Beyond Ohba’s Conjecture: A bound on the choice number of $k$-chromatic graphs with $n$ vertices

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slides available on DBW preprint page

Joint work with
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Def. A list assignment $L$ assigns each $v \in V(G)$ a list $L(v)$ of available colors.
List Coloring and Choosability

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- Since lists may be equal at all vertices, \( ch(G) \geq \chi(G) \).
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**Ex.** $\text{ch}(K_{4,2}) > 2 = \chi(K_{4,2})$. 

\[
\begin{array}{c}
\{1,2\} & \{3,4\} \\
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Let \( K_{1\ast k_1, 2\ast k_2, \ldots} \) denote the one with \( k_i \) parts of size \( i \).

**Sharpness** for Ohba’s Conjecture: When \( k \) is even, \( K_{2\ast(k-1), 4\ast1} \) and \( K_{1\ast \left(\frac{k}{2}-1\right), 3\ast \left(\frac{k}{2}+1\right)} \) are not \( k \)-choosable.
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**Sharpness** for Ohba’s Conjecture: When \( k \) is even, \( K_{2*(k-1),4*1} \) and \( K_{1*(\frac{k}{2}-1),3*(\frac{k}{2}+1)} \) are not \( k \)-choosable.

Always \( K_{2*(k-1),5*1} \) is not \( k \)-choosable (EOOS [2002]).
Beyond $2k + 2$ Vertices

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**Thm.** (Yang [2003]) \( \left\lfloor \frac{3k}{2} \right\rfloor \leq \text{ch}(K_{4 \times k}) \leq \left\lceil \frac{7k}{4} \right\rceil \). 
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**Thm.** (N–W–W–Z [2013+]) If \( G \) has \( n \) vertices and chromatic number \( k \), then \( \text{ch}(G) \leq \max \{ k, \left\lceil \frac{n+k-1}{3} \right\rceil \} \).

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Our result improves this to \( \text{ch}(K_{4*k}) \leq \left\lceil \frac{5k-1}{3} \right\rceil \).

Also, \( \left\lfloor \frac{8k}{5} \right\rfloor \leq \text{ch}(K_{5*k}) \leq 2k \) and \( \left\lfloor \frac{5k}{3} \right\rfloor \leq \text{ch}(K_{6*k}) \leq \left\lceil \frac{7k-1}{3} \right\rceil \).
Lower Bound Constructions for $\text{ch}(K_{m^*k})$

**Constr 1:** Split $2k - 1$ colors into $X_1, \ldots, X_m$. Assign all but $X_i$ to the $i$th vertex in each part. $L$-coloring uses at least two colors on each part, in disjoint pairs. Hence it uses $2k$ colors, but only $2k - 1$ exist. The list sizes are at least $\left\lceil \frac{m-1}{m} \cdot 2k \right\rceil$. ■
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This lower bound never exceeds $2k$, and our upper bound is $\frac{m+1}{3} k$; both are weak for large $m$. 
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**Constr 2:** Let $m = \binom{k-1}{(k-1)j}$. Assign all $(k-1)j$-sets from $kj - 1$ colors as lists on each part. Any $j - 1$ colors avoid some list, so $j$ colors must be used on each part. Thus $kj$ colors needed, but only $kj - 1$ exist. The list sizes are about $c \frac{k}{\log k} \log m$. 

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**Thm.** (Alon [1992]) $\text{ch}(K_{m \times k}) = \Theta(k \log m)$. 
Lower Bound Constructions for $\text{ch}(K_{m \ast k})$

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**Thm.** (Alon [1992]) $\text{ch}(K_{m \ast k}) = \Theta(k \log m)$.

**Conj.** (Noel [2012]) $K_{m \ast k}$ has largest choice number among graphs with $\chi(G) = k$ and $n \leq mk$. 

Thm. (N–W–W–Z [2013+]) If $G$ has $n$ vertices and chromatic number $k$, then $\text{ch}(G) \leq \max \left\{ k, \left\lceil \frac{n+k-1}{3} \right\rceil \right\}$.
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Outline

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- Properties of a minimal counterexample $(G, L)$:
  1. The union of all lists has size less than $n$.
  2. All parts have size at most 4.
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- Coloring procedure:
  1. Break \( V(G) \) into stable sets of size at most 2 by splitting some parts.
  2. Produce an \( L \)-coloring whose color classes are these sets.
First Reduction

**Lem.** (Kierstead [2000], Reed–Sudakov [2002]) If $G$ is not $r$-choosable, then $G$ has no $L$-coloring for some $r$-uniform list assignment $L$ with $|\bigcup_v L(v)| < |V(G)|$. 
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If \( |L(X)| \geq |X| \ \forall \ X \subseteq V(G) \), then Hall’s Theorem yields distinct colors for vertices. Pick \( X \) maximal with \( |L(X)| < |X| \).
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Let $L'(\nu) = L(\nu)$ for $\nu \in X$ and $L'(\nu) \subseteq L(X)$ for $\nu \notin X$.

By construction, $|L'(V(G))| = |L(X)| < |X| < |V(G)|$.

Now $G$ is $L'$-colorable; restricts to $L$-coloring of $G[X]$. 

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For $Y \subseteq V(G) - X$, choice of $X$ yields $|L(X \cup Y)| \geq |X \cup Y|$.

Hence $|L(X \cup Y) - L(X)| > |X \cup Y| - |X| = |Y|$.
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Hall’s Theorem picks distinct colors for the vertices outside $X$ using colors outside $L(X)$. \hfill ∎
Reductions (for minimal c/ex \((G, L)\))

**Prop.** If \(A\) is a stable set in \(G\) having common color \(c\) in lists, then 
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\left\lfloor \frac{|V(G')| + \chi(G') - 1}{3} \right\rfloor = \left\lfloor \frac{n + k - 1}{3} \right\rfloor,
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**Pf.** Form \(L'\) on \(G'\) by deleting \(c\) from each list in \(L\). Since \(|L(v)| \geq k + 1\), we have \(|L'(v)| \geq k \geq \chi(G')\) for \(v \in V(G')\).
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Reductions (for minimal c/ex \((G, L)\))

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**Lem.** Each part $A$ in $G$ has size at most 4.

**Pf.** Since each color appears at most twice on $A$,

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Thus $|A| \leq 6 \frac{n-1}{n+k-1}$, which yields $|A| \leq 5$. If $|A| = 5$, then $n \geq 5k + 1$, which requires a part of size at least $6$. ■
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**Summary:** (properties of minimal counterexample)

**Obs.** $G$ is a complete multipartite graph.

**Thm.** (Noel–Reed–Wu) $n \geq 2k + 2$, so $|L(v)| = \left\lceil \frac{n+k-1}{3} \right\rceil$.

**Lem.** (Kierstead) $|\bigcup_{v} L(v)| < n$.

**Cor.** Parts of size 2 have disjoint lists.

**Cor.** Colors appear in at most two lists in each part.

**Lem.** Parts have size at most 4.
The Merging Idea

Since each color appears at most twice in each part, color classes in an $L$-coloring will have size at most 2.
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**(1)** Obtain conditions that are sufficient for Hall’s Theorem to guarantee the SDR.

**(2)** Define a procedure to make merges that guarantee these conditions.
A Sufficient Condition

**Def.** Let $G$ have $k_i$ parts of size $i$, for $i \in \{1, 2, 3, 4\}$. Let $A^*$ be what remains of part $A$ after the merges. Let $Z_3$ be a fixed set of $\left\lfloor \frac{2}{3}k_3 \right\rfloor$ 3-parts. Let $Z_4$ be a fixed set of $\max\{0, \frac{k_4+k_1-k_3+1}{3}\}$ 4-parts.
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(P5) $A \in Z_3$ and $x, y \in A^*$ $\Rightarrow |L(x) \cup L(y)| \geq k + t_3 + k_4$.
(P6) $|A| = 3$ and $x, y \in A^*$ $\Rightarrow |L(x) \cup L(y)| \geq k + \left\lceil \frac{k_3}{3} \right\rceil + k_4$.
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Sufficiency

**Lem.** The lists left after good merges have an SDR.

**Pf.** We check $|L(S)| \geq |S|$ for each set $S$ of vertices.
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If $S$ has three vertices from one part, then

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Now $S$ is restricted to merged vertices, and (P8) suffices. ■
Merges in $\mathbb{Z}_3$ and $\mathbb{Z}_4$

**Idea:** Merge vertices with many common colors in lists.
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**Def.** $\ell(A) = \max_{u,v \in \binom{A}{2}} |L(u) \cap L(v)|$.

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Let $t_3$ be the largest integer such that $\ell(A) \geq \left\lceil \frac{k+t_3-1}{3} \right\rceil$ for at least $t_3 - \lceil k_3/3 \rceil$ parts in $Z_3$. (Note $t_3 \geq \lceil k_3/3 \rceil$.)
Merge a pair achieving $\ell(A)$ in each of these parts.

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Also merge $\{x, y\} \in A$ if $|L(x) \cap L(y)| \geq \frac{k_1+3k_2+5k_3+4k_4+1}{3}$.
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**Lem.** These merges guarantee (P1)-(P7) if also each 3-part and each 4-part outside $Z_3 \cup Z_4$ has one merge.


SDR for the Merged Vertices (Property P8)

**Idea:** For the set $Y$ of 3- and 4-parts outside $Z_3 \cup Z_4$, choose a merge in each part so that the SDR exists!
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**Pf.** $|S| \leq k_3+k_4+|Z_4|$. Restrict $S$ to ensure $|L(S)| \geq |S|$. If $S$ has two for $A$, then $A \in Z_4$; (P7) $\Rightarrow$ $|L(S)| \geq k+k_4 \geq |S|$. If $S$ has $L_A$ with $|A|=4$, then Lem $\Rightarrow$ $|S| \leq k_3+|Z_4|$. ...