed values for n from 4 through 8. Subsequently, addition of u_q, v_q, x_q, z_q raises $w_{q-1}, y_{q-1}, u_q, v_q$, respectively, to degree 5. Addition of w_q and y_q does not introduce a 5-vertex, and x_q and z_q never exceed degree 2.

When $n = 6 \cdot 2 - 4$, we have two 5-vertices; note that $2 = \lfloor 2(8-4)/3 \rfloor$. For $6q - 3 \leq n \leq 6q + 2$, the values required by the stated formula for the number of 5^+ -vertices are

$4q - 5, 4q - 4, 4q - 4, 4q - 3, 4q - 3, 4q - 2$, respectively. The induction hypothesis for an induction on q states that when $n = 6q - 4 = 6(q - 1) + 2$, the graph has $4(q - 1) - 2$ vertices of degree 5. Starting from this point, we augmented one vertex to degree 5 when n is congruent to each of $\{-3, -2, 0, 2\}$ modulo 6, matching the formula. \square

The lower bound in Lemma 2.1 and upper bound in Corollary 1.2 are equal except when $n \equiv 1 \pmod{6}$. In this case we improve the upper bound by showing that there is no outerplanar graph having the vertex degrees required to achieve equality in the upper bound. The construction shows $\beta_5(n) \geq \lfloor 2(n - 5)/3 \rfloor$. We used the existence of two vertices with degree 3 or 4 to improve the upper bound from $\lfloor 2(n - 3)/3 \rfloor$ to $\lfloor 2(n - 4)/3 \rfloor$, which differs from $\lfloor 2(n - 5)/3 \rfloor$ by 1 when $n \equiv 1 \pmod{6}$. We begin by showing that slightly stronger hypotheses further reduce the bound.

Lemma 2.2. *If G is a MOP having a 6^+ -vertex, or a 3-vertex and a 4-vertex, or two 4-vertices, or at least three 3-vertices, then G has at most $\lfloor (2n - 9)/3 \rfloor$ vertices of degree at least 5.*

Proof. We have proved that $\beta \leq (2n - 8)/3$. If we cannot improve the upper bound to $(2n - 9)/3$, then equality must hold in all the inequalities that produced the upper bound $(2n - 8)/3$. Thus $n_2 = n - \beta - 2$, which forbids a third vertex with degree 3 or 4. Also, the sum of the degrees must equal $2n_2 + 3(n - n_2 - \beta) + 5\beta$ (see (1)). This requires that the two vertices of degree 3 or 4 both have degree 3 and that no vertex has degree at least 6. \square

Let T be the subgraph of the dual graph of G induced by the vertices corresponding to bounded faces of G ; we call T the *dual tree* of G . Since G is a MOP, T is a tree. A triangular face in G having j edges on the external cycle corresponds to a $(3 - j)$ -vertex in T . The next lemma will reduce the proof of the theorem to the case where T is a special type of tree.

Lemma 2.3. *If in the dual tree T the neighbor of a leaf t has degree 2, then G has a 3-vertex on the triangle corresponding to t . If two leaves in T have a common neighbor, then the common vertex of the corresponding triangles in G is a 4-vertex in G .*

Proof. Let x, y, z be the vertices of the triangle in G corresponding to t , with x having degree 2 in G . The neighboring triangle t' raises the degree of y and z to 3. If t' has degree 2 in T , then in G only one of $\{y, z\}$ can gain another incident edge.

Two leaves in T having a common neighbor \hat{t} correspond to two triangles in G having a common vertex z . The vertex \hat{t} in T corresponds to a triangle in G that shares an edge with each of them. No further edges besides the four in these triangles are incident to z . \square

A triangle in a MOP is *internal* if none of its edges lie on the external cycle.

Lemma 2.4. *In a MOP with n vertices, let n_2 be the number of 2-vertices and t be the number of internal triangles. If $n \geq 4$, then $t = n_2 - 2$.*

Proof. Let T be the dual tree. Note that T has $n - 2$ vertices, of which n_2 have degree 1, t have degree 3, and the rest have degree 2. By counting the edges in terms of degrees, $2(n - 3) = n_2 + 3t + 2(n - 2 - n_2 - t)$, which yields $n_2 - 2 = t$. \square

Theorem 2.5. *If $n \geq 4$, then $\beta_5(n) = \begin{cases} \lfloor 2(n - 5)/3 \rfloor & \text{if } n \equiv 1 \pmod{6}, \\ \lfloor 2(n - 4)/3 \rfloor & \text{otherwise.} \end{cases}$*

Proof. It suffices to prove the upper bound when $n \equiv 1 \pmod{6}$. Let $n = 6q + 1$. Corollary 1.2 yields $\beta \leq 4q - 2$, and we want to improve this to $4q - 3$, which equals $\lfloor (2n - 9)/3 \rfloor$. If the upper bound cannot be improved, then by the computation in the proof of Lemma 2.2 we may assume that G has exactly two 3-vertices, exactly $4q - 2$ vertices of degree 5, and exactly $2q + 1$ vertices of degree 2.

A MOP with n vertices has $n - 2$ bounded faces, so T has $n - 2$ vertices. Since G has $2q + 1$ vertices of degree 2, there are $2q + 1$ leaves in T ; by Lemma 2.4, T has $2q - 1$ vertices of degree 3. The remaining $2q - 1$ vertices of T have degree 2.

A *caterpillar* is a tree such that deleting all the leaves yields a path, called its *spine*. A tree that is not a caterpillar contains as a subtree the graph Y obtained by subdividing each edge of the star $K_{1,3}$. If $Y \subseteq T$, then consider longest paths in T starting from the central vertex v of Y along each of the three incident edges. Each such path reaches a leaf. By Lemma 2.3, each such path generates a vertex of degree 3 or 4 in G . Since G has at most two such vertices, T is a caterpillar. Furthermore, since G has no 4-vertices, Lemma 2.3 implies that each endpoint of the spine of T has degree 2 in T .

Consider vertices $a, b, c \in V(T)$ such that $ab, bc \in E(T)$. The corresponding three triangles in G have a common vertex x . If a and c are 3-vertices in T , then x has degree 6 in G . Hence no two 3-vertices in T have a common neighbor. This implies that along the spine of T (which starts and ends with 2-vertices), there are at most two consecutive 3-vertices, and nonconsecutive 3-vertices are separated by at least two 2-vertices.

In particular, every run of 3-vertices has at most two vertices, every run of 2-vertices has at least two vertices (except possibly the runs at the ends), and the number of runs of 2-vertices is one more than the number of runs of 3-vertices. The only way this can produce the same number of 2-vertices and 3-vertices is $2, 3, 3, 2, 2, 3, 3, \dots, 2, 2, 3, 3, 2$. However, in this configuration the number of vertices of each type is even and cannot equal $2q - 1$.

We have proved that no outerplanar graph has the required vertex degrees. \square

Theorem 2.6. *If $n \geq 7$, then $\beta_4(n) = n - 4$.*

Proof. When the dual tree T is a path, the graph G has two 2-vertices, two 3-vertices, and $n - 4$ vertices of degree 4; this proves the lower bound. For the upper bound, since leaves of T correspond to triangles in G having 2-vertices, we may assume that T has at most three leaves. If T has only two leaves, then the neighbor of each has degree 2 in T , and Lemma 2.3 provides two 3-vertices in G , matching the construction.

If T has exactly three leaves, then T has one 3-vertex and at least four other vertices, since T has $n - 2$ vertices and $n \geq 7$. Hence at least one leaf in T has a neighbor of degree 2, and Lemma 2.3 provides one 3-vertex in addition to the three 2-vertices in G . (In fact, this case yields another construction having exactly $n - 4$ vertices with degree at least 4.) \square

3 The solution for $k \geq 6$

Trivially, $\beta_k(n) = 1$ when $k = n - 1$, which does not satisfy the general formula. We restrict our attention to $n \geq k + 2$ and begin with the construction. Fix k with $k \geq 6$.

Form a graph B' from B in Section 2 by respectively replacing edges yz and ux with paths P and Q , each having $k - 6$ internal vertices. Make v adjacent to all of $V(P)$ and w adjacent to all of $V(Q)$ (see Fig. 2). Since P and Q have $k - 4$ vertices each, B' has $2k - 6$ vertices; also, B' is a MOP. Its vertices v and w of maximum degree have degree $k - 2$.

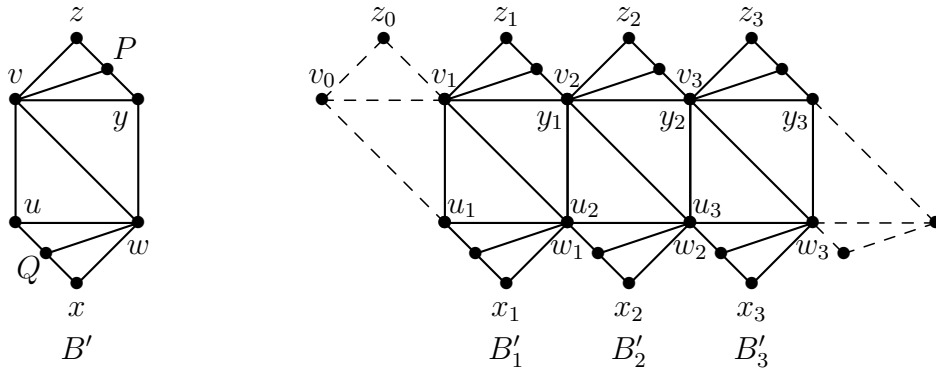


Figure 2: The graph B' and its use in constructing F' .

Lemma 3.1. *If $n \geq k + 2$, then $\beta_k(n) \geq \left\lfloor \frac{n - 6}{k - 4} \right\rfloor$.*

Proof. Letting $n = 2(k - 4)q + 6 + r$ with $0 \leq r < 2(k - 4)$, the claim is equivalent to $\beta_k(n) \geq 2q$ when $0 \leq r < k - 4$ and $\beta_k(n) \geq 2q + 1$ when $k - 4 \leq r < 2(k - 4)$.

Let F' be the union of q copies B'_1, \dots, B'_q of B' , modifying the names of vertices in B'_i by adding the subscript i and taking $y_i = v_{i+1}$ and $w_i = u_{i+1}$ for $i \geq 1$ (see the solid graph

in Fig. 2). Note that F' is a MOP with $2 + 2q(k - 4)$ vertices, and the $2q - 2$ vertices that lie in two copies of B' have degree k . To obtain the lower bound, we will add vertices one by one as in Lemma 2.1, but we will start with the appended vertices z_0 and v_0 in order to raise v_1 to degree k quickly.

Each vertex when added will be a 2-vertex appended to an edge of the external cycle, so we always have a MOP. Begin with the triangle on $\{v_1, z_0, v_0\}$. Add u_1, w_1, y_1 and all of P in B'_1 in order. We now have $k + 1$ vertices, and v_1 has degree k . There remains only one k -vertex as we add the rest of Q to complete B'_1 , at which point w_1 has degree $k - 2$.

Having completed B'_q , we have $2q(k - 4) + 4$ vertices (including v_0 and z_0), of which $2q - 1$ vertices have degree k (including v_1). We show that adding $V(B'_{q+1})$ in the right order produces two more k -vertices at the right times.

With two more vertices, $n = 2q(k - 4) + 6$ and $r = 0$, and the number of k -vertices should be $2q$. Add w_{q+1} and then the first vertex of Q from B'_{q+1} , raising the degree of w_q to k .

The next k -vertex should arrive when $n = 2q(k - 4) + 6 + (k - 4)$, with $r = k - 4$. Adding the $k - 4$ vertices from y_{q+1} to z_{q+1} along P in B'_{q+1} raises the degree of y_q from 4 to k . Finally, we add the rest of Q to complete B'_{q+1} . No k -vertex appears, but the degree of w_{q+1} rises to $k - 2$ to be ready for the next iteration. \square

In order to prove the upper bound, we consider a related problem. Let $D(n, s)$ be the maximum sum of the degrees of s vertices in an n -vertex outerplanar graph. Of course, these will be s vertices of largest degrees, and the maximum will be achieved by a MOP. We will obtain an upper bound on $D(n, s)$ by considering the structure of the subgraph induced by the vertices with largest degrees. Subsequently, we will apply the bound on $D(n, s)$ to prove that $\beta_k(n) \leq \lfloor \frac{n-6}{k-4} \rfloor$ when $k \geq 6$.

Lemma 3.2. *Fix s with $1 \leq s \leq n$. If G is a MOP in which the sum of the degrees of some s vertices is $D(n, s)$, then each set of s vertices with largest degrees in G induces a MOP.*

Proof. Let C be the external cycle in an outerplanar embedding of G , with vertices v_1, \dots, v_n in order. Let S be a set of s vertices with largest degrees.

We show first that the outer boundary of the subgraph induced by some such set S is a cycle. For $x \in S$, let y be the next vertex of S along C . Let P be the path from x to y along C . If x is not adjacent to y , then let u be the last vertex of P adjacent to x . Since G is a MOP, xu lies in two triangles, and by the choice of u there is a triangle containing xu whose third vertex is not on P ; let z be this third vertex. Let v be the next vertex of P after u . Let U be the set of neighbors of u not on the x, u -subpath of P ; note that $z \in U$. Replace the edges from u to U with edges from x to $U \cup \{v\}$. Each edge moved increases the degree of x , and hence the sum of the s largest degrees does not decrease. (If we removed

an edge from a neighbor of x that is in S and its degree is no longer among the s largest, then it was replaced by a vertex of the same degree; thus we have increased the sum of the s largest degrees, which contradicts the choice of G .)

The last neighbor of x is now farther along P . When it reaches y the sum of the m largest degrees increases. Since we started with a MOP maximizing this sum, the edge xy must have been present initially.

We have shown that the outer boundary of $G[S]$ is a cycle. Every bounded face is a face of G , since there are no vertices of G inside it. Hence $G[S]$ is a MOP. \square

In fact, when $s < n - 1$ there is always a unique set of s vertices with largest degrees in a graph maximizing the sum of those degrees.

Theorem 3.3. *The maximum value $D(n, s)$ of the sum of s vertex degrees in an n -vertex outerplanar graph is given by*

$$D(n, s) = \begin{cases} n - 1 & \text{if } s = 1, \\ n - 6 + 4s & \text{if } s < n/2, \\ 2n - 6 + 2s & \text{if } s \geq n/2. \end{cases}$$

Proof. Let G be a MOP in which some set S of s vertices has degree-sum $D(n, s)$. If $s = 1$, then $n - 1$ is clearly an upper bound, achieved by a star. For $s \geq 2$, let G be a MOP achieving the maximum; we know that $G[S]$ is also a MOP and hence has $2s - 3$ edges. The question then becomes how the remaining $n - s$ vertices can be added to produce the maximum sum of the degrees in S .

Consider an outerplanar embedding of G . The subgraph induced by S is also an outerplanar embedding of S . Since $G[S]$ has $2s - 3$ edges, the outer boundary of the subgraph is a cycle. In the embedding of G , no vertex of $V(G) - S$ appears inside this cycle. Also, vertices outside S can be adjacent to only two vertices of S , and they can be adjacent to two only if those two are consecutive on the outer boundary of $G[S]$. This implies that at most s vertices of $V(G) - S$ can have two neighbors in S , and the rest have at most one neighbor in S . Furthermore, the vertices outside S can be added to achieve this bound.

If $s \geq n/2$, then we add $2(n - s)$ to the degree-sum within $G[S]$, obtaining $D(n, s) = 2n - 6 + 2s$. If $s \leq n/2$, then we add $2s + 1(n - 2s)$, obtaining $D(n, s) = n - 6 + 4s$. \square

Theorem 3.4. *If $k \geq 6$, then $\beta_k(n) \leq \left\lfloor \frac{n - 6}{k - 4} \right\rfloor$.*

Proof. In an extremal graph, the $\beta_k(n)$ vertices with degree at least k have the largest degrees. With $s = \beta_k(n)$, we have $sk \leq D(n, s)$. Using the bound obtained in Theorem 3.3,

we have

$$sk \leq \begin{cases} n - 6 + 4s & \text{if } s < n/2, \\ 2n - 6 + 2s & \text{if } s \geq n/2. \end{cases}$$

If $k \geq 6$ and $s \geq n/2$, then

$$6s \leq ks \leq 2n - 6 + 2s \leq 6s - 6.$$

Hence $k \geq 6$ implies $s < n/2$, and therefore $ks \leq n - 6 + 4s$, which simplifies to $s \leq \frac{n-6}{k-4}$. \square

Finally, we consider the maximum sum of the degrees of the vertices with degree at least k . Essentially, the point is that we cannot increase this sum by using fewer than $\beta_k(n)$ vertices with degrees larger than k .

Corollary 3.5. *For $k \geq 6$, the maximum sum of the degrees of the vertices with degree at least k in an n -vertex outerplanar graph is $n - 6 + 4 \lfloor \frac{n-6}{k-4} \rfloor$.*

Proof. Let G be an n -vertex outerplanar graph, and let $S = \{v \in V(G) : d(v) \geq k\}$. Let $s = |S|$. Since these are the vertices of largest degree, $\sum_{v \in S} d(v) \leq D(n, s)$.

For $k \geq 6$, since $D(n, s)$ is monotone increasing in s , we obtain a bound on the sum by using the bound on $\beta_k(n)$ obtained in Theorem 3.4. Since $\beta_k(n) < n/2$, Theorem 3.3 yields $\sum_{v \in S} d(v) \leq D(n, s) \leq D(n, \beta_k(n)) = n - 6 + 4 \lfloor \frac{n-6}{k-4} \rfloor$.

We show that a modification of the construction in Lemma 3.1 achieves equality in the bound. When $n = (k - 4)q' + 6$ for some integer q' , let G be the graph constructed in Lemma 3.1, having $\beta_k(n)$ vertices of degree at least k ; here $\beta_k(n) = q'$. All q' of these vertices have degree exactly k , so the sum of their degrees is $q'k$, equaling the upper bound here. For each increase in n over the next $k - 5$ vertices, adding one vertex of degree 2 can increase the degree-sum of these vertices by 1, again equaling the upper bound here. When the $(k - 4)$ th addition is reached, start over with the construction from Lemma 3.1 for the new value of q' . \square

Remark 3.6. Similar analysis solves the problem of maximizing the sum of the degrees of the vertices with degree at least 5. We remark that the maximum value when $n \not\equiv 1 \pmod{6}$ is $2n - 8 + 4\beta_k(n)$, which is smaller by 2 than the former in terms of $\beta_k(n)$ when $k \geq 6$. Writing the expression as $2(n - \beta_k(n)) - 2 + (4\beta_k(n) - 6)$, we begin with degree-sum $4\beta_k - 6$ within the MOP H induced by the vertices of degree at least 5. Since $\beta_5(n)$ is roughly $2n/3$, we should be able to augment the sum of the degrees by 2 for each of the remaining $n - \beta_k(n)$ vertices. However, H has at least two vertices of degree 2. For each such vertex, raising its degree to 5 requires one of the added vertices to contribute only 1 instead of 2. With this adjustment, the improved upper bound meets the construction. When $n \equiv 1 \pmod{6}$, there are additional technicalities we leave to the reader.

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