On-line Size Ramsey Number for Ordered Tight Paths

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slides available on DBW preprint page

Joint work with
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Def. size Ramsey number $\hat{R}_t(G) = \min\{|E(H)|: H \to_t G\}$.

(Other parameter Ramsey numbers have been studied, minimizing $\omega(H)$, $\chi(H)$, $\Delta(H)$, genus, etc.)

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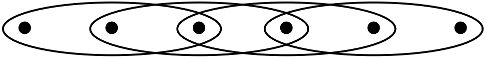
The k-uniform ordered tight path $P_r^{(k)}$ has vertex set $[r]$; its edges are the sets of k consecutive vertices.



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These are applications of $R_t(P_r^{(k)})$ (number of vertices).

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$2n-3 \le \tilde{R}_2(P_n) \le 4n-7$ (GKP [2008])

Ordinary $R_t(P_3) = t+2$, but ordered path $R_t(P_3) = 2^t+1$.
General ordered path $R_t(P_n) = (n-1)^t+1$.

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$$\mathrm{tow}_{k-2}\left(\frac{m^{t-1}}{2\sqrt{t}}\right) \leq R_t(P_r^{(k)}) \leq \mathrm{tow}_{k-2}(2m^{t-1}).$$

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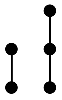
- Trivial upper bound $\binom{|Q_k|+1}{k}$.

Enumerative Problem

Def. For any poset Q, the poset $J(Q)$ is the set of all down-sets in Q, ordered by inclusion.



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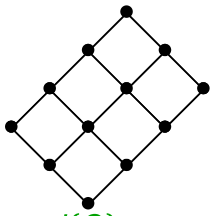

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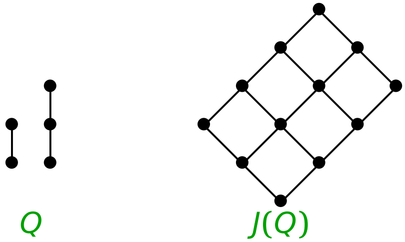

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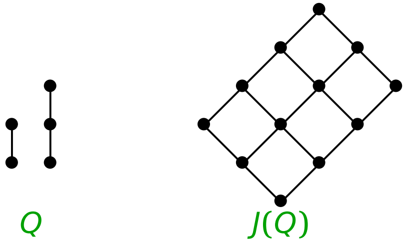

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- This iteration leads to the upper and lower bounds.

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Hence in at most $(m^t - 1)[(m-1)t - 1] + 1$ rounds some label reaches Λ, and the next play wins. ■

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Since B is the middle (largest) level of a chain product with m^t elements and $(m-1)t+1$ levels, $|B| \ge m^{t-1}/t$. ∎

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- When vertices enter only from left to right, Painter can use all of M^t as labels, via a linear extension, and the lower bound is $m^t/2$ even with edges anywhere.

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Being a downset in Q_{k-j}, by definition $g_j(Y) \in Q_{k-j+1}$.

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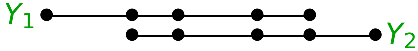

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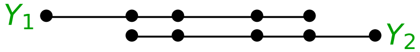

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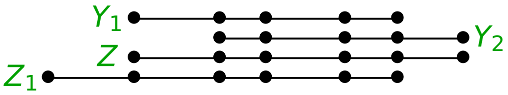

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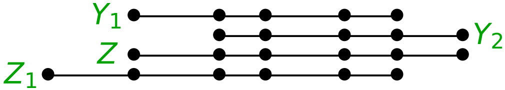

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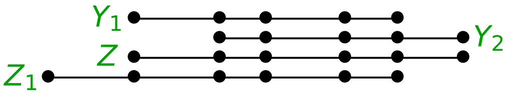

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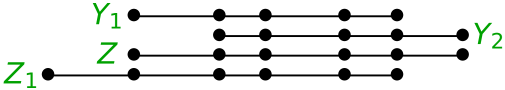

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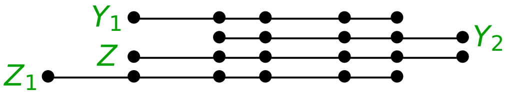

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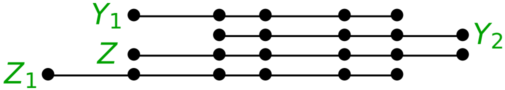

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But $Z \in \overleftarrow{Y_2}$, so $g_{j+1}(Z) \in g_j(Y_2)$. $\therefore g_j(Y_2) \not\subseteq g_j(Y_1)$ in Q_{k-j}. ∎

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Lem. A j-set Y_2 follows a j-set Y_1 if and only if $Y_1^+ = Y_2^-$ and there is an instance of $U(Y_1)$ such that for every edge W assigned to a leaf, replacing the first vertex of W with the last vertex of Y_2 yields an edge Z in G.

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Builder then plays $Y \cup \{n\}$ to win.

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When $|Y| = j < k$ and no precursor of Y has a label,

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Ranks 0 through $j-2$ of Q_j form one chain.

For $i \leq k-j$, the label of a j-set with least vertex i is always the only element of rank $i-1$ in Q_{k-j+1}.

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When $|Y| = j < k$ and no precursor of Y has a label,

down$(\overleftarrow{Y_1}) = \emptyset$, so $g_j(Y) =$ bottom element of Q_{k-j+1}.

This holds always for $(k-1)$-sets whose precursors are not edges and for j-sets with least vertex 1.

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Pigeonhole!

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When Λ_k is not the label of any vertex before n, some such x and y have the same label, by pigeonhole.

Since $g_1(x) = g_1(y)$, by 1st Lemma y does not follow x.

Builder plays edges to make y follow x. By 1st Lemma, this makes $g_1(y)$ go up in Q_k. How to make y follow x?

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In an instance of $U(\{x\})$, assigned to each leaf is an edge Z ending with x. By 2nd Lemma, it suffices that deleting $\min Z$ and adding y at the end yields an edge.

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Since $\text{ht}(Q_k) = |Q_{k-1}|$ and the posets grow by iterated exponentiation, #moves $\le |Q_k|(\lg |Q_k|)^{2+\epsilon}$. ■

Lower Bound - Painter Strategy

Thm. $\tilde{R}_t(P_r^{(k)}) \geq |Q_k|/(k \lg |Q_k|)$.

Lower Bound - Painter Strategy

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Pf. For $1 \le j \le k$, Painter assigns a label $f_j(Y) \in Q_{k-j+1}$ to each j-set Y of vertices; labels never change.

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Painter chooses any such element as the label $f_{j+1}(Y)$.

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Hence $f_{j+1}(Y_2)$ is not below anything in the downset $f_j(Y_2^-)$, including $f_{j+1}(Y_1) \in f_j(Y_1^+) = f_j(Y_2^-)$.

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Since chains have only $m-1$ elements, no $P_r^{(k)}$ occurs. ■

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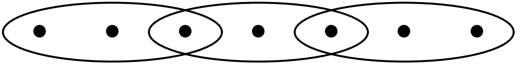
- Similar ideas for digraph Ramsey problems yield some slight improvements to bounds on size Ramsey numbers in Ben-Eliezer–Krivelevich–Sudakov [2012].

A Generalization

Def. In the ℓ-loose k-uniform monotone path $P_r^{k,\ell}$, each edge consists of k consecutive vertices, but each edge starts ℓ vertices after the start of the previous edge. (Note $P_r^{k,1} = P_r^{(k)}$.)

$P_7^{3,2}$



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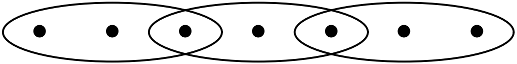
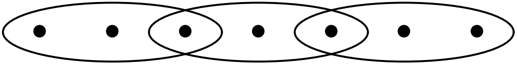


Generalizing to $P_r^{k,\ell}$ is not hard. The off-line value $R_t(P_r^{k,\ell})$ was first obtained in Cox–Stolee [2016].

A Generalization

Def. In the ℓ-loose k-uniform monotone path $P_r^{k,\ell}$, each edge consists of k consecutive vertices, but each edge starts ℓ vertices after the start of the previous edge. (Note $P_r^{k,1} = P_r^{(k)}$.)

$P_7^{3,2}$



Generalizing to $P_r^{k,\ell}$ is not hard. The off-line value $R_t(P_r^{k,\ell})$ was first obtained in Cox–Stolee [2016].

Thm. For $k, \ell, r, t \in \mathbb{N}$, let $h = \lceil k/\ell \rceil$ and $s = k - (h-1)\ell$. With $Q_1, \ldots, Q_h$ defined using k, r, t as before,

$$R_t(P_r^{k,\ell}) = \ell|Q_h| + s \text{ and}$$

$$|Q_h|/(k \lg |Q_h|) \le \tilde{R}_t(P_r^{k,\ell}) \le \ell|Q_h|(\lg |Q_h|)^{2+\epsilon}$$

(given fixed ϵ and large $t(r-k)$).

Further Directions

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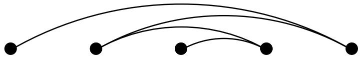

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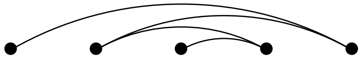

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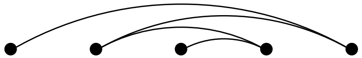

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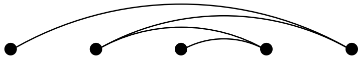

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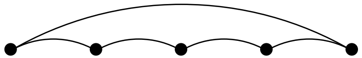

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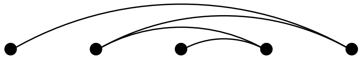

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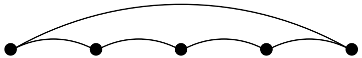

Thm. (BCKK [2015])
$R(\vec{C}_r, \vec{C}_s) = (r-1)(s-1) + (r-2)(s-2) + 1$.