Beyond Ohba's Conjecture: A bound on the choice number of k-chromatic graphs with n vertices

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Abstract

Let $\mathrm{ch}(G)$ denote the choice number of a graph G (also called "list chromatic number" or "choosability" of G). Noel, Reed, and Wu proved the conjecture of Ohba that $\mathrm{ch}(G) = \chi(G)$ when $|V(G)| \leq 2\chi(G) + 1$. We extend this to a general upper bound: $\mathrm{ch}(G) \leq \max\{\chi(G), \lceil(|V(G)| + \chi(G) - 1)/3\rceil\}$. Our result is sharp for $|V(G)| \leq 3\chi(G)$ using Ohba's examples, and it improves the best-known upper bound for $\mathrm{ch}(K_{4,\ldots,4})$.

1 Introduction

Choosability is a variant of classical graph coloring; it models limited availability of resources. Each vertex v in a graph G is assigned a list $L(v)$ of available colors. An *L-coloring* is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$, and G is *k-choosable* if G has an L-coloring whenever $|L(v)| \geq k$ for all $v \in V(G)$. The *choice number* of G, denoted $\mathrm{ch}(G)$, is the least k such that G is k-choosable. Introduced by Vizing [25] and by Erdős, Rubin, and Taylor [8], choosability is now a well-studied topic (surveyed in [2, 14, 22, 23]).

Since k-choosability requires an L-coloring when $L(v) = \{1, \ldots, k\}$ for all $v \in V(G)$, always $\mathrm{ch}(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number. However, there is no upper bound on $\mathrm{ch}(G)$ in terms of the chromatic number $\chi(G)$ (even for bipartite graphs). Such bounds exist when the number of vertices is specified, since always $\mathrm{ch}(G) \leq |V(G)|$, so it is

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natural to seek the maximum of $\mathrm{ch}(G)$ among k-chromatic graphs with n vertices. Ohba [17] conjectured $\mathrm{ch}(G) = \chi(G)$ for $n \le 2\chi(G) + 1$. Several papers proved partial results in this direction (see [13, 17, 19, 20, 21]), and the conjecture has now been proved:

Theorem 1.1 (Noel, Reed, and Wu [16])**.** *If* $|V(G)| \le 2\chi(G) + 1$, *then* $\mathrm{ch}(G) = \chi(G)$.

With n and k fixed, it suffices to study complete k-partite graphs, since adding an edge cannot reduce the choice number. In discussing complete k-partite graphs, we use *part* rather than the formal term "partite set" to mean a maximal stable set (also called "independent set", a *stable set* is a set of pairwise nonadjacent vertices). We write $K_{1*k_1,2*k_2,\ldots}$ for the complete multipartite graph with k_i parts of size i.

When k is even, $K_{2*(k-1),4*1}$ and $K_{1*(k/2-1),3*(k/2+1)}$ are not k-choosable [7], making Theorem 1.1 sharp. However, when k is odd, $K_{2*(k-1),4*1}$ is k-choosable [7]. Since $\mathrm{ch}(K_{2*(k-1),5*1}) > k$ for all k [7], the only unsettled case is whether $n = 2k + 2$ implies $\mathrm{ch}(G) = k$ when k is odd. Noel [15] conjectured that the only complete k-partite graphs on $2k + 2$ vertices that are not k-choosable are $K_{1*(k/2-1),3*(k/2+1)}$ and $K_{2*(k-1),4*1}$ for even k.

Moving to larger n, Ohba [18] determined the choice number for a family of complete k-partite graphs with at most $3k$ vertices.

Theorem 1.2 (Ohba [18])**.** $\mathrm{ch}(K_{1*k_1,3*k_3}) = \max\left\{k, \left\lceil\frac{n+k-1}{3}\right\rceil\right\}$, *where* $k = k_1 + k_3$ *and* $n = k_1 + 3k_3$.

Earlier, Kierstead [11] computed the special case $\mathrm{ch}(K_{3*k}) = \left\lceil\frac{4k-1}{3}\right\rceil = \left\lceil\frac{n+k-1}{3}\right\rceil$. Our main result relies on Theorem 1.1 and extends Ohba's upper bound to all graphs. It is sharp when $n \le 3k$ (with $n - k$ even) and useful when n is bounded by a small multiple of k, but it is weak when n/k is large:

Theorem 1.3. *For any graph G with n vertices and chromatic number k,*

$$\mathrm{ch}(G) \le \max\left\{k, \left\lceil\frac{n+k-1}{3}\right\rceil\right\}.$$

For use in bounding $\mathrm{ch}(G)$ for the random graph, [8] suggested finding good bounds on $\mathrm{ch}(K_{m*k})$. By our result, K_{3*k} has the largest choice number among k-chromatic graphs with at most $3k$ vertices. For $m = 4$, Yang [24] proved $\left\lfloor\frac{3k}{2}\right\rfloor \le \mathrm{ch}(K_{4*k}) \le \left\lceil\frac{7k}{4}\right\rceil$; our result improves the upper bound to $\left\lceil\frac{5k-1}{3}\right\rceil$. Since the writing of our paper, Kierstead, Salmon, and Wang [12] have determined that $\mathrm{ch}(K_{4*k})$ equals the easy lower bound $\left\lfloor\frac{3k}{2}\right\rfloor$. Our result also yields $\left\lfloor\frac{8k}{5}\right\rfloor \le \mathrm{ch}(K_{5*k}) \le 2k$ and $\left\lfloor\frac{5k}{3}\right\rfloor \le \mathrm{ch}(K_{6*k}) \le \left\lceil\frac{7k-1}{3}\right\rceil$. The bounds $\left\lfloor\frac{2k(m-1)}{m}\right\rfloor \le \mathrm{ch}(K_{m*k}) \le \left\lceil\frac{k(m+1)-1}{3}\right\rceil$ are valid for all m, but they are interesting only for small m. The lower bound arises from the following construction.

Construction 1.4. Consider a universe U of $2k-1$ colors, split into sets $X_1, \dots, X_m$ of sizes $\lfloor \frac{2k-1}{m} \rfloor$ and $\lceil \frac{2k-1}{m} \rceil$. Assign list $U - X_i$ to the ith vertex of each part. An L-coloring must use at least two colors on each part, and these pairs must be disjoint. Hence at least $2k$ colors must be used, but $|U| < 2k$, so there is no L-coloring. The list sizes are all at least $2k-1-\lceil \frac{2k-1}{m} \rceil$, which is at least $\left\lfloor \frac{2k(m-1)}{m} \right\rfloor - 1$. (For sharpness of Theorem 1.3 when $n = 3k-2i$, use $2k-1-i$ colors, with $k-i$ parts of size 3 and i singleton parts whose list is the full color set.)

Problem 1.5. *Determine* $\mathrm{ch}(K_{m*k})$ *for small* m*, beginning with* $m = 5$.

Noel [15] conjectured that in fact K_{m*k} always has the largest choice number among k-chromatic graphs with at most mk vertices. More generally:

Conjecture 1.6 (Noel [15])**.** *For* $n \ge k \ge 2$*, among* n*-vertex* k*-chromatic graphs the choice number is maximized by a complete* k*-partite graph with independence number* $\lceil n/k \rceil$*.*

Theorem 1.1 implies Conjecture 1.6 for $n \le 2k+1$. Theorems 1.3 and 1.2 together imply Conjecture 1.6 when $n \le 3k$ and $n-k$ is even.

Construction 1.4 never yields lists of size exceeding $2k$. When m is large in terms of k, a simple explicit construction generalizing an example in [8] gives a good lower bound.

Construction 1.7. Let $m = \binom{kj-1}{(k-1)j}$. For K_{m*k}, assign all $(k-1)j$-sets from a set U of $kj-1$ colors as lists on each part. Each list omits $j-1$ colors. If fewer than j colors are used on some part, then this part has a list from which no color is chosen. Hence kj colors are needed, but $|U| < kj$. When k is large, the leading behavior of m is a constant times $k^j/\sqrt{j}$, which yields $j \approx \log_k m + \frac{1}{2} \log_k \log_k m$. Thus $\mathrm{ch}(K_{m*k}) \ge c(k-1)\frac{\log m}{\log k}$.

For intermediate m, Constructions 1.4 and 1.7 can be combined. Alon [1] improved the latter, proving $c_1 k \log m \le \mathrm{ch}(K_{m*k}) \le c_2 k \log m$ for some constants c_1 and c_2. This yields choice number $O(\frac{n \log\log n}{\log n})$ for the random graph. More precise asymptotic bounds were later obtained by Gazit and Krivelevich [9]: $\mathrm{ch}(K_{m*k}) = (1+o(1))\frac{\log m}{\log(1+1/k)}$. With Conjecture 1.6, the general upper bound would be $\mathrm{ch}(G) \le (1+o(1))\frac{\log(n/k)}{\log(1+1/k)}$.

Our proof of Theorem 1.3 begins with several restrictions on minimal counterexamples. First, Theorem 1.1 verifies the claim in the most difficult range ($n \le 2k+1$), which will serve as a basis. In that range the lists have size only k; when the problem is restricted to $n \ge 2k+2$, the lists will always have size at least $k+1$. As noted, we may assume that G is a complete k-partite graph. We prove that in a minimal counterexample, all parts have size at most 4 and no color appears in more than two lists on one part. We then produce an L-coloring, contradicting the assumption of a counterexample.

Step 1. Break $V(G)$ into stable sets of size at most 2 by splitting some parts.

Step 2. Produce an L-coloring whose color classes are the sets obtained in Step 1.

In Section 3, we prove that if a partition of the type in Step 1 satisfies several special properties, then Hall's Theorem [10] on matchings in bipartite graphs produces an L-coloring to complete Step 2. In Sections 4 and 5, we show that $V(G)$ admits a partition satisfying these special properties, thereby completing Step 1 and the proof of Theorem 1.3.

2 Preliminary Reductions

If Theorem 1.3 is not true, then there is a minimal counterexample.

Remark 2.1. If Theorem 1.3 fails, then by Theorem 1.1 there is an n-vertex complete k-partite graph G with list assignment L such that G has no L-coloring, $n \geq 2k+2$, $|L(v)| \geq \lceil \frac{n+k-1}{3} \rceil \geq k+1$ for all $v \in V(G)$, and the conclusion of Theorem 1.3 holds for all graphs with fewer vertices.

Henceforth, G and L will have the properties stated in Remark 2.1. We will derive additional properties, after which we will produce an L-coloring of G. For example, we may assume $\left|\bigcup_{v \in V(G)} L(v)\right| < n$ due to the following lemma proved independently by Kierstead [11] and by Reed and Sudakov [20, 19].

Lemma 2.2 ([11, 20, 19])**.** *If G is not r-choosable, then there is a list assignment L such that G has no L-coloring, all lists have size at least r, and their union has size less than $|V(G)|$.*

The reduction $\left|\bigcup_{v \in V(G)} L(v)\right| < n$ is a standard reduction for minimal counterexamples in choosability problems, so much so that it has a name: the "Small Pot Lemma". It has been applied in diverse situations, including [3, 4, 5, 6, 16, 12].

Next, we obtain more specific restrictions on G and L for our problem. These use the following key proposition.

Proposition 2.3. *If A is a stable set in G whose lists have a common color, then*

$$\left\lceil \frac{|V(G-A)| + \chi(G-A) - 1}{3} \right\rceil = \left\lceil \frac{n+k-1}{3} \right\rceil .$$

Proof. Let $c \in \bigcap_{v \in A} L(v)$. Let $G' = G - A$, and let L' be the list assignment for G' obtained from L by deleting c from each list containing it.

Since $|L(v)| \geq k+1$, we have $|L'(v)| \geq k \geq \chi(G')$ for all $v \in V(G')$. Also $|L'(v)| \geq \lceil \frac{n+k-1}{3} \rceil - 1$. If $\left\lceil \frac{|V(G')|+\chi(G')-1}{3} \right\rceil < \lceil \frac{n+k-1}{3} \rceil$, then $|L'(v)| \geq \max\left\{\chi(G'), \left\lceil \frac{|V(G')|+\chi(G')-1}{3} \right\rceil\right\}$. By the minimality of G, we obtain an L'-coloring of G', which extends to an L-coloring of G by giving color c to A. Hence $\left\lceil \frac{|V(G')|+\chi(G')-1}{3} \right\rceil \geq \lceil \frac{n+k-1}{3} \rceil$, and $G' \subseteq G$ yields equality. $\square$

Corollary 2.4. *The lists on a part of size 2 in G are disjoint.*

Proof. A shared color in a part A of size 2 contradicts Proposition 2.3 via $\left\lceil\frac{V(G-A)+\chi(G-A)-1}{3}\right\rceil = \left\lceil\frac{(n-2)+(k-1)-1}{3}\right\rceil < \left\lceil\frac{n+k-1}{3}\right\rceil$. □

Corollary 2.5. *Each color appears in at most two lists in each part in G.*

Proof. Having three vertices with a common color in a part A contradicts Proposition 2.3 via $\left\lceil\frac{|V(G-A)|+\chi(G-A)-1}{3}\right\rceil \le \left\lceil\frac{(n-3)+k-1}{3}\right\rceil < \left\lceil\frac{n+k-1}{3}\right\rceil$. □

Lemma 2.6. $\alpha(G) \le 4$.

Proof. Let A be a stable set in G. By Lemma 2.5, each color appears in at most two lists on A, so $\sum_{v\in A}|L(v)| \le 2\left|\bigcup_{v\in V(G)} L(v)\right| \le 2(n-1)$, by Lemma 2.2. Also, $\sum_{v\in A}|L(v)| \ge |A|\left\lceil\frac{n+k-1}{3}\right\rceil$. Together, the inequalities yield $|A| \le 6\frac{n-1}{n+k-1}$, so $|A| \le 5$. If equality holds, then $n \ge 5k+1$, which requires a part of size at least 6 and is already forbidden. □

The restrictions so far simplify the main approach. The remaining restrictions in this section are technical statements used to simplify the arguments in Sections 4 and 5 that $V(G)$ admits a partition satisfying the properties specified in Section 3. We obtain them here as further consequences of Proposition 2.3.

Lemma 2.7. $\frac{n+k-1}{3}$ *is an integer.*

Proof. Let A be a largest part, so $n \le k|A|$. If the lists on A are disjoint, then

$$n \le k|A| \le \sum_{v\in A}|L(v)| = \left|\bigcup_{v\in A} L(v)\right| \le \left|\bigcup_{v\in V(G)} L(v)\right| < n.$$

Hence A contains a 2-set A' with intersecting lists, which by Corollary 2.5 and $n > 2k$ is not all of A. Now

$$\left\lceil\frac{|V(G-A')|+\chi(G-A')-1}{3}\right\rceil \le \left\lceil\frac{(n-2)+k-1}{3}\right\rceil = \left\lfloor\frac{n+k-1}{3}\right\rfloor.$$

If $\frac{n+k-1}{3} \notin \mathbb{Z}$, then this contradicts Proposition 2.3. □

Henceforth let k_i be the number of parts with size i, for $i \in \{1,2,3,4\}$. Note that $k = \sum_{i=1}^4 k_i$ and $n = \sum_{i=1}^4 ik_i$.

Corollary 2.8. *The parameters k_1, k_2, k_3, k_4 satisfy the following relationships.*
(a) $\frac{n+k-1}{3} = k + k_4 - \frac{k_1-k_3+k_4+1}{3}$, *with both fractions being integers.*
(b) $\frac{n+k-1}{3} + \frac{k}{3} \ge k + k_4 + \frac{2k_3-1}{3}$.
(c) $\frac{2(n+k-1)}{3} = n + \frac{k_1-k_3-2k_4-2}{3} = k + k_3 + 2k_4 + \frac{k+2k_2+k_3-2}{3}$.

Proof. (a) $n+k = 2k_1+3k_2+4k_3+5k_4 = 3k - k_1 + k_3 + 2k_4$. Integrality was shown in Lemma 2.7.

(b) Use $k \ge k_1+k_3+k_4$ in the right side of (a).

(c) $2(n+k) = 4k_1+6k_2+8k_3+10k_4 = 4k+2k_2+4k_3+6k_4$. □

3 A Sufficient Condition for L-Coloring

By Corollary 2.5, each color appears in at most two lists in each part. Therefore, an L-coloring must refine the partition of $V(G)$ into stable sets of size at most 2. To find an L-coloring, we must determine which pairs will form the color classes of size 2.

Definition 3.1. *Merging* non-adjacent vertices u and v in G means replacing u and v by one vertex w with list $L(w)$ equal to $L(u) \cap L(v)$. We then say that w is a *merged* vertex. Given a set S of vertices, each of which may be merged or unmerged, let $L(S) = \bigcup_{v \in S} L(v)$.

A *system of distinct representatives (SDR)* for a family $\{X_1, \ldots, X_m\}$ of sets is a set $\{x_1, \ldots, x_m\}$ of distinct elements such that $x_i \in X_i$ for $i \in \{1, \ldots, m\}$. Our goal is to perform some merges so that the resulting color lists have an SDR; assigning each vertex the chosen representative of its list (with the color chosen for a merged vertex used on both original vertex comprising it) then yields an L-coloring of G. To facilitate finding an SDR, it is natural to merge vertices whose lists have many common colors. Merged vertices come from the same part in G and will only merge two previously unmerged vertices, since by Corollary 2.5 a merge of three vertices would have an empty list of colors.

In this section we prove that if the merging procedure satisfies the properties in Definition 3.2, then the desired SDR exists. The proof uses Hall's Theorem [10], which states that $\{X_1, \ldots, X_m\}$ has an SDR if and only if $\left|\bigcup_{i \in S} X_i\right| \geq |S|$ for all $S \subseteq \{1, \ldots, m\}$. The proof of Theorem 1.3 will be completed in Sections 4 and 5 by showing that merges can be performed to establish these properties.

Definition 3.2. Let a *j-part* be a part having size j in G (before merges). After performing merges, let A^* denote the set of vertices resulting from the part A. Let t_3 denote the number of 3-parts having merged vertices. Let Z_3 be some fixed set of $\left\lfloor \frac{2}{3} k_3 \right\rfloor$ 3-parts, and let Z_4 be some fixed set of $\max\{0, \frac{k_1 - k_3 + k_4 + 1}{3}\}$ 4-parts. For a set of merges, we define properties (P1)–(P8) below. Note that (P3)-(P7) can be considered for individual parts.

(P1) $t_3 \geq k_3/3$.
(P2) In every 4-part, at least one merge occurs.
(P3) If $x, y, z \in A^*$ are distinct, then $|L(x) \cup L(y) \cup L(z)| \geq n - t_3 - k_4$.
(P4) If $|A^*| = |A| = 3$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + k_3 + k_4$.
(P5) If $A \in Z_3$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + t_3 + k_4$.
(P6) If $|A| = 3$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + \frac{k_3}{3} + k_4$.
(P7) If $A \in Z_4$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + k_4$.
(P8) The set of lists of merged vertices has an SDR.

In the specification of Z_4, note that $\frac{k_1 - k_3 + k_4 + 1}{3}$ is an integer, by Corollary 2.8(a). Property (P3) applies to 3-parts without merges and to 4-parts with one merge. To understanding the intuition behind using Hall's Theorem to show that these properties are sufficient, note that the lower bounds in (P3)–(P7) are successively weaker (the comparison of the bounds in (P5) and (P6) uses (P1)). A large set S must contain vertices whose lists are large, thereby satisfying $|L(S)| \geq |S|$ and allowing such sets to be excluded. As the remaining sets to be considered become smaller by eliminating such vertices, smaller lower bounds on the list sizes become sufficient. Property (P8) can then be viewed as reducing the problem to finding an SDR of a smaller family (generated by the merged vertices) when (P1)–(P7) hold.

Lemma 3.3. *When the merges satisfy (P1)-(P8), the family of all resulting lists has an SDR.*

Proof. By Hall's Theorem, it suffices to prove $|L(S)| \geq |S|$ for each vertex set S (after the merges). We use (P1)–(P7) to restrict S until it consists only of merged vertices, and then (P8) guarantees $|L(S)| \geq |S|$ for such S.

By (P2), the merges leave at most $n - t_3 - k_4$ vertices, so $|S| \leq n - t_3 - k_4$. Thus (P3) yields $|L(S)| \geq |S|$ whenever S has three vertices from one part. We may thus restrict S to having at most two vertices from each part, yielding $|S| \leq k + k_2 + k_3 + k_4 \leq 2k$.

If S contains both vertices from a part of size 2 (unmerged), then by Corollary 2.4 their lists are disjoint and $|L(S)| \geq 2k + 2 > |S|$; hence $|S| \leq k + k_3 + k_4$. If S contains two vertices from a 3-part with no merged vertices, then (P4) yields $|L(S)| \geq k + k_3 + k_4$; hence $|S| \leq k + t_3 + k_4$. If S contains two vertices from a 3-part in Z_3, then (P5) yields $|L(S)| \geq k + t_3 + k_4$; hence $|S| \leq k + \lceil \frac{k_3}{3} \rceil + k_4$, since $\lceil \frac{k_3}{3} \rceil = k_3 - |Z_3|$. If S contains two vertices from any 3-part, then (P6) yields $|L(S)| \geq k + \lceil \frac{k_3}{3} \rceil + k_4$; hence $|S| \leq k + k_4$. If S contains two vertices from a 4-part in Z_4, then (P7) yields $|L(S)| \geq k + k_4$; hence $|S| \leq k + k_4 - |Z_4| \leq \frac{n+k-1}{3}$, by Corollary 2.8(a) and the formula for $|Z_4|$. Now $|L(S)| \geq |S|$ if S contains any unmerged vertex. Finally, (P8) applies. □

In the rest of the paper, we describe an explicit procedure to obtain such merges.

4 Greedy Merges

In order to guarantee (P3)–(P8), it is helpful to merge vertices whose lists have large intersection. Our first task is to make the meaning of "large" precise.

Definition 4.1. For a part A in G, let $\ell(A) = \max\{|L(u) \cap L(v)|\colon\ u, v \in A\}$.
If $|A| \geq 3$, then a pair $\{u, v\} \subseteq A$ is a *good pair* for A if
$|A| = 3$ and $|L(u) \cap L(v)| \geq \frac{k_1+k_4+1}{3}$, or if
$|A| = 4$ and $|L(u) \cap L(v)| \geq |L(w) \cap L(z)|$, where $\{w, z\} = A - \{u, v\}$.
The merge of a good pair is a *good merge*. A part A is *good* if a good merge is made in it.

When $|A| = 4$, a pair in A whose lists have largest intersection is good by definition. With a lower bound on $\ell(A)$, this will also hold when $|A| = 3$.

Lemma 4.2. *If $T \subseteq V(G)$ is a stable set of size 3, then $\sum_{\{u,v\}\in\binom{T}{2}} |L(u) \cap L(v)| \geq k$.*

Proof. Since colors appear at most twice in each part, and $|L(V(G))| < n$, we have

$$\sum_{\{u,v\}\in\binom{T}{2}} |L(u) \cap L(v)| = \sum_{u\in T} |L(u)| - |L(T)| \geq 3\left(\frac{n+k-1}{3}\right) - (n-1) = k. \qquad \square$$

Corollary 4.3. *If $|A| \geq 3$, then $\ell(A) \geq \frac{k}{3}$.*

Corollary 4.4. *When $|A| \geq 3$, a pair $\{u, v\} \subseteq A$ maximizing $|L(u) \cap L(v)|$ is good for A.*

Proof. When $|A| = 4$, the conclusion is immediate from Definition 4.1. If $|A| = 3$, then $k_3 \geq 1$. Thus $|L(u) \cap L(v)| = \ell(A) \geq \frac{k}{3} \geq \frac{k_1+k_4+1}{3}$, so $\{u, v\}$ is good for A. $\square$

Lemma 4.5. *Every good 3-part A satisfies (P6).*

Proof. We have $A^* = \{x, y\}$ and may assume that y is merged and x is not. By Corollary 2.5, $L(x) \cap L(y) = \varnothing$, and forming y by a good merge yields

$$|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq \frac{n+k-1}{3} + \frac{k_1+k_4+1}{3} = k + \frac{k_3}{3} + k_4,$$

by Corollary 2.8(a). Thus, the desired inequality holds. $\square$

Lemma 4.6. *If (P1) holds and A is a good 4-part, then A satisfies (P3).*

Proof. Let A be a 4-part, with $x, y, z \in A^*$. Since A is good, we may assume that x is merged and that y and z are not. Since x arises from a good merge, $|L(x)| \geq |L(y) \cap L(z)|$. Also, $L(x) \cap L(y) = L(x) \cap L(z) = \varnothing$ by Corollary 2.5. Therefore,

$$\begin{aligned}|L(x) \cup L(y) \cup L(z)| &= |L(x)| + |L(y)| + |L(z)| - |L(y) \cap L(z)| \\ &\geq |L(y)| + |L(z)| \geq \frac{2(n+k-1)}{3} \geq n + \frac{k_1 - k_3 - 2k_4 - 2}{3},\end{aligned}$$

by Corollary 2.8(c). By (P1), we have $t_3 \geq k_3/3$, and clearly $k_1 \geq 0$ and $2k_4/3 < k_4$, so $|L(y)| + |L(z)| \geq n + \frac{k_1-k_3-2k_4-2}{3} \geq n - t_3 - k_4$, and the desired conclusion holds. $\square$

Finally, we are ready to specify merges. We will specify merges in special sets Z_3 of 3-parts and Z_4 of 4-parts. These will make each such part good, though when we make two merges in a 4-part in Z_4 they need not both be good.

We will do this in subsequently specified in each 3-part or 4-part outside $Z_3 \cup Z_4$, then the set of merges will satisfy properties (P1)–(P7) and Q1. In Section 5, we will show that those remaining good merges can then be chosen so that (P8) is also satisfied.

4.1 Parts of Size 3

Specify a fixed set Z_3 of exactly $\lfloor \frac{2k_3}{3} \rfloor$ 3-parts; exactly $\lceil \frac{k_3}{3} \rceil$ 3-parts lie outside Z_3. For every 3-part outside Z_3, we will choose a good merge in the next section. Here we choose merges in some members of Z_3 based on intersection sizes; only good pairs will be merged. Note that we have not yet specified the value t_3 giving the number of 3-parts that will have merges.

Construction 4.7. Set t_3 to be the largest integer for which there exists a set $Z_3' \subseteq Z_3$ of size $t_3 - \lceil \frac{k_3}{3} \rceil$ such that $\ell(A) \ge \frac{k+t_3-1}{3}$ for all $A \in Z_3'$. For $A \in Z_3$, merge a pair in A achieving $\ell(A)$ if and only if $A \in Z_3'$.

By Corollary 4.4, a pair whose lists have largest intersection in a part is always a good pair. Possibly no member of Z_3 has such a large intersection size, in which case $t_3 = \lceil \frac{k_3}{3} \rceil$ and Z_3' is empty. In any case, $t_3 \ge k/3$.

Lemma 4.8. *If one merge is made in each 3-part outside Z_3, then (P1) holds, (P3) holds for 3-parts, (P4) and (P5) hold, and (P6) holds for 3-parts in Z_3.*

Proof. If we later merge one pair in each 3-part outside Z_3 (there are $\lceil \frac{k_3}{3} \rceil$ such parts), then the total number of merges in 3-parts will be t_3, and (P1) holds.

For 3-parts, (P3) and (P4) apply only to those without merges, all lying in $Z_3 - Z_3'$. Membership in $Z_3 - Z_3'$ requires $\ell(A) \le \frac{k+t_3-2}{3}$. By Corollary 2.5,

$$\begin{aligned} |L(A)| &= \sum_{v \in A} |L(v)| - \sum_{\{u,v\} \in \binom{A}{2}} |L(u) \cap L(v)| \\ &\ge \frac{3(n+k-1)}{3} - \frac{3(k+t_3-2)}{3} > n - t_3 \ge n - t_3 - k_4, \end{aligned}$$

which proves (P3). For (P4), we take just two vertices $x, y \in A$. We compute

$$\begin{aligned} |L(x) \cup L(y)| &= |L(x)| + |L(y)| - |L(x) \cap L(y)| \\ &\ge \frac{2(n+k-1)}{3} - \frac{k+t_3-2}{3} \ge k + k_3 + 2k_4 + \frac{2k_2}{3} \ge k + k_3 + k_4 \end{aligned}$$

using Corollary 2.8(c) and $k_3 \ge t_3$.

Since $k_3 \ge t_3$ and $k_3 \ge k_3/3$, (P4) implies (P5) and (P6) for 3-parts containing no merge. Since (P5) is imposed only for parts in Z_3, it therefore suffices to consider $A \in Z_3'$. By Corollary 4.4, the merge in A is good, so by Lemma 4.5 A satisfies (P6). For (P5), we may assume that y is merged and x is not. Since $L(x) \cap L(y) = \varnothing$ by Corollary 2.5, Construction 4.7 yields

$$\begin{aligned} |L(x) \cup L(y)| = |L(x)| + |L(y)| &\ge \frac{n+k-1}{3} + \frac{k+t_3-1}{3} \\ &\ge k + \frac{k_2 + 2k_3 + t_3}{3} + k_4 - \frac{2}{3} \ge k + t_3 + k_4 - \frac{2}{3}, \end{aligned}$$

using Corollary 2.8(b) and $k_3 \ge t_3$. Hence $|L(x) \cap L(y)| \ge k + t_3 + k_4$, as desired. □

To complete the proof of (P6) for all parts, it suffices by Lemma 4.5 to specify a good merge in each 3-part outside Z_3.

4.2 Parts of Size 4

Corollary 2.8(a) and $n \geq 2k+2$ imply that $\frac{k_1-k_3+k_4+1}{3}$ is an integer less than k_4. Hence we can specify a fixed set Z_4 consisting of exactly $\max\left\{0, \frac{k_1-k_3+k_4+1}{3}\right\}$ of the 4-parts, chosen arbitrarily. In the next section we will choose one good pair to merge in each 4-part not in Z_4. Here we specify one or two merges in each member of Z_4, based on intersection sizes.

Construction 4.9. For $A \in Z_4$, merge a pair $\{u, v\}$ such that $|L(u) \cap L(v)| = \ell(A)$. Also merge the remaining pair $\{w, z\}$ if $|L(w) \cap L(z)| \geq s$, where $s = \frac{2n-k+1}{3} - k_4$.

Remark 4.10. Lemma 2.7 implies that s is an integer. Also $s \geq \frac{k}{3}$, since $s - \frac{k}{3} > \frac{2(n-k)}{3} - k_4 \geq \frac{6k_4}{3} - k_4$. Since also $\ell(A) \geq \frac{k}{3}$ (Corollary 4.3), the list of any merged vertex in a member of Z_4 has size at least $\frac{k}{3}$.

Lemma 4.11. *The merging procedure satisfies (P7).*

Proof. Property (P7) applies only for $A \in Z_4$. Given $A \in Z_4$ and $x, y \in A^*$, we need $|L(x) \cup L(y)| \geq k + k_4$. We consider three cases, depending on the merges in A.

Case 1: *Neither x nor y is merged.* By Construction 4.9, $|L(x) \cap L(y)| \leq s - 1$. Thus

$$|L(x) \cup L(y)| \geq |L(x)| + |L(y)| - |L(x) \cap L(y)| \geq \frac{2(n+k-1)}{3} - \frac{2n-k-2}{3} + k_4 = k + k_4.$$

Case 2: *Exactly one of $\{x, y\}$, say y, is merged.* By Corollary 2.5, $L(x) \cap L(y) = \varnothing$. Thus, by Remark 4.10 and Corollary 2.8(b),

$$|L(x) \cup L(y)| \geq |L(x)| + |L(y)| \geq \frac{n+k-1}{3} + \frac{k}{3} \geq k + k_4 - \frac{1}{3},$$

and $|L(x) \cup L(y)|$ is an integer.

Case 3: *Both x and y are merged.* By Construction 4.9 and symmetry, we may assume $\ell(A) = |L(x)| \geq |L(y)| \geq s$. Since $L(x) \cap L(y) = \varnothing$ by Corollary 2.5, we compute

$$|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq 2s = n + \frac{n-2k+2}{3} - 2k_4 > \left(\sum ik_i\right) - 2k_4 \geq k + k_4.$$

In each case, the desired inequality holds. □

Note that the verification of (P7) did not use any property of merges in parts outside Z_4.

5 The Remaining Merges

At this point, we can reduce the proof of Theorem 1.3 to one main task.

Lemma 5.1. *If in addition to the merges previously specified, it is possible to specify one good merge in each 3-part outside Z_3 and each 4-part outside Z_4 in such a way that (P8) holds, then Theorem 1.3 is true.*

Proof. Specifying any merge in each 3-part outside Z_3 completes the proofs of (P1), (P4), and (P5), and it completes the proof of (P3) for 3-parts (by Lemma 4.8). If those merges are good, then by Lemma 4.5 we also have proved (P6) completely.

Furthermore, we have previously proved (P7) completely, since it applies only to parts in Z_4. Specifying any merge in each 4-part outside Z_4 completes the proof of (P2). If those merges are good and we have specified merges outside Z_3, then by Lemma 4.6 we have (P3) also for 4-parts.

If the merges also satisfy (P8), then Lemma 3.3 completes the proof. □

Hence our task is to merge a good pair in every 3-part outside Z_3 and every 4-part outside Z_4 in such a way that (P8) holds. In fact, we specify the merges among the possible good merges *by requiring that (P8) holds.* Note that since we have already proved (P7) completely, we can use it in this section.

Let T denote the set of all merged vertices in parts in $Z_3 \cup Z_4$. Let Y denote the set of parts of size 3 or 4 outside $Z_3 \cup Z_4$. To complete the proof, we need to find distinct colors, one for each vertex of T and one for each part in Y, such that the color chosen for each $A \in Y$ belongs to both lists for a good pair in A, and the color chosen for a merged vertex w in T belongs to $L(w)$. To obtain such a set of colors (and thereby define the remaining merges), we again apply Hall's Theorem.

Definition 5.2. For $A \in Y$, let L_A be the set of all colors c such that $c \in L(u) \cap L(v)$ for some good pair $\{u, v\} \subseteq A$. Let X be the family of sets consisting of L_A for all $A \in Y$ and $L(w)$ for all $w \in T$.

We seek an SDR for X. We start with lower bounds on $|L_A|$ for $A \in Y$. Note that this special list L_A differs from $L(A)$, which we defined to be $\bigcup_{v \in A} L(v)$.

Lemma 5.3. *If $A \in Y$ and $|A| = 3$, then $|L_A| \geq k_3 + \frac{k_1+k_4}{3}$.*

Proof. By Corollary 4.4, some pair in A is good. If $\{u, v\}$ is not good, then $|L(u) \cap L(v)| \leq \frac{k_1+k_4}{3}$, by Definition 4.1. At most two pairs are not good, so Lemma 4.2 and $k = \sum k_i$ yield

$$|L_A| \geq k - \frac{2(k_1 + k_4)}{3} \geq k_3 + \frac{k_1 + k_4}{3}.$$

□

Lemma 5.4. *If $A \in Y$ and $|A| = 4$, then $|L_A| \geq k_3 + k_4$.*

Proof. By Corollary 2.5 and the definition of a good pair for A,

$$|L_A| \geq \frac{1}{2} \sum_{\{u,v\} \in \binom{A}{2}} |L(u) \cap L(v)| = \frac{\sum_{u \in A} |L(u)|}{2} - \frac{\left|\bigcup_{u \in A} L(u)\right|}{2}.$$

Since the union of all lists has fewer than n colors,

$$|L_A| \geq \frac{1}{2} \left(\frac{4(n+k-1)}{3} - (n-1) \right) = \frac{n+4k-1}{6} > k \geq k_3 + k_4,$$

since $n > 2k + 1$. $\square$

As argued in Lemma 5.1, the following lemma completes the proof of Theorem 1.3.

Lemma 5.5. *There is an SDR for X.*

Proof. As in Lemma 3.3, we check Hall's Condition by verifying for successively restricted $S \subseteq X$ that the union of the lists indexed by S has size at least $|S|$. By construction, each 3-part contributes at most one list to X, each 4-part outside Z_4 also contributes at most one, and each part in Z_4 contributes at most two. Hence $|S| \leq |X| \leq k_3 + k_4 + |Z_4|$.

If S contains two lists for a part A, then $A \in Z_4$. By (P7), the union of these two lists has size at least $k + k_4$, which exceeds $|X|$. Hence S has at most one list from each part, so $|S| \leq k_3 + k_4$. By Lemma 5.4, we are now finished if S contains a list for a 4-part outside Z_4, so we may assume $|S| \leq k_3 + |Z_4|$.

If S contains the list for a 3-part A outside Z_3, then $|L_A| \geq k_3 + \frac{k_1+k_4}{3}$, by Lemma 5.3. Since A is a 3-part, $k_3 \geq 1$, so $\frac{k_1+k_4}{3} \geq \max\{0, \frac{k_1-k_3+k_4+1}{3}\} = |Z_4|$, and $|L_A| \geq |S|$.

Thus we may assume that S contains lists only for parts in $Z_3 \cup Z_4$, and at most one list for each such part. These are lists for vertices merged in Section 4. Within Z_3, we performed such merges only for parts in Z_3', so

$$|S| \leq |Z_3'| + |Z_4| = t_3 - \left\lceil \frac{k_3}{3} \right\rceil + \max\left\{0, \frac{k_1 - k_3 + k_4 + 1}{3}\right\}.$$

By Construction 4.7, the list for any merged vertex from a part in Z_3' has size at least $\lceil \frac{k+t_3-1}{3} \rceil$. Whether Z_4 is empty or not, $t_3 \leq k_3$ yields $\lceil \frac{k+t_3-1}{3} \rceil \geq |Z_3'| + |Z_4|$.

Hence S contains lists only for at most one merged vertex from each part in Z_4. That is, $|S| \leq |Z_4| = \max\left\{0, \frac{k_4+k_1-k_3+1}{3}\right\}$. By Remark 4.10, each such list has size at least $\frac{k}{3}$. Since $k \geq k_1 - k_3 + k_4$ and $\frac{k_1-k_3+k_4+1}{3}$ is an integer (by Corollary 2.8(a)), always the size of the union of the lists in S is at least $|S|$. $\square$

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