Ordered Ramsey Theory and Track Representations of Graphs

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slides available on DBW preprint page

Joint work with
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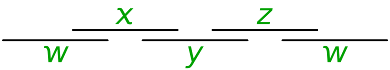
Our Motivation

Def. interval number $i(G)$ - the least t such that G is the intersection graph of unions of t real intervals.
interval graph - a graph G with $i(G) = 1$.

Ex. $i(C_4) = 2$.



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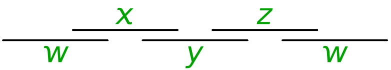

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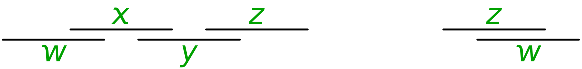

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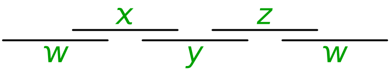

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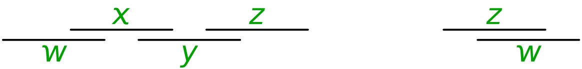

- Always $i(G) \leq \tau(G)$, but $i(K_{5,3}) = 2 < 3 = \tau(K_{5,3})$.

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Thm. (Milans–Stolee–West [2015])

$$\Omega\left(\frac{\lg\lg n}{\lg\lg\lg n}\right) \leq \tau(L(K_n)) \leq O(\lg\lg n).$$

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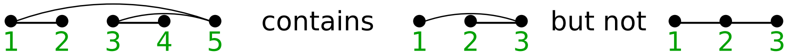

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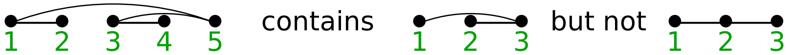

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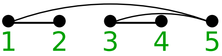

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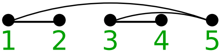

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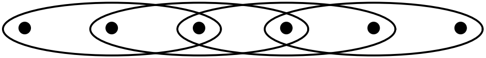
(General: $n = \mathrm{OR}(H_1, \ldots, H_t)$ forces H_i in color i for some i.)

Tight Paths

The k-uniform ordered path P_r^k has vertices $1, \ldots, r$; the edges are the sets of k consecutive vertices.



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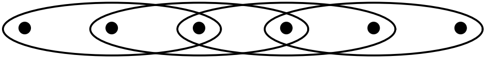

Ex. $R_t(P_3^2) = t + 2$.

To avoid unordered monochr. P_3 in $E(K_n)$, each color is a matching. With $|E(K_n)| = \binom{n}{2}$ edges, needs $t \geq n - 1$.

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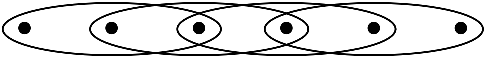

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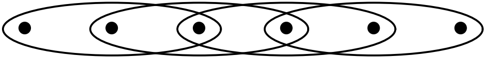

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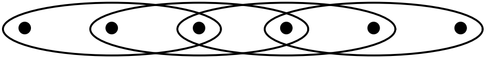

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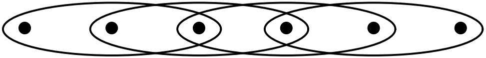

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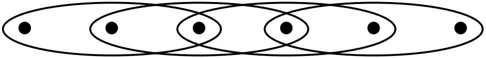

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With $\Pi_{i=1}^{t}(r_i - 1)$ vertices as t-tuples, c can be constructed to avoid the threshold in each color. ■

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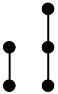
Thm. (Moshkovitz–Shapira [2014]) For $t \geq 2$, $k \geq 3$, and $m = r - k + 1 \geq 2$,

$$\mathrm{tow}_{k-2}(m^{t-1}/2\sqrt{t}) \leq \mathrm{OR}_t(P_r^k) \leq \mathrm{tow}_{k-2}(2m^{t-1}).$$

Equivalent Enumerative Problem

Def. For any poset Q, the poset $J(Q)$ is the set of all down-sets in Q, ordered by inclusion.



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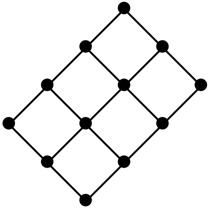

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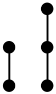

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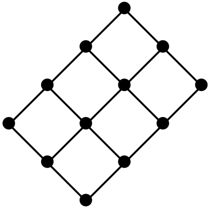

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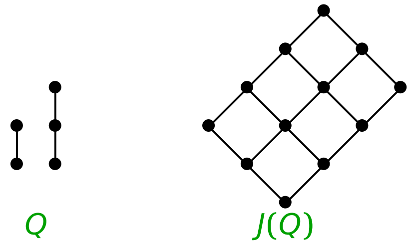

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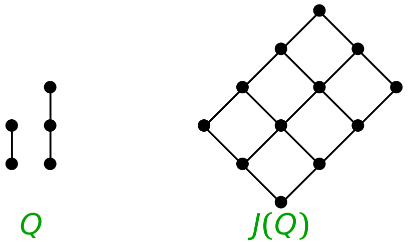

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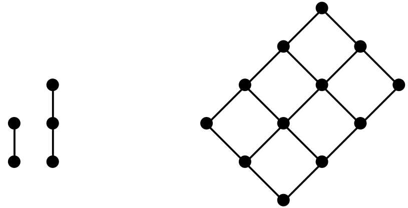

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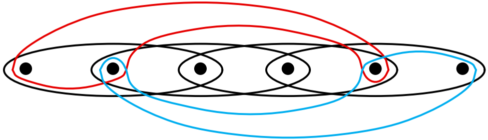
- Proof already given for $k = 2$. Full proof one page.

Track Number - Lower Bound

Thm. Obtain P' from the ordered P_6^3 by adding edges $\{1,2,5\}$ and $\{2,5,6\}$. If $n \geq \mathrm{OR}_t(P')$, then $\tau(L(K_n)) > t$.



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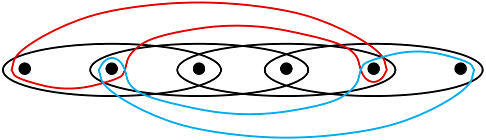

Pf. Given a t-track rep'n of $L(K_n)$, and $x < y < z$ in $[n]$, let $\phi(x, y, z)$ be the index of a track where intervals for xy and yz intersect. (Note ϕ is a t-coloring of $E(K_n^3)$.)

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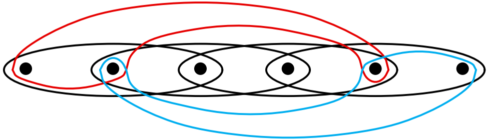

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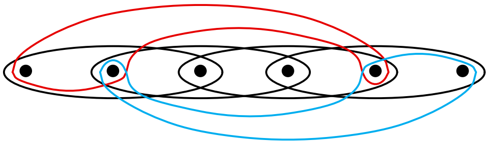

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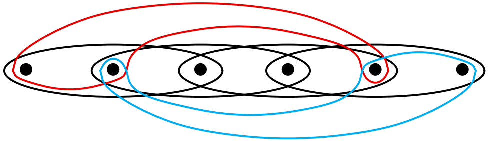
In track s, interval for x_2x_5 must meet those for x_1x_2 and x_5x_6 but not for x_3x_4, which lies between them.

Track Number - Lower Bound

Thm. Obtain P' from the ordered P_6^3 by adding edges $\{1,2,5\}$ and $\{2,5,6\}$. If $n \geq \mathrm{OR}_t(P')$, then $\tau(L(K_n)) > t$.



Pf. Given a t-track rep'n of $L(K_n)$, and $x < y < z$ in $[n]$, let $\phi(x,y,z)$ be the index of a track where intervals for xy and yz intersect. (Note ϕ is a t-coloring of $E(K_n^3)$.)

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Thus $L(K_n)$ has no t-track representation. ■

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Thm. If $n < \mathrm{OR}_t(P_4^3)$, then $\tau(L(K_n)) \leq t + 2$.

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In track i for $1 \leq i \leq t$: For $xy \in E(K_n)$,
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Edge $\{xy, xz\}$ of $L(K_n)$ is represented in the left track, and $\{wy, xy\}$ is represented in the right track. ∎

Numerical Bounds

Erdős–Rado [1952]: $R_t(K_6^3) \le 2^{2^{O(t \lg t)}}$.

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- Since $\mathrm{OR}_t(P_4^3) = |J^2(Q)| + 1$, finding $\mathrm{OR}_t(P_4^3)$ is the famous Dedekind Problem: counting the antichains in the subset lattice of order t.

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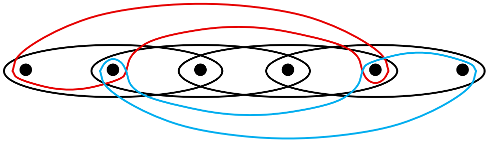

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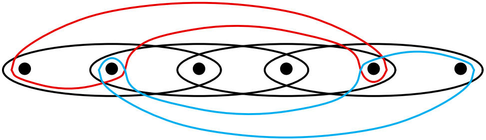

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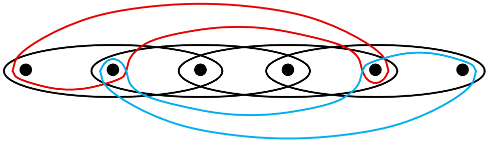

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(**Ex.** is the linear ordering worst for the path P_n?)

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Thm. (B–J–V [2016+]) There is a constant c such that for all graphs G with maximum degree 2, there is an ordering such that $\mathrm{OR}(G,G) \leq c|V(G)|$.

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Pettie [2011] constructed larger examples adding a $\log\log n$ factor, Fox et al. added another $\log\log\log n$ factor, etc., but it is not known whether any grows as fast as $n(\log n)^2$.

Plan for "Computing" $\mathrm{OR}_t(P^k_r)$

Thm. Given $r_1, \ldots, r_t > k$, let Q consist of disjoint chains with sizes $r_1 - k, \ldots, r_t - k$. With $Q_1 = Q$ and $Q_i = J(Q_{i-1})$ for $i > 1$,

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Pf. $k = 1$ is just the pigeonhole principle; any t-coloring of 1-sets from $1+\sum_{i=1}^{t}(r_i-1)$ vertices has r_i with color i for some i, forming an ordered copy of $P^1_{r_i}$ in color i.

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Thm. Given $r_1, \ldots, r_t > k$, let Q consist of disjoint chains with sizes $r_1 - k, \ldots, r_t - k$. With $Q_1 = Q$ and $Q_i = J(Q_{i-1})$ for $i > 1$,

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For $k \geq 2$, use the inductive definition of Q_k.

Lower Bound: For $n = |Q_k|$, we t-color $E(K_n^k)$ to avoid having $P_{r_i}^k$ in color i for any i.

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If $x \not\geq y$ in Q_j, then by definition $x \not\supseteq y$ in Q_{j-1};
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Call a list $x_1, \ldots, x_s$ descent-free if $x_i \not\geq x_{i+1}$ for all i;
in that case let $f(x_1, \ldots, x_s) = (f(x_1, x_2), \ldots, f(x_{s-1}, x_s))$.

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- Under f, the image of a descent-free s-list in Q_j is a descent-free $(s-1)$-list in Q_{j-1}.

Lower Bound continued

Let $y_1, \ldots, y_n$ be a linear extension of Q_k.
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iterating f for $k - 1$ steps yields a 1-element list in Q_1;
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Since $n = |Q_k|$, we want to give $\{x_1, \ldots, x_k\}$ (in $E(K_n^k)$) a color in $[t]$. Color it with the index i of the chain in Q containing $f^{k-1}(x_1, \ldots, x_k)$.

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An i-colored copy of P_s^k has vertices $x_1, \ldots, x_s$ with color i on each consecutive k-list. Thus $f^{k-1}(x_1, \ldots, x_s)$ is a descent-free $(s-k+1)$-list in chain i of Q.

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Thus $s < r_i$, so we avoid $P_{r_i}^k$ in color i for each i. ■

Upper Bound: $\mathrm{OR}_t(P^k_{r_1}, \ldots, P^k_{r_t}) \leq |Q_k| + 1$

Given $\phi\colon E(K^k_n) \to [t]$ that avoids $P^k_{r_i}$ in color i for each i, we define an injection from $[n]$ to Q_k.

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We have $1 \le h \le r_i - k$, since ϕ yields no i-colored $P^k_{r_i}$.

If $\phi(Y^-) = \phi(Y^+)$ for $Y \in \binom{[n]}{k+1}$, then $g_k(Y^+) > g_k(Y^-)$.
Else, $g_k(Y^+)$ and $g_k(Y^-)$ are on different chains in Q.
Hence $g_k(Y^-) \not\ge g_k(Y^+)$, and $(*)$ holds.

Upper Bound continued

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For $X \in \binom{[n]}{j}$, let $S = \{g_{j+1}(Y) : Y^+ = X\}$, and define $g_j(X)$ to be the down-set in Q_{k-j} generated by S.

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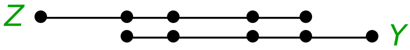
If also $g_{j+1}(Y) \in g_j(Y^-)$, then the down-set $g_j(Y^-)$ in Q_{k-j} has a $g_{j+1}(Z)$ with $Z^+ = Y^-$ and $g_{j+1}(Y) \leq g_{j+1}(Z)$ in Q_{k-j}.



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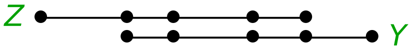

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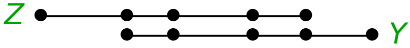

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Thus $g_{j+1}(Y) \in g_j(Y^+) - g_j(Y^-)$, so $g_j(Y^-) \ngeq g_j(Y^+)$. ∎