On-Line Ramsey Theory in Bounded Degree Graphs

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Joint work with
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This defines the on-line Ramsey game $(G, \mathcal{H})$.
Can Builder playing on $\mathcal{H}$ force a monochromatic G?

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Grytczuk–Kierstead–Prałat [2008?]

$\tilde{r}(G) = \min\{k :$ Builder wins on graphs with $\le k$ edges$\}$.

$\tilde{r}(P_n) \le 4n - 7$, but for trees it can be quadratic.

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Thm. $\mathrm{odr}(G) \leq 8$ if $\Delta(G) \leq 2$ (maybe less).

Warmup - Paths

Thm. $\mathrm{odr}(P_n) \leq 3$.

Pf. Stronger statement: In $\mathcal{S}_3$, Builder forces a monochr. P_m with $m \geq n$ whose ends have no other edges.

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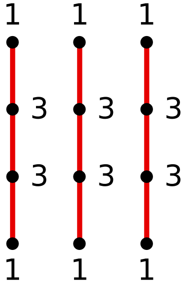

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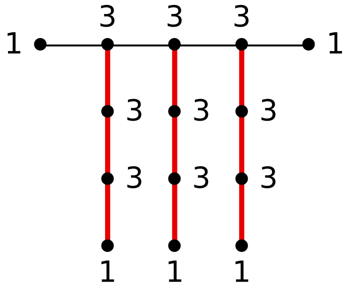

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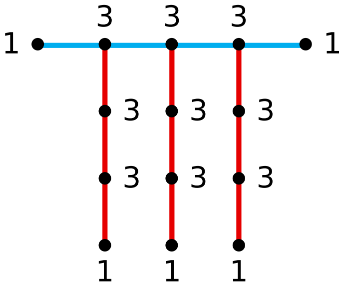

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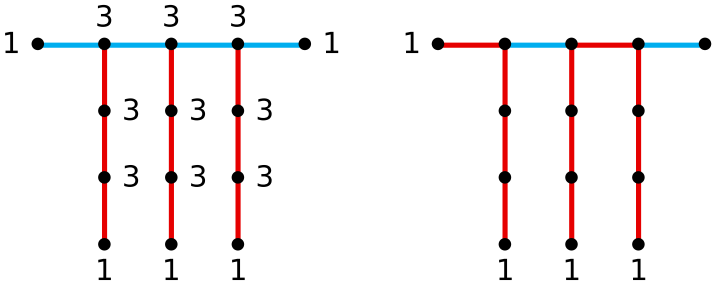

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Obs. If Builder can force a copy of G with weight at most k at each vertex when no restrictions are imposed on edges played, then $\mathrm{odr}(G) \le k$.

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Obs. If Builder wins $(G, \mathcal{H})$ (weighted G), then Builder wins $(mG, \mathcal{H})$.

Pf. Play $2m - 1$ times, then pigeonhole on two colors. ∎

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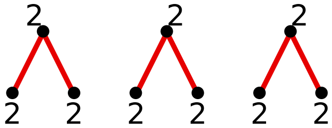

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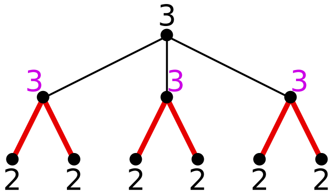

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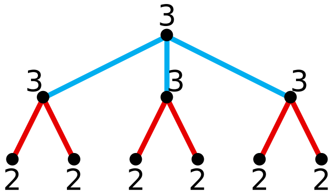
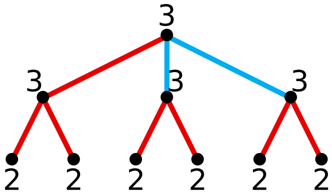


One red .



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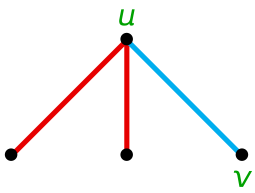

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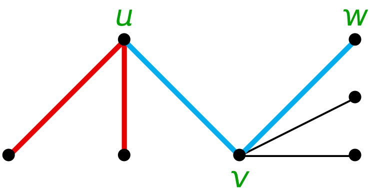

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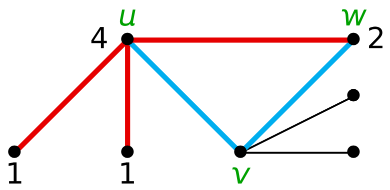
3) Present uw. Done if blue ; otherwise, red $(4,2,2,2)$-claw.



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Pf. Edge gets blue $\Leftrightarrow$ an endpt already has $m-1$ red.

Painter never makes a red $K_{1,m}$. If a blue $K_{1,m}$, then center has $m-1$ red or each leaf has $m-1$ red . ■

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Pf. Let $m = \Delta(G)$. $\mathcal{S}_{m-1}$-Painter never makes red G.

Let xy be an edge with maxmin degree in G.

A blue G has an edge for xy; it has $m-1$ red at one endpt and at least $\min\{d_G(x), d_G(y)\}$ blue there. ■

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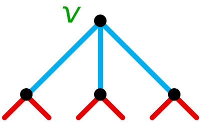
To create degree 3 in blue at v, each neighbor must already have two red and hence no more blue (in $\mathcal{S}_3$).
$\therefore$ blue component is $K_{1,3}$.



odr ≤ 3, cont.

Long blue path unforceable against greedy $\mathcal{L}$-Painter .

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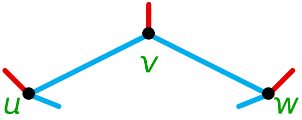

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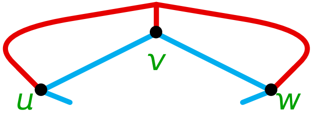
uv, vw blue $\Rightarrow$ $\exists$ red u, v-path and red v, w-path. This violates red $\subseteq \mathcal{L}$.



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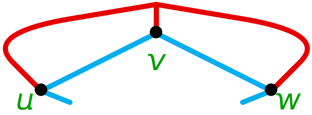
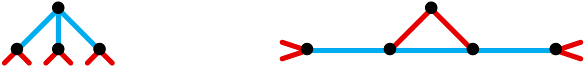


Builder forces blue $K_{1,3} + P_4$ against greedy $\mathcal{L}$-Painter .



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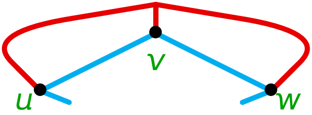

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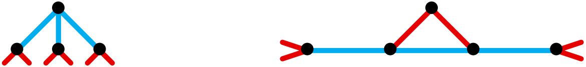

However, greedy $\mathcal{S}_2$-Painter yields
$\mathrm{odr}(K_{1,3} + P_4) \geq 3 - 1 + \min\{2, 2\} = 4.$

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Pf. Idea: Builder forces an arbitrarily large monochr. complete k-ary tree, where $k = \Delta(G) - 1$.

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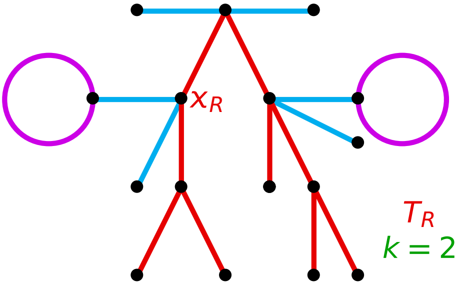

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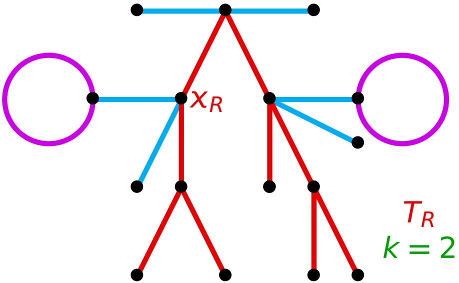
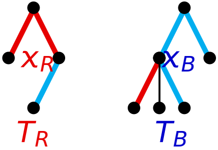


An active vertex is
satisfied if it has k children via its own color.
dangerous if it has k incident edges of the other color.

Builder Strategy

Builder plays pendant edges at active vertices (in T_R or T_B) until Painter makes one satisfied or dangerous.



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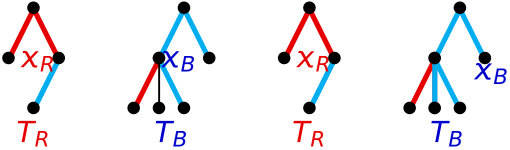

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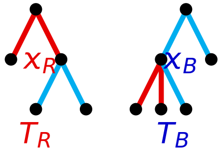

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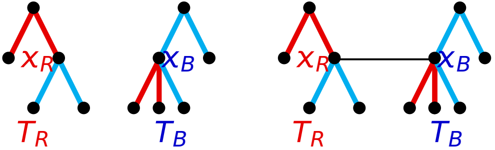

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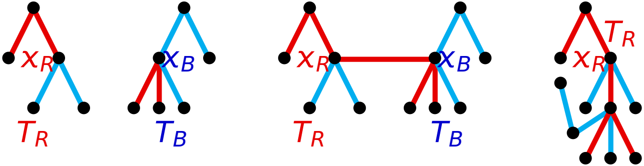

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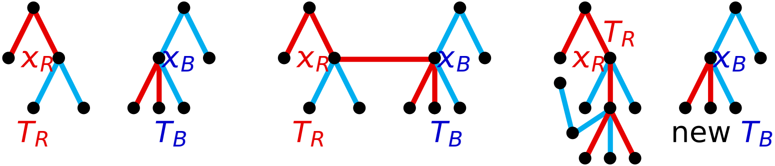
Then Builder regenerates the other tree.



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Pf. $\forall$ Painter $\mathcal{A}$, $\exists$ consistent Painter $\mathcal{A}'$ s.t. Builder can force against $\mathcal{A}$ whatever he can force against $\mathcal{A}'$.

Even Cycles

Assume Builder plays on $\mathcal{S}_k$ and Painter is consistent.

Lem. Let F_1, F_2 be weighted graphs Builder can force in red , with vertices u_1, u_2. Form F from $F_1 + F_2$ by adding u_1u_2 and increasing weights on u_1 and u_2 by 2. If q is even, then Builder can force a red F or a blue C_q.

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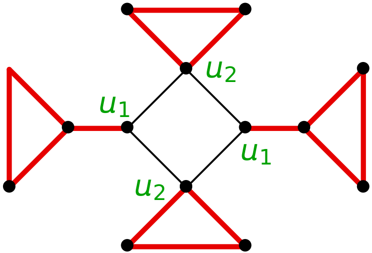
Pf. Builder forces $q/2$ copies of F_1 and F_2 and then adds a cycle alternating between the copies of u_1 and u_2. ■



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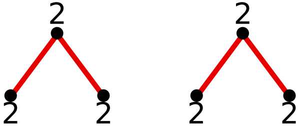

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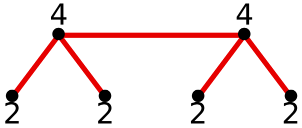

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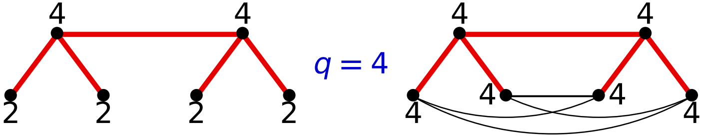

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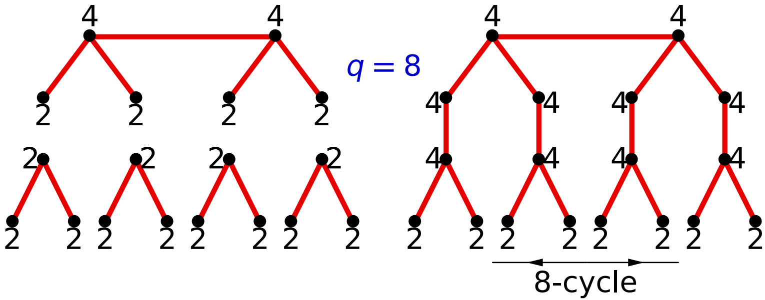

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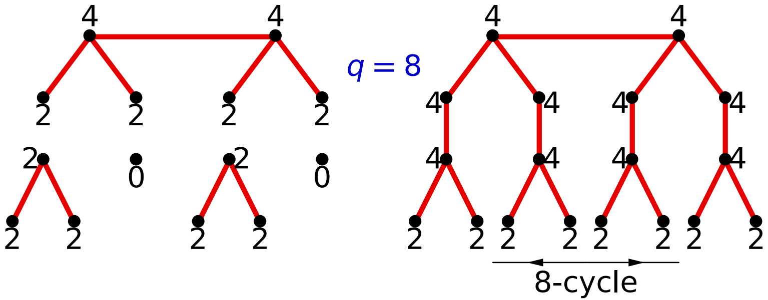

Extension by P_3 lengthens both sides by 2.
Extension by K_1 lengthens both sides by 1.

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Consistent Painter makes consistent triangles.

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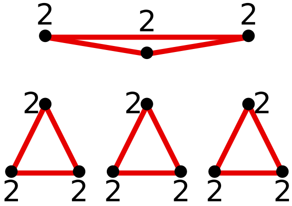

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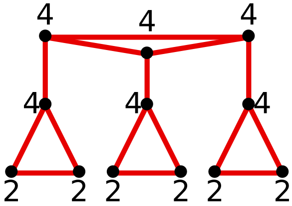

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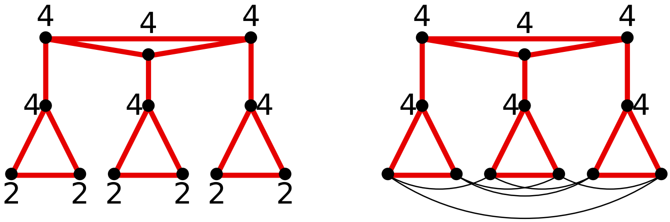

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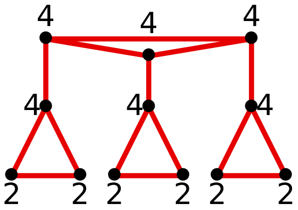

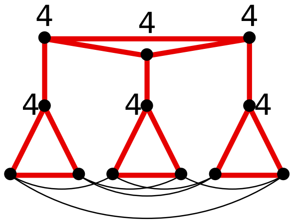

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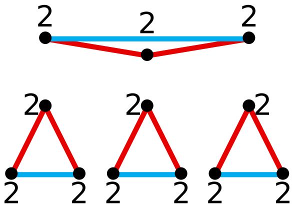

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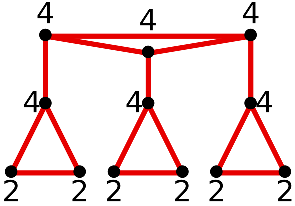

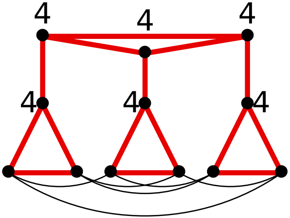

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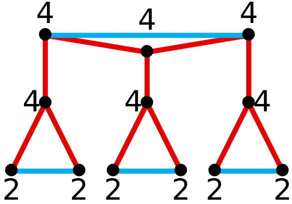

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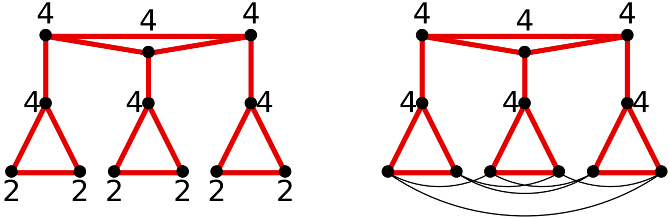

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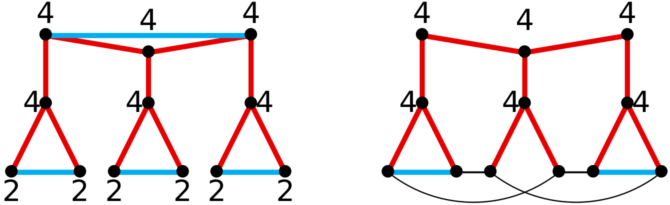

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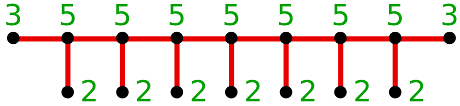

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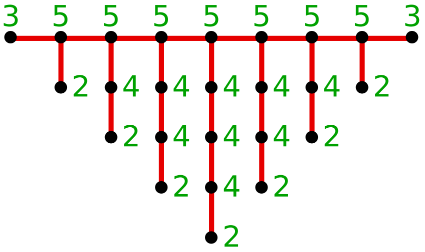

Leaf distances $q - 1$ (opposite halves or to middle).
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Thm. If a tree T with $\Delta(T) = a$ has exactly one vertex of degree a, and all others have degree at most b (where $b \le a$), then $\mathrm{odr}(T) \le a + b - 1$.

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Like $\mathrm{odr}(G)$, some results are known for $\mathrm{dr}(G)$ when G is a path, star, cycle, or tree.

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Thm. Thomassen [1999] Every 3-regular graph has a 2-edge-coloring such that each monochromatic component is contained in P_6.

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Thm. Thomassen [1999] Every 3-regular graph has a 2-edge-coloring such that each monochromatic component is contained in P_6.

Cor. $\mathrm{dr}(P_n; 2) = 4$ and $\mathrm{dr}(P_n; s) \ge 3s/2$ for $n > 6$.

Degree Ramsey for Paths

Thm. Alon–Ding–Oporowski–Vertigan[2003] $\mathrm{dr}(P_n; s) \le 2s$.

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Pf. If $\Delta(H) < 3s/2$, then H is $3s/2$-edge-colorable.
Triples of colors form subgraphs with maxdeg 3.
Use two colors on each subgraph. ■

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- If G is connected and $\mathrm{dr}(G) \leq 4$, then G is a path or an even cycle.

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Ques. What is $\mathrm{dr}(C_n)$?
What is $\mathrm{dr}(K_{1,2k}; s)$ when s is odd?