(5, 2)-Coloring of Sparse Graphs

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May 22, 2007

Abstract

We prove that every triangle-free graph whose subgraphs all have average degree less than 12/5 has a $(5,2)$-coloring. This includes planar and projective-planar graphs with girth at least 12. Also, the degree result is sharp; we construct a minimal non-$(5,2)$-colorable triangle-free graph with 10 vertices that has average degree 12/5.

1 Introduction

The famous theorem of Grötzsch [6] states that triangle-free planar graphs are 3-colorable. Perhaps coloring becomes "easier" with stronger restrictions on girth. Long odd cycles may still occur, so we cannot reduce the chromatic number below 3. However, we can improve the bound on a finer measure of coloring difficulty than the chromatic number.

Vince [12] introduced a parameter now called the "circular chromatic number". A (k,d)-*coloring* of a graph G is a map $\phi\colon V(G) \to \{0,\dots,k-1\}$ such that $d \le |\phi(u)-\phi(v)| \le k-d$ for every edge $uv \in E(G)$. An equivalent "circular" phrasing is that $\phi\colon V(G) \to \mathbb{Z}_k$ so that the images of adjacent vertices differ (cyclically) by at least d. A graph having a (k,d)-coloring is (k,d)-*colorable*. The *circular chromatic number* $\chi_c(G)$ of a graph G is $\inf\{k/d\colon G$ is (k,d)-colorable$\}$. The infimum is a minimum: if $\chi_c(G)=k/d$, then G has a (k,d)-coloring.

An ordinary proper k-coloring is a $(k,1)$-coloring, so $\chi_c(G) \le \chi(G)$, where χ denotes the ordinary chromatic number. We view χ_c as a refined measure of coloring difficulty

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because $\chi_c(G) > \chi(G) - 1$, proved by Vince [12] and again by Bondy and Hell [1]. Thus $\chi(G) = \lceil \chi_c(G) \rceil$, and knowing $\chi_c(G)$ also gives $\chi(G)$. Zhu [13] surveys early results on χ_c.

A 3-chromatic graph is not 2-colorable, but if its circular chromatic number is near 2, then in some sense it is "just barely" not 2-colorable. For odd cycles, $\chi_c(C_{2k+1}) = 2 + \frac{1}{k}$, and $C_{2k+1} \subseteq G$ implies $\chi_c(G) \geq 2 + \frac{1}{k}$.

The *girth* of a graph is the length of its shortest cycles. It is conjectured that $\chi_c(G) \leq 2 + \frac{1}{k}$ when G is a planar graph with girth at least $4k$ (this is implied by a restriction of a conjecture of Jaeger [7] on integer flows). When $k = 1$, the statement reduces to Grötzsch's Theorem. The conjectured threshold $4k$ is sharp; DeVos [3] constructed a planar graph G with girth $4k - 1$ and $\chi_c(G) > 2 + \frac{1}{k}$ (add a cycle of length $4k - 1$ through the leaves of a tree consisting of $4k - 1$ paths of length $2k - 1$ sharing just a common endpoint).

Nešetřil and Zhu [10] and Galuccio, Goddyn, and Hell [5] proved that girth at least $10k - 4$ suffices for $\chi_c(G) \leq 2 + \frac{1}{k}$ when G is planar. Zhu [14] reduced the threshold to $8k - 3$. Borodin, Kim, Kostochka, and West [2] further lowered it to $\frac{20k-2}{3}$.

The case $k = 2$ deserves particular attention, because the case $k = 2$ of the full statement of Jaeger's Conjecture [7] implies Tutte's 5-Flow Conjecture [11] (see [8, p. 209]). When $k = 2$, the result of [2] or [14] implies that every planar graph with girth at least 13 is $(5, 2)$-colorable. Our main result implies that girth 12 suffices.

The "Folding Lemma" of Klostermeyer and Zhang [9] strengthens some results to give $(2k + 1, k)$-colorings when the length of the shortest *odd* cycle (called the *odd-girth*) reaches the given threshold. For example, Zhu [14] proved that $\chi_c(G) \leq 2 + \frac{1}{k}$ when G is a planar graph with odd-girth at least 13. The argument of [2] does not yield a threshold for odd-girth. Like our argument here, it relies solely on a sparseness condition. The result in [2] is that $\chi_c(G) \leq 2 + \frac{1}{k}$ whenever G has girth at least $6k - 2$ and every subgraph of G has average degree at most $2 + \frac{6}{10k-4}$, regardless of whether G is planar.

When G is planar, a lower bound on girth imposes an upper bound on the average degree of every subgraph. It is well known from Euler's Formula that an n-vertex planar graph G with girth g has at most $(n - 2)\frac{g}{g-2}$ edges, and for a graph in the projective plane the bound is $(n - 1)\frac{g}{g-2}$. Since deleting edges cannot reduce girth, every subgraph of G thus has average degree strictly less than $\frac{2g}{g-2}$. Therefore, girth at least $\frac{20k-2}{3}$ enforces average degree less than $2 + \frac{6}{10k-4}$ and yields $\chi_c(G) \leq 2 + \frac{1}{k}$. Our corollary that girth 12 yields $\chi_c(G) \leq 5/2$ when G embeds in the plane or projective plane follows in the same way from our main theorem:

Theorem 1.1 *If G is triangle-free and every subgraph of G has average degree less then $12/5$, then G is $(5, 2)$-colorable.*

In fact, the bound on average degree in this theorem is sharp.

Example 1.2 The graph on the left in Figure 1 has 10 vertices and 12 edges, yielding average degree $12/5$. Furthermore, all its proper subgraphs have smaller average degree. Theorem 1.1 thus implies that all proper subgraphs are $(5, 2)$-colorable.

This graph can be expressed as $G_1 \cup G_2$, where G_1 consists of two 5-cycles sharing the edge xy, and G_2 is a copy of P_4 whose endpoints u and v are the vertices in G_1 that are non-neighbors of both x and y.

In a $(5,2)$-coloring of C_5, all five colors must be used, being $0, 2, 4, 1, 3$ in order around the cycle in one direction (see Figure 1). Thus G_1 requires u and v to have the same color. However, in a $(5,2)$-coloring of P_4, the endpoints cannot have the same color, so G_2 requires u and v to have different colors. □

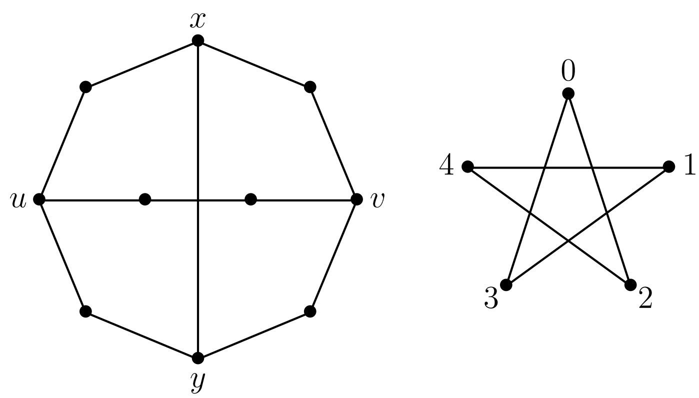

Figure 1: Sharpness of the main result.

The *maximum average degree* of a graph G is the maximum, over all subgraphs of G, of the average vertex degree. We use the notation $\mathrm{mad}(G)$; thus $\mathrm{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. Specifying $\mathrm{mad}(G) < t$ is a condition of "sparseness" for G. As noted above, girth g implies $\mathrm{mad}(G) < 2g/(g-2)$ when G is planar. For nonplanar G, girth and $\mathrm{mad}(G)$ can both be arbitrarily large. This follows from the existence of regular graphs with arbitrary degree and girth (the smallest such are called *cages*), as shown by Erdős and Sachs [4].

When studying a family $\mathbf{F}$ such that $\mathrm{mad}(G) < t$ for all $G \in \mathbf{F}$, the "Discharging Method" is a natural technique. If the desired conclusion Q does not always hold, then we consider a graph $G \in \mathbf{F}$ that is minimal among those failing Q. After using minimality to derive properties of G, we view the degree of each vertex as an initial "charge" on the vertex. "Discharging" is the process of moving charge among vertices to "smooth it out". If the structural properties allow discharging to produce a state where every vertex has charge at least t, then the existence of a counterexample has been disproved.

Many discharging arguments for planar graphs also assign charge to faces, since average degree is bounded also in the dual graph. We assign charge only to vertices, since we assume only $\mathrm{mad}(G) < 12/5$, not planarity. A point of interest is that our discharging rules may move charge arbitrarily far in the graph; such global discharging is unusual. The discharging argument in [2] moves charge along paths consisting solely of vertices with degree 2, but not past vertices of higher degree.

We use $d(v)$ for the degree of a vertex v and call v a *j-vertex* when $d(v) = j$. As in [2] and elsewhere, a *thread* in G is a path whose internal vertices are 2-vertices. A *maximal thread* is a thread whose endpoints are not 2-vertices. If G is 2-connected and not a cycle, then G is the union of its maximal threads (note that adjacent 3-vertices form a maximal thread of length 1). We use *j-thread* to mean a thread with exactly j internal 2-vertices (and length $j+1$). A *partial* $(5,2)$*-coloring* of G is a $(5,2)$-coloring of an induced subgraph of G.

2 Obstructions and Coloring Extensions

Let $\mathbf{F} = \{G\colon \operatorname{mad}(G) < 12/5\}$. The 3-cycle C_3 is a minimal graph in $\mathbf{F}$ having no $(5,2)$-coloring. We use the term *obstruction* for every other minimal such graph, if any exists.

Remark 2.1 *If G is an obstruction, then G is triangle-free and 2-connected and decomposes into maximal threads. Also, G has no 4-cycle through a 2-vertex.*

Proof. The first conclusion holds because $\chi_c(G) \ge 3$ whenever G contains C_3. Because components can be colored separately, and because names of colors on two subgraphs can be permuted (cyclically) to agree at a cut-vertex, we may assume that G is 2-connected, and hence $\delta(G) \ge 2$. Since $\chi_c(C_n) \le 5/2$ when $n > 3$, we have that G is not a cycle, and hence G decomposes into maximal threads.

Finally, if G has a 4-cycle H through a 2-vertex x, then a $(5,2)$-coloring of $G - x$ extends to G by copying onto x the color of the vertex opposite x on H. □

We view the set $\mathbb{Z}_5$ of colors as a set of values on a circle modulo 5. We write $|a-b| = 0$ if a and b are (in) the same class, $|a-b| = 1$ if they are (in) consecutive classes, and otherwise $|a-b| = 2$. For example, $|4-0| = 1$ and $|4-1| = 2$.

When G is an obstruction, a partial $(5,2)$-coloring does not extend to a $(5,2)$-coloring of G. We use this to prove structural properties of obstructions: if the desired property fails, then a $(5,2)$-coloring of some subgraph extends to a $(5,2)$-coloring of G. Our first lemma discusses extension along threads and implies that an obstruction has no thread with at least three internal vertices. Prohibiting long threads increases average degree, which supports our plan to eliminate all obstructions by forcing them to have average degree at least 12/5.

Lemma 2.2 *Let P be a j-thread in G with endpoints u and v. If ϕ is a partial $(5,2)$-coloring of the graph obtained by deleting the internal vertices of P (and is defined on $\{u,v\}$), then ϕ extends along P in the following cases:*

(a) $j \ge 3$.
(b) $j = 2$ and $\phi(u) \ne \phi(v)$.
(c) $j = 1$ and $|\phi(u) - \phi(v)| \ne 2$.
(d) $j = 0$ and $|\phi(u) - \phi(v)| = 2$.

More generally, extension along a path from a colored vertex allows two consecutive colors *at its neighbor, three consecutive colors at the next vertex, and all but one color at the third vertex, if there is no interference from other colored vertices.*

Proof. Part (d) is a degenerate case where we are not actually adding new vertices; it merely asserts that the colors on adjacent vertices must differ by 2.

Let the vertices of P be $x_0, \ldots, x_{j+1}$ in order, with $x_0 = u$ and $x_{j+1} = v$. When $\phi(u) = a$, the coloring extends along P with the color at x_i allowed to be $a+2$ or $a-2$ when $i = 1$, any color in $\{a-1, a, a+1\}$ when $i = 2$, any of $\mathbb{Z}_5 - \{a\}$ when $i = 3$, and any of $\mathbb{Z}_5$ when $i \geq 4$. The extension can be made consistent for all of P if $\phi(v)$ belongs to the set allowed when $i = j + 1$. This yields the claim in each case. □

The extension conditions are also necessary; for given j, extension is possible along P only when $\phi(u)$ and $\phi(v)$ are related as described.

Lemma 2.2 suggests a local measure of "extendibility" of a partial $(5, 2)$-coloring ϕ. We define the *ϕ-set* of v, denoted $\hat{\phi}(v)$, to be the set of colors allowed at v by threads from v to colored vertices. A colored vertex is equivalently a vertex whose ϕ-set has size 1. If v is not within distance 3 of a colored vertex, then $\hat{\phi}(v) = \mathbb{Z}_5$. Otherwise, threads to colored vertices reduce $\hat{\phi}(v)$ in accordance with Lemma 2.2. The set $\hat{\phi}(v)$ can be viewed as a list of colors from which the color at v must be chosen in order to extend ϕ to all of G.

Lemma 2.3 *Let uv be an edge in G, let ϕ be a partial $(5, 2)$-coloring ϕ of G, and let $A = \hat{\phi}(u)$ and $B = \hat{\phi}(v)$. If $|A| + |B| \geq 4$, then ϕ extends to $\{u, v\}$, unless there exists $a \in \mathbb{Z}_5$ such that $A = B = \{a, a+1\}$ or the two sets are $\{a\}$ and $\{a-1, a, a+1\}$. If $|A| + |B| \geq 5$ with $|B| \geq |A|$, then extension to $\{u, v\}$ allows two choices of the color at v unless B consists of three colors and A consists of two consecutive colors contained in B.*

Proof. When a vertex receives color a, a neighboring vertex is colorable if and only if its ϕ-set contains $a+2$ or $a-2$. When $|A| + |B| \geq 4$, only the two configurations described fail.

When $|A| + |B| \geq 5$, there are two choices for the extension at v unless for each element of A at most one element of B differs from it by 2, and always the same element of B if any. There are only two configurations for three colors (in B), and in each case this condition forces A to consist of two consecutive elements in B. A third element of A leads to a second option for the extension to v. □

A k-vertex is *full* if every incident maximal thread has at least one internal vertex.

Lemma 2.4 *An obstruction has no cycle that consists of 2-vertices and full 3-vertices.*

Proof. Let G be an obstruction having such a cycle, and let H be such a cycle in G having the fewest 3-vertices. Each 3-vertex on H has exactly one incident edge not in H, and its neighbor along that edge is a 2-vertex and hence not on H. Let $v_1, \ldots, v_k$ be the 3-vertices of H in order around H, indexed cyclically. Let w_i be the neighbor of v_i that is not on H, and let s_i be the neighbor of w_i other than v_i.

By Remark 2.1 and the minimality of H, the vertices $w_1, \dots, w_k$ are distinct and nonadjacent, with one possible exception. The exception is when H is a 5-cycle with two 3-vertices, and w_1w_2 is an edge. In this case, the entire graph G consist of a 6-cycle plus one additional vertex adjacent to two opposite vertices on the 6-cycle; this graph has a $(5, 2)$-coloring. We may therefore assume that $w_1, \dots, w_k$ are distinct and nonadjacent, and each s_i is neither on H nor in $w_1, \dots, w_k$.

Let ϕ be a $(5, 2)$-coloring of $G - V(H) - \{w_1, \dots, w_k\}$. We complete a contradiction by extending ϕ to G. Let $a_i = \phi(s_i)$; note that $\hat{\phi}(v_i) = \{a_i - 1, a_i, a_i + 1\}$. For each i, we will choose a color $f(v_i)$ from $\hat{\phi}(v_i)$; every such partial $(5, 2)$-coloring extends to w_i. It suffices to choose $f(v_1), \dots, f(v_k)$ so that the coloring extends to all of H.

By Lemma 2.2 and the fullness of v_i, the thread Q_i from v_i to v_{i+1} along H has one or two 2-vertices. Any choice of $f(v_i)$ permits a set S of at least three *consecutive* colors at v_{i+1} in extending the coloring along Q_i. Since we have only five colors, S intersects the set $\{a_{i+1} - 1, a_{i+1} + 1\}$ contained in $\hat{\phi}(v_{i+1})$. Hence we may choose $f(v_{i+1}) \in \{a_{i+1} - 1, a_{i+1} + 1\}$ and continue. The aim is to choose $f(v_1)$ so that when this procedure reaches $f(v_k)$, the coloring will also extend along Q_k and be compatible with $f(v_1)$.

Case 1: Some Q_i is a 2-thread. Shift the indices so that this thread ends at v_1. If $a_1 \neq a_k$, then choose $f(v_1) \in \hat{\phi}(v_1) - \hat{\phi}(v_k)$; if $a_1 = a_k$, then choose $f(v_1) = a_1$. Following the plan above, we reach v_k with $f(v_k) \in \{a_k - 1, a_k + 1\}$. We have arranged that $f(v_k) \neq f(v_1)$. By Lemma 2.2, the coloring therefore extends also along Q_k.

Case 2: Every Q_i is a 1-thread. In this case, arranging for $f(v_k)$ to differ by at most 1 from $f(v_1)$ will permit the extension along Q_k to complete the proof. If there exists i such that $|a_i - a_{i+1}| \leq 1$, then shift the indices down by i. Now $a_k \in \hat{\phi}(v_1)$; let $f(v_1) = a_k$. When we reach v_k and choose $f(v_k) \in \{a_k - 1, a_k + 1\}$, the color on v_k will differ by 1 from $f(v_1)$.

We may therefore assume that a_i and a_{i+1} always differ by 2. If they alternate between a and $a + 2$, then we simply set $f(v_i) = a + 1$ for all i. In the remaining case, we may assume by symmetry that $a_{k-1} = a$, that $a_k = a + 2$, and that $a_1 = a + 4$. Let $f(v_1) = a + 3$. When the process reaches v_{k-1}, it chooses $f(v_{k-1}) \in \{a - 1, a + 1\}$. If $f(v_{k-1}) = a + 1$, then set $f(v_k) = a + 2$. If $f(v_{k-1}) = a - 1$, then set $f(v_k) = a + 3$. Since $|(a + 3) - (a - 1)| = 1$, the coloring extends along both Q_{k-1} and Q_k. □

Definition 2.5 A *y, z-path* is a path with endpoints y and z. We write $\langle u_0, \dots, u_k\rangle$ to denote a path with vertices $u_0, \dots, u_k$ in order. An *alternating path* is a path whose internal vertices alternate between 2-vertices $x_0, \dots, x_k$ and full 3-vertices $v_1, \dots, v_k$ (the degrees of the endpoints are unspecified). A *transmitting path* (from y to z) is an alternating y, z-path P whose internal 3-vertices $v_1, \dots, v_k$ have distinct nonadjacent neighbors outside P. These neighbors are 2-vertices $w_1, \dots, w_k$. Let $U(P) = V(P) \cup \{w_1, \dots, w_k\}$. Let s_i denote the neighbor of w_i other than v_i. We call $s_1, \dots, s_k$ the *side vertices* of P.

For a shortest alternating path, $k = 0$, and such a path has only three vertices: y, x_0, z. It has no side vertices, but y and z both have a neighbor of degree 2 on the path. Note that in general the side vertices need not be distinct.

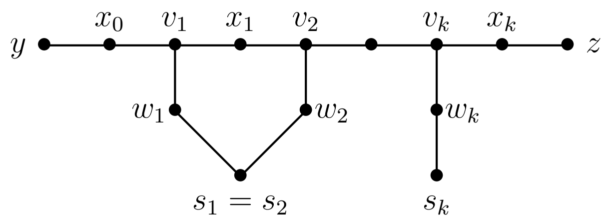

Figure 2: A typical transmitting path.

Lemma 2.6 *In an obstruction, every alternating path is a transmitting path.*

Proof. Let P be an alternating path, with notation as in Definition 2.5. If w_i and w_j are equal or adjacent, then the v_i, v_j-path along P together with the v_i, v_j-path through $\{w_i, w_j\}$ forms a cycle forbidden by Lemma 2.4. □

Definition 2.7 Let P be a transmitting path. A coloring ϕ is *partial for P* if it is a partial $(5, 2)$-coloring of $G - U(P)$ and colors the side vertices of P but not the neighbors of z outside P. Extending ϕ *along* P means also coloring $U(P)$ to form a $(5, 2)$-coloring of a larger subgraph. We allow neighbors of the initial vertex y to be colored by ϕ, but we assume that neighbors of the final vertex z are not colored, in order to discuss the restrictions imposed along P.

Lemma 2.8 *Let P be a transmitting y, z-path in G, and let ϕ be partial for P. If $|\hat{\phi}(y)| = t \le 3$, then ϕ extends along P with at least $t + 2$ choices for the color at z. Furthermore, if $t = 1$, then the extension allows three consecutive choices for the color at z.*

Proof. In the notation of Definition 2.5, extension from s_i to v_i yields $|\hat{\phi}(v_i)| = 3$, for $1 \le i \le k$, and for any choice among these three colors the intervening vertex w_i is colorable.

If $k = 0$, then P is a 1-thread, and Lemma 2.2 applies. For $k > 0$, extension along P from y to v_1 allows at least $t + 2$ choices at v_1, forbidding at most $3 - t$. Deleting $3 - t$ from $\hat{\phi}(v_1)$ leaves at least t choices for the color of v_1 in extending along P to v_1.

Applying this argument k times leaves at least t colors available at v_k before extending along the thread from v_k to z. Two more steps to z leave $t + 2$ colors available there. For the final statement, extension along a thread from a vertex with a fixed color always leaves a consecutive set of colors available at each vertex of the thread. □

Definition 2.9 Often we specify one endpoint of a transmitting path P as its *origin* (y) and the other as its *target* (z). When the origin is a 2-vertex, we call its neighbor outside P the *anchor* of P. Two transmitting paths are *independent* if no side vertex of either is an internal vertex of the other.

Lemma 2.10 *Given two independent transmitting paths with a common target z and distinct origins of degree 2, a coloring ϕ that is partial for both and colors their anchors extends along both with at most two colors forbidden at z.*

Proof. We apply Lemma 2.8 to both paths. A vertex that is a side vertex for both causes no problem; it plays the same role with fixed color for both. The specification of independence ensures that extending along one path has no effect on extending along the other. Since the anchors are colored, the ϕ-sets of the origins have size 2. Using Lemma 2.8 with $t = 2$, each extension forbids at most one color at z. □

We call a vertex of degree k a $(j_1, \ldots, j_k)$-*vertex* when the maximal threads starting at v have $j_1, \ldots, j_k$ internal vertices, respectively. When we study obstructions, Lemma 2.2 forbids j-threads with $j \geq 3$, and all j_i lie in $\{0, 1, 2\}$.

Lemma 2.11 *In an obstruction G,*

(a) there is no $(2,2,1)$-vertex and no $(2,2,2)$-vertex.

(b) there is no alternating path P whose endpoints are $(2,1,1)$-vertices and whose internal 3-vertices are $(1,1,1)$-vertices.

Proof. If any such structure occurs, let z be the $(2,2,1)$-vertex or $(2,2,2)$-vertex or the $(2,1,1)$-vertex at an end of P. Along each 2-thread at z is a transmitting path starting with a 2-vertex and ending at z. In case (a), call two such paths Q and Q', with origins y and y'.

In case (b), there is only one 2-thread ending at z, and it is not along P. Let u be the other end of P. Since u is a $(2,1,1)$-vertex, adding two vertices from the 2-thread at u creates a longer transmitting path Q to z starting at a 2-vertex y (see Figure 3). By Lemma 2.4, $y \notin N(z)$, and the threads at z not along P do not reach $V(P)$. Let Q' be the transmitting path to z that lies along the 2-thread ending at z, with origin y'.

In both cases, Q and Q' are independent. Let w be the neighbor of z not on $Q \cup Q'$, and let ϕ be a $(5,2)$-coloring of $G - (U(Q) \cup U(Q') \cup \{w\})$. Since $|\hat{\phi}(y)| = |\hat{\phi}(y')| = 2$, Lemma 2.10 allows extension along either Q or Q' with at most one color forbidden at z. Since w is an uncolored 2-vertex, $|\hat{\phi}(w)| = 2$, and extension along this thread forbids at most two more colors at z. Hence a color remains available for z in extending ϕ to G. □

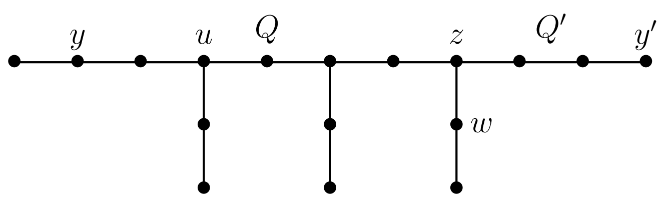

Figure 3: Part b of Lemma 2.11.

Lemma 2.11a implies that every full 3-vertex is a $(2, 1, 1)$-vertex or a $(1, 1, 1)$-vertex; Lemma 2.11b will be useful in the discharging argument.

The discharging argument will need further results about coloring extension for delicate cases where two transmitting paths have the same initial edge. First we extend Lemma 2.8.

Definition 2.12 A path that differs from an alternating path only by subdividing some edges, thereby adding r extra internal 2-vertices, is an *r-augmented alternating path.* If the outside neighbors of its internal 3-vertices are distinct and nonadjacent, then it is an *r-augmented transmitting path.*

Lemma 2.13 *In an obstruction, every r-augmented alternating y, z-path P is an r-augmented transmitting path. If ϕ is partial for P and $|\hat{\phi}(y)| = t$, then ϕ extends along P allowing $\min\{5, t + 2 + r\}$ colors at z.*

Proof. Lemma 2.4 applies as in Lemma 2.6 for the first statement. The argument of Lemma 2.8 also applies to extension along an r-augmented transmitting path, gaining an extra choice whenever an extra 2-vertex is encountered. □

Lemma 2.14 *In an obstruction G, let Q be an alternating y, z-path and Q' be an alternating y', z'-path, such that yy' is the first edge of each. Let ϕ be partial for both Q and Q'.*

(a) If $z \neq z'$ and $|\hat{\phi}(z)| = t$, where $t \in \{1, 2\}$, then ϕ extends along Q (backwards) and along Q' with at most $2 - t$ colors forbidden (that is, $t + 3$ allowed) at z'.

(b) If $zz' \in E(G)$ and $|\hat{\phi}(z)| \geq 3$ and $|\hat{\phi}(z')| \geq 4$, then ϕ extends along Q and Q'.

(c) If $z = z'$, then ϕ extends along Q and Q' with at most two colors forbidden at z.

Proof. In the notation of Definition 2.5, the internal 3-vertices of Q and Q' are $v_1, \ldots, v_k$ and $v'_1, \ldots, v'_{k'}$, respectively, with x_0 and x'_0 preceding v_1 and v'_1. We have $y = x'_0$ and $y' = x_0$, so $d(y) = d(y') = 2$, and $\langle v_1, y', y, v'_1 \rangle$ is a 2-thread. See Figure 4.

(a): The targets are distinct. Since y and y' are 2-vertices, $Q \cup Q'$ is a 1-augmented alternating z, z'-path. Since G is an obstruction, Lemma 2.13 applies to complete the desired extension (ignoring the influence of a possible edge zz'.)

(b): Again $Q \cup Q'$ is a 1-augmented transmitting path. Each choice of a color from $\hat{\phi}(z)$ allows two consecutive colors at z', and these pairs are distinct for distinct colors in $\hat{\phi}(z)$. Since $|\hat{\phi}(z')| \geq 4$, there are only two pairs of consecutive colors that are not contained in $\hat{\phi}(z')$. Hence we may choose $b \in \hat{\phi}(z)$ to color z leaving two colors available at z'. By Lemma 2.13, with b at z we can extend along $Q \cup Q'$ so that only one color is forbidden at z'. Hence we can extend simultaneously along $Q \cup Q'$ and along the edge zz'.

(c): The targets are equal, and the neighbor(s) of z outside $Q \cup Q'$ need not have degree 2. Nevertheless, "splitting" z and z' would turn $Q \cup Q'$ into a 1-augmented transmitting path. Thus extensions along Q and Q' do not conflict before reaching z, if they are compatible on yy'. We seek simultaneous extensions that agree at z, with at most two colors forbidden.

Case 1: Q or Q' is a 1-thread. By symmetry, we may assume that $Q' = \langle y', y, z' \rangle$ and $k' = 0$. Since G is triangle-free, $k \geq 1$. We have $|\hat{\phi}(v_i)| = 3$ for all i.

Let $\hat{Q}$ denote the v_1, z-path along Q; note that $\hat{Q}$ is also a transmitting path. For any $a \in \hat{\phi}(v_1)$ chosen as a color on v_1, Lemma 2.8 guarantees extension along $\hat{Q}$ allowing a set $S(a)$ of three consecutive colors at z. An extension along Q that assigns z the color $b \in S(a)$ extends to all of $Q \cup Q'$ if $b \neq a$, since Q' is a z, v_1-path of length 3. The proof is complete if $a \notin S(a)$. If $a \in S(a)$, then the values in $S(a) - \{a\}$ are allowed at z.

Since $|\hat{\phi}(v_1)| = 3$, we may choose $a, a' \in \hat{\phi}(v_1)$ with $|a - a'| = 1$. If $a \in S(a')$, then a becomes allowed in extension to z (using a' at v_1). We may thus assume that $a \notin S(a')$, and similarly $a' \notin S(a)$. Since each set consists of three consecutive colors, when $|a - a'| = 1$ we conclude that all of $\mathbb{Z}_5 - \{a, a'\}$ is allowed at z by extensions along $Q \cup Q'$.

Case 2: Neither Q nor Q' is a 1-thread. Now $k \geq 1$ and $k' \geq 1$. If $\hat{\phi}(v_1) \neq \hat{\phi}(v_1')$, then since each has size 3 we can choose disjoint pairs A and A' with $A \subseteq \hat{\phi}(v_1)$ and $A' \subseteq \hat{\phi}(v_1')$. The v_1, z-path $\hat{Q}$ along Q and the v_1', z-path $\hat{Q}'$ along Q' are transmitting paths; by Lemma 2.8, extension along $\hat{Q}$ or $\hat{Q}'$ with A or A' available at the initial vertex forbids at most one color each at z. Hence three colors can be chosen for z consistent with both some color from A at v_1 and some color from A' at v_1'. Since $A \cap A' = \varnothing$, each such choice extends along $\langle v_1, y', y, v_1' \rangle$.

Hence we may assume that $\hat{\phi}(v_1) = \hat{\phi}(v_1') = \{a-1, a, a+1\} = S$. If $k = k' = 1$, then $\hat{Q}$ and $\hat{Q}'$ are 1-threads and $Q \cup Q'$ is a 7-cycle. Given $b \in S$, there exists $b' \in S$ with $|b - b'| = 1$. We assign b to z and to v_1 and assign b' to v'; now the coloring extends along each thread.

Otherwise, we may assume by symmetry that $k \geq 2$. Since $|\hat{\phi}(v_1)| = |\hat{\phi}(v_2)| = 3$, we may choose $b \in \hat{\phi}(v_2) \cap \hat{\phi}(v_1)$. Assign color b to v_1, and delete b from $\hat{\phi}(v_1')$. On v_2 we can use b or a color in $\hat{\phi}(v_2)$ that differs from b by 1 (since $\hat{\phi}(v_2)$ consists of three consecutive colors). Thus two colors are allowed in extensions from v_1 to each of v_2 and v_1'. The paths from these vertices to z along Q and Q' are transmitting paths. By Lemma 2.8, ϕ extends along each, with each extension forbidding at most one color at z. □

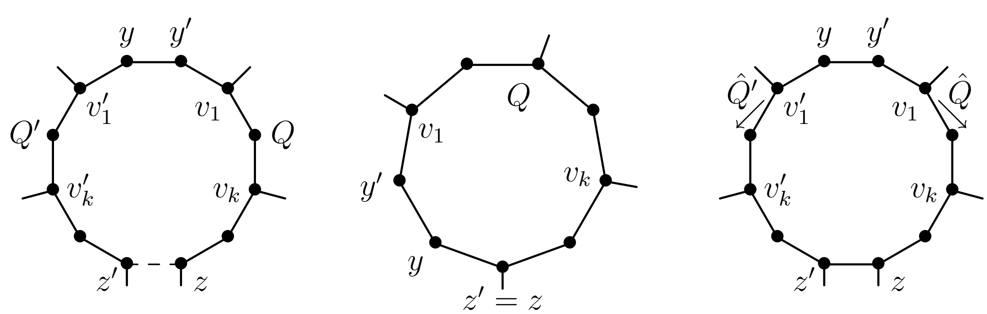

Figure 4: Parts a/b, c(i), and c(ii) of Lemma 2.14.

3 Initial Discharging: Feeding Paths

Assume the existence of an obstruction G; we obtain a contradiction by using discharging. There are two ways to phrase the argument. We can give each vertex initial charge equal to its degree and move charge around to reach a situation where each charge is at least $12/5$. Instead, we give each vertex v initial charge $\mu(v)$ equal to $5d(v) - 12$. Now $\text{mad}(G) < 12/5$ implies that the total charge is negative. Discharging to make each vertex "happy" by reaching nonnegative charge produces the contradiction. The two phrasings are equivalent; the second leaves the computations in integers and simplifies some language.

With $\mu(v) = 5d(v) - 12$, the initial charges are -2 for 2-vertices, 3 for 3-vertices, 8 for 4-vertices, etc. Only the 2-vertices are unhappy, and we must pull charge from the other vertices to make them happy. For this we introduce discharging rules.

Rule 1: Each 2-vertex in a 2-thread pulls charge 2 from its neighbor of degree exceeding 2. Each 2-vertex not in a 2-thread pulls charge 1 from each neighbor.

Since an obstruction has no 3-thread, each 2-vertex in a 2-thread has a neighbor of degree exceeding 2. Each 2-vertex not in a 2-thread is in a 1-thread and has two such neighbors. Thus application of Rule 1 increases the charge of each 2-vertex by 2 without pulling charge from any 2-vertex. Hence all the 2-vertices are now happy.

Meanwhile, we have pulled charge from every vertex of degree at least 3 that has a neighbor of degree 2. The charge that we have pulled from a $(j_1, \ldots, j_k)$-vertex is exactly $\sum_{i=1}^{k} j_i$, since all $j_i \in \{0, 1, 2\}$. Since $\mu(v) = 5d(v) - 12$, the new charge is at least $3d(v) - 12$, which is nonnegative for $d(v) \geq 4$. We need only worry about 3-vertices that have lost charge at least 4. By Lemma 2.11a, no 3-vertex loses charge more than 4, and the only ones that have lost 4 (and remain unhappy) are the $(2, 1, 1)$-vertices and the $(2, 2, 0)$-vertices.

For each $(2, 1, 1)$-vertex v, we construct a path along which v will receive charge 1.

Definition 3.1 A *feeding path* for a $(2, 1, 1)$-vertex v is chosen as any shortest path F formed by concatenating threads in the following way. First, F starts along either 1-thread at v. After F has traversed some number of thread, let v' be the last vertex reached. If v' is a $(1, 1, 1)$-vertex, then F continues along one of the other 1-threads incident to v'; otherwise, F ends at v'. The *sponsor* of a $(2, 1, 1)$-vertex v is the other endpoint of its feeding path; v is the *feeding vertex* of the path. The *anchor* of the feeding path is the other endpoint of the 2-thread incident to the feeding vertex.

A feeding path is a union of maximal threads that are 1-threads. Associated with each feeding path is a transmitting path obtained by adding the two 2-vertices on the 2-thread incident to the feeding vertex (Lemma 2.6 makes it a transmitting path, since it is an alternating path in an obstruction). We refer to the side vertices of the associated transmitting path as the side vertices of the feeding path. We view the sponsor of the feeding vertex as the target of the associated transmitting path. The anchor of the feeding path is what we called the anchor of the associated transmitting path, since its origin has degree 2. Coloring the anchor allows two possible colors on the origin. With this association, our results on transmitting paths apply to the discussion of feeding paths.

Lemma 3.2 *In an obstruction G, every $(2,1,1)$-vertex starts one feeding path. The target (sponsor) of a feeding path cannot be one of its side vertices. The sponsor of a feeding path is not a $(2,1,1)$-vertex. Any two feeding paths are independent.*

Proof. From a $(2,1,1)$-vertex, a path grown by Definition 3.1 must terminate, since if it repeats a vertex it creates a cycle forbidden by Lemma 2.4. Hence each $(2,1,1)$-vertex chooses one feeding path. Minimality in the construction of feeding paths (and the absence of 4-cycles) prevents the sponsor from being a side vertex; the path would be shortened.

If a side vertex or internal vertex of one feeding paths lies on another, then G has a path joining $(2,1,1)$-vertices that is forbidden by Lemma 2.11b. Hence two feeding paths are independent. □

Since feeding paths do not intersect internally, a $(j_1,\ldots,j_k)$-vertex is the sponsor for at most l vertices, where l is the number of 1s in $(j_1,\ldots,j_k)$.

Rule 2: Every $(2,1,1)$-vertex pulls charge 1 from its sponsor.

Lemma 3.3 *After applying Rules 1 and 2, the charge of every $(2,1,1)$-vertex vanishes, the charge of every $(2,2,0)$-vertex is -1, and every other vertex has nonnegative charge except for $(2,1,0)$-vertices that are sponsors and $(1,1,0)$-vertices that are sponsors for two vertices.*

Proof. Let v be a vertex of degree at least 3. Using Rules 1 and 2, each incident maximal thread pulls from v at most two units of charge, with equality only when it is a 2-thread or is a 1-thread that ends a feeding path. As before, $5d(v)-12 \geq 2d(v)$ when $d(v) \geq 4$.

If $d(v)=3$, then v has initial charge 3. Since $(1,1,1)$-vertices lose exactly 3, and $(2,1,1)$-vertices have lost 4 and regained 1, these vertices now have charge 0. For any other 3-vertex (by Lemma 2.11a) some incident maximal thread is a 0-thread.

Hence a 3-vertex now has negative charge only if among its incident maximal threads, one is a 0-thead and the other two each take away two units of charge, being 2-threads or being 1-threads that lie on feeding paths. □

Definition 3.4 An *overloaded vertex* is a 3-vertex that has negative charge after Rules 1 and 2. A *loaded path* is a transmitting path that (1) is associated with a feeding path, or (2) is contained in a 2-thread and has as its target a 3-vertex that is not a feeding vertex.

A transmitting path contained in a 2-thread consists of three consecutive vertices in the thread; its end that is not a 2-vertex is its target. We do not count it as a loaded path if its target is fed by a feeding path, because then its target does not have negative charge after Rules 1 and 2. An overloaded vertex is a 3-vertex that is the target of two loaded paths. Each loaded path takes two units of charge from its target.

Lemma 3.5 *Two loaded paths share no edges and no internal vertices, except that they may have the same initial edge. No side vertex of one lies on another. An overloaded vertex u cannot be a side vertex for a loaded path.*

Proof. If a loaded path lies in a 2-thread, then its target is not a feeding vertex; hence only the initial edge can be shared. Such a loaded path has no side vertex.

By Lemma 3.2, feeding paths are disjoint, and no side vertex of one is internal to another. They may share side vertices. The anchor of a feeding path cannot be internal to another feeding path; it has an incident 2-thread and cannot be a $(1,1,1)$-vertex or a 2-vertex.

Let u be an overloaded vertex. By Lemma 3.3, each maximal 1-thread incident to u belongs to a feeding path with target u. If u were a side vertex for a loaded path, then the vertex at the other end of the relevant 1-thread would lie in two feeding paths. □

Lemma 3.5 states that the transmitting paths that are loaded paths are independent, except when they share initial edges. We say that two loaded paths with the same initial edge *splice*. Always the shared edge is the central edge of a 2-thread, and the number of endpoints of that 2-thread that are feeding vertices may be 0, 1, or 2. The union of loaded transmitting paths that splice is a longer path or a cycle, depending on whether their targets are the same. When two loaded paths splice, we will apply Lemma 2.14.

Lemma 3.6 *In an obstruction G, overloaded vertices are nonadjacent.*

Proof. Let u and w be adjacent overloaded vertices. Let ϕ be partial for all the loaded paths to u and w.

By Lemma 3.5, the two loaded paths with target u are independent unless they splice. In either case, by Lemma 2.10 or Lemma 2.14c, ϕ extends along both to u with at most two colors forbidden at u. The same holds for w. Now Lemma 2.3 completes the proof if these extensions to u and w can be accomplished independently (see Figure 5). They can unless some loaded path U having target u splices with some loaded path W having target w. Let U' and W' be the other loaded paths to u and w.

If U' and W' are not also spliced, then extensions along U' and W' forbid at most one color each at u and w, by Lemma 2.8. Extending along U' and W' forbids at most one color each at u and w. Now Lemma 2.14b applies to complete the extension along $U \cup W$, satisfying also the edge uw.

If U' and W' are spliced, then we start at u. Let v be the first 3-vertex reached when following $U \cup W$ from u. Note that $\hat{\phi}(v)$ consists of three consecutive colors. We choose b at

u to avoid reducing $\hat{\phi}(v)$. If v has distance 3 from u, then we choose b outside $\hat{\phi}(v)$. If v has distance 2 from u, then we let b be the central color in $\hat{\phi}(v)$. Now the v,w-path along $U \cup W$ is a transmitting path or a 1-augmented transmitting path, and Lemma 2.8 or Lemma 2.13 guarantees extension along it with no color forbidden at w.

With such a color b at u, Lemma 2.14a allows us to extend along $U' \cup W'$ forbidding at most one color at w. Also the edge uw itself forbids three colors at w. Hence a color is available at w to complete simultaneous extension along the three paths. □

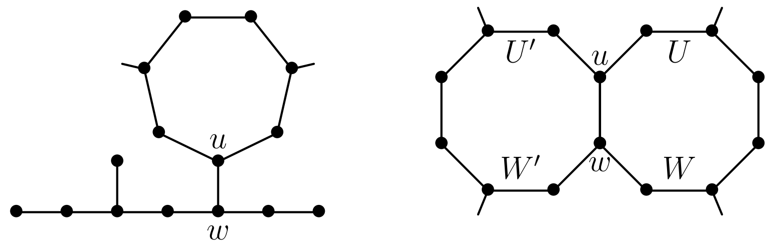

Figure 5: Sample cases for Lemma 3.6.

4 Final discharging

At this point, only the overloaded vertices are unhappy. Our final discharging rule makes them happy. By Lemma 3.3, every overloaded vertex has one neighbor of degree at least 3.

Rule 3: Every overloaded vertex pulls charge 1 from its neighbor of degree at least 3.

Lemma 3.6 ensures that no overloaded vertices are adjacent. Hence Rule 3 makes the overloaded vertices happy. We must check that the vertex from which an overloaded vertex pulls charge does not become unhappy. Before proving this, we will need one more lemma.

Lemma 4.1 *An obstruction G cannot have three transmitting paths whose targets are distinct vertices on a 5-cycle H, with the other vertices of H having degree* 2.

Proof. Let ϕ be a coloring that is partial for each of the three paths and leaves $V(H)$ uncolored. If no two of the paths splice, then extension along each forbids at most one color at its target vertex, by Lemma 2.8. In this case it suffices to show that a 5-cycle H is $(5,2)$-colorable when three of the vertices have four available colors and the others have five. Those with five may be adjacent or not, yielding two cases. In each case, the vertices with five available colors are the internal vertices of threads along which we apply Lemma 2.2. We must choose colors at the other vertices that permit extension along the threads.

Case $(4,4,4,5,5)$. Let $\langle u,v,w\rangle$ be the path of vertices with four available colors. We want to give u and w distinct colors to allow extension along the 2-thread joining them. For any color a chosen at v, we can extend to distinct colors on u and w unless one of them is missing both $a+2$ and $a-2$, or both of them are missing one of those two colors. Since both u and w lack only one color, we may therefore assume that both are missing the same color b. Now since v has more than two available colors, a color outside $\{b-2, b+2\}$ is available at v; it can serve as the color a.

Case $(4,4,5,4,5)$. Let u and v be the adjacent vertices with four available colors, and let x be the third such vertex. A coloring of the endpoints of a 1-thread extends to the interior if their colors differ by at most 1. Hence it suffices to give u, v, and x colors a, b, and c, respectively, such that $|a-b|=2$ and c is the unique color consecutive to both. If u and v lack the same color, then two colors differ from it by 2. One of them is available at x; let this be c, and let the colors next to c be a and b. Otherwise, let a be the color not available at v. Now we can choose $a-1$ or $a+1$ for c (whichever is available at x and choose $a-2$ or $a+2$, respectively, for b (both are available at v.

Having completed these two cases, we may henceforth assume that two of the three transmitting paths splice. Let their targets be x and y; together they form an x,y-path P. By symmetry, we may assume that 0 is the color (if any) forbidden at the third target z by extensions along its transmitting path.

Case 1: z is adjacent to x (or y, by symmetry). Give x color 1. Now Lemma 2.14a allows extension along P to y with at most one color forbidden at y. If y is not adjacent to x, then we put one of $\{0,2\}$ at y. If y is adjacent to x, then we put one of $\{3,4\}$ at y. In either case, completion of the $(5,2)$-coloring on H puts 3 or 4 at z, so all the extensions are consistent (Figure 6 shows the choices).

Case 2: z is nonadjacent on H to both x and y. Give x color 0. Extension along P allows putting one of $\{2,3\}$ at y, which is adjacent to x. Under either possibility, the further extension to H does not put 0 at z, so again all extensions are consistent. □

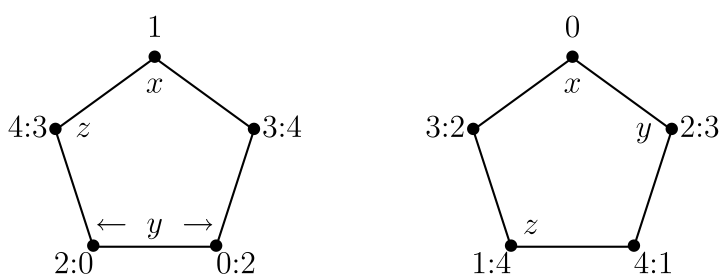

Figure 6: Cases 1 and 2 for Lemma 4.1.

Lemma 4.2 *In an obstruction, Rules 1–3 give every vertex nonnegative charge.*

Proof. By Lemma 3.6, overloaded vertices are nonadjacent, so it suffices to show that no vertex having charge 0 after Rule 2 is adjacent to an overloaded vertex.

When vertex u is adjacent to an overloaded vertex, the edge joining them is a 0-thread. Such a thread takes no charge from u via Rules 1 and 2. If u has t overloaded neighbors, then its final charge $\mu^*(u)$ satisfies $\mu^*(u) \geq 5d(u) - 12 - 2[d(u) - t] + t$. The lower bound is nonnegative if $d(u) \geq 4$. Hence we may assume that $d(u) = 3$.

With initial charge 3, u must lose four units to become negative. Charge travels along threads away from u. Each thread pulls at most two units, and no two threads can pull two units each, since u is not overloaded. Hence one thread pulls two units along a loaded path U_2, and two threads pull one unit each. Again we show that in each case some partial $(5, 2)$-coloring extends to all of G.

Let w be an overloaded neighbor of u, and let W_1 and W_2 be the loaded paths at w. Let U_1 be the remaining maximal thread at u; it pulls charge 1 from u and hence cannot be a 2-thread. Let y be the vertex at the other end of U_1. If U_1 is a 1-thread, then it is not on a feeding path, since u is not overloaded. If U_1 is a 0-thread, then $\mu^*(u) < 0$ requires y to be overloaded to make $\mu^*(u)$ negative. We consider two cases when U_1 is a 1-thread and two when U_1 is a 0-thread, sketched in Figures 7 and 8.

Recall that the loaded path U_2 is a transmitting path with target u, associated with a 2-thread or a feeding path. The origin of any loaded path P is a 2-vertex, so extension along P of a coloring that is partial for P forbids at most one color at its target, by Lemma 2.8.

If U_1 is a 1-thread, then y lying on U_2 would allow a shorter feeding path by substituting U_1 for the y, x-path in U_2; hence $y \notin V(U_2)$. Let z be the neighbor of u on U_1. Let ϕ be a coloring that is partial for U_2, W_1, and W_2 and leaves z uncolored.

Case 1: U_1 is a 1-thread and $y \notin V(W_1) \cup V(W_2)$. If U_2 does not splice with W_1 or W_2, then we extend ϕ along U_1 and U_2 forbidding at most two and one colors at u, respectively. Whether W_1 and W_2 splice or not, Lemma 2.10 or Lemma 2.14c implies that extension along W_1 and W_2 forbids at most two colors at w. Now Lemma 2.3 completes the extension.

Now suppose (by symmetry) that U_2 splices with W_2. Extension from y to u forbids two colors at u, and extension along W_1 forbids at most one color at w (Lemma 2.8). Now Lemma 2.14c applies to complete the extension along $U_2 \cup W_2 \cup uw$.

Case 2: U_1 is a 1-thread and $y \in V(W_1) \cup V(W_2)$. By symmetry, assume $y \in V(W_1)$. Substituting $\langle y, z, u \rangle$ for the y, w-path on W_1 yields another possible feeding path for the anchor of W_1; minimality thus implies that W_1 ends with a 1-thread $\langle y, x, w \rangle$. Now u, w, x, y, z in order induce a 5-cycle.

Since W_1 is a loaded path that ends with $\langle y, x, w \rangle$ and does not lie in a 2-thread, $W_1 - \{w, x\}$ is a transmitting path with target y. Now U_2, W_2, and $W_1 - \{w, x\}$ are transmitting paths to distinct vertices on a 5-cycle, which is forbidden by Lemma 4.1.

Case 3: U_1 is a 0-thread, no splicing except W_1 with W_2 and/or Y_1 with Y_2. Since u loses four units to become overloaded, y is an overloaded vertex. Let Y_1 and Y_2 be the loaded paths at y. Let ϕ be partial for all of U_2, W_1, W_2, Y_1, and Y_2. (Dashed lines in Figures 7 and 8 indicate optional splicing.)

If these paths are all independent, then by Lemma 2.8 extensions along the paths exist with at most one color forbidden at u, two at w, and two at y. If W_1 splices with W_2 and/or Y_1 with Y_2, then by Lemma 2.14 such extensions still exist. With three colors allowed at both w and y, we can achieve a common color for them. It may forbid three colors at u, but this leaves an allowed color at u to complete the extensions simultaneously.

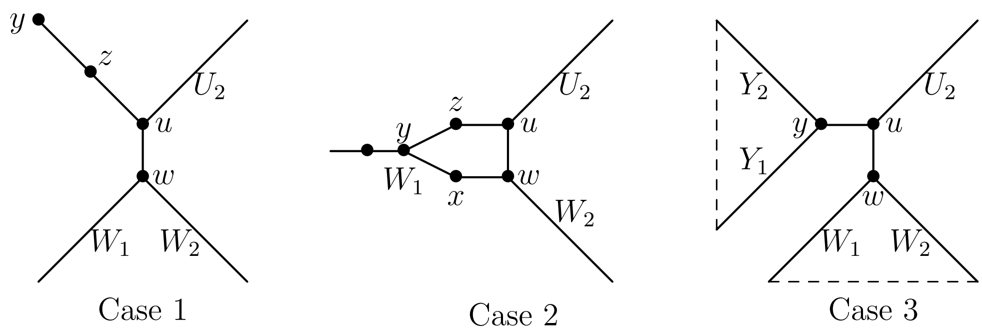

Figure 7: Initial cases for Lemma 4.2.

Case 4: U_1 is a 0-thread and nontrivial splicing occurs. Again we try to put the same color on w and y, but this does not always work. If U_2 splices with W_2, and W_1 is unspliced, then we extend along W_1 and along $Y_1 \cup Y_2$ (spliced or not) to put the same fixed color on w and y. Two colors remain available for u. By Lemma 2.14a, we can extend from w along $W_2 \cup U_2$ to put one of those two colors on u (see Figure 8).

In the remaining cases, at least one of $\{W_1, W_2\}$ splices with one of $\{Y_1, Y_2\}$; we may assume that W_1 splices with Y_1. The cases are that U_2 is unspliced or splices with W_2 (by symmetry in y and w, since they are both overloaded).

If U_2 is unspliced, then W_2 may or may not splice with Y_2. We start with a color a at w. There are at least four choices for a if W_2 is unspliced and we extend ϕ along W_2 to reach w; if W_2 splices with Y_2, then we choose any fixed a.

Given a, we extend along $W_1 \cup Y_1$ with at most one color forbidden at y. Extending along Y_2 (or along $W_2 \cup Y_2$ if W_2 splices with Y_2) also forbids at most one color at y. Extending along U_2 forbids at most one color at u. If color a is not the forbidden color at y, then we perform the extensions with a at y and one of $\{a+2, a-2\}$ at u.

There are (at least) four choices for a. If any can be used at both w and y, then we are finished. Otherise, for each choice of a we can perform the extensions to put one of $\{a-1, a+1\}$ at y. With four choices for a, there are two choices that differ by 2. Pairing each of these with a color consecutive to it yields two distinct pairs of consecutive colors that can be used on w and y. Each of them is compatible with one color at u. Since U_2 is unspliced, the color that may be forbidden at u by extension along U_2 does not depend on a. Thus we can extend along U_2 to put a color at u that is compatible with a pair of colors that can be placed at $\{w, y\}$.

Finally, suppose that U_2 splices with W_2. Again we start at w, but this time we want to choose a color b at w so that extension along $W_2 \cup U_2$ imposes no restriction on the color at u. The trick to do this was used at the end of the proof of Lemma 3.6; we repeat the argument with the present notation.

Let v be the first 3-vertex reached when following $W_2 \cup U_2$ from w. By Remark 2.1, $v \neq u$. Note that $\hat{\phi}(v)$ consists of three consecutive colors. If v has distance 3 from w, then choose b outside $\hat{\phi}(v)$. If v has distance 2 from w, then let b be the central color in $\hat{\phi}(v)$. Now the v, u-path along $W_2 \cup U_2$ is a transmitting path or a 1-augmented transmitting path, and Lemma 2.8 or Lemma 2.13 guarantees extension along it with no color forbidden at w. Also the coloring extends along the w, v-thread.

By Lemma 2.14a, extension along $W_1 \cup Y_1$ forbids at most one color at y. By Lemma 2.8, extension along Y_2 also forbids at most one color at y. Hence we can complete these extensions simultaneously so that the color at y is in $\{b-1, b, b+1\}$. Now a color differing by 2 from the colors at w and y can be chosen for u, and the coloring can extend along $W_2 \cup U_2$ to be consistent with that. □

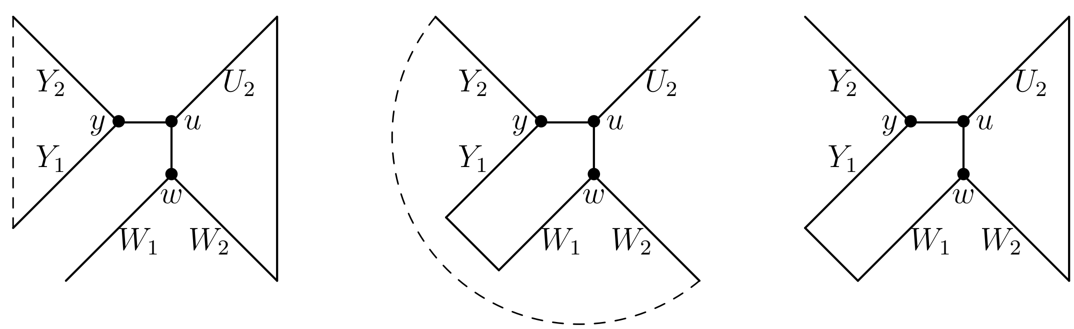

Figure 8: Case 4 of Lemma 4.2.

Lemma 4.2 shows that no obstruction is possible, completing the proof of Theorem 1.1.

Acknowledgment

The authors thank Aleksey Glebov for useful remarks on the proof.

References

[1] J. A. Bondy and P. Hell, A note on the star chromatic number. *J. Graph Theory* 14 (1990), no. 4, 479–482.

[2] O. V. Borodin, S.-J. Kim, A. V. Kostochka, and D. B. West, Homomorphisms from sparse graphs with large girth. *J. Combin. Theory (B)* 90 (2004), 147–159.

[3] M. DeVos, communication at Workshop on Flows and Cycles, Simon Fraser Univ., June 2000.

[4] P. Erdős and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 12 (1963), 251–257.

[5] A. Galuccio, L. Goddyn, and P. Hell, High girth graphs avoiding a minor are nearly bipartite. *J. Combin. Theory* (B) 83 (2001), 1–14.

[6] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 8 (1959), 109–120.

[7] F. Jaeger, On circular flows in graphs, in: *Finite and Infinite Sets (Eger, 1981). Colloq. Math. Soc. J. Bolyai* 37 (1984), North–Holland, 391–402.

[8] T. R. Jensen and B. Toft, *Graph coloring problems*. John Wiley & Sons, New York, 1995.

[9] W. Klostermeyer and C.-Q. Zhang, $(2+\epsilon)$-coloring of planar graphs with large odd-girth. *J. Graph Theory* 33 (2000), 109–119.

[10] J. Nešetřil and X. Zhu, On bounded tree-width duality of graphs. *J. Graph Theory* 23 (1996), 151–162.

[11] W. T. Tutte, A contribution to the theory of chromatic polynomials. *Canad. J. Math.* 6 (1954), 80–91.

[12] A. Vince, Star chromatic number. *J. Graph Theory* 12 (1988), 551–559.

[13] X. Zhu, Circular chromatic number: a survey. Combinatorics, graph theory, algorithms and applications. *Discrete Math.* 229 (2001), 371–410.

[14] X. Zhu, Circular chromatic number of planar graphs of large odd girth. *Electronic J. Combin.* 8 (2001), Article #R25.