Oriented Diameter of Graphs with Diameter 3

Peter K. Kwok*, Qi Liu†, Douglas B. West‡

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Abstract

In 1978, Chvátal and Thomassen proved that every 2-edge-connected graph with diameter 2 has an orientation with diameter at most 6. They also gave general bounds on the smallest value $f(d)$ such that every 2-edge-connected graph G with diameter d has an orientation with diameter at most $f(d)$. For $d = 3$, their general bounds reduce to $8 \le f(3) \le 24$. We improve these bounds to $9 \le f(3) \le 11$.

1 Introduction

The One-Way Street Problem was solved by Robbins [8] in 1939: if a connected graph has no cut-edge, then it is possible to orient the edges so that in the resulting directed graph every vertex remains reachable from every other. It is natural to seek an orientation so that the maximum distance to reach some vertex from another is not much larger than the maximum in the original graph. Chvátal and Thomassen [1] made these notions precise, studying the problem in terms of diameter.

A vertex v is *reachable* from a vertex u if there is a path from u to v, called a *u,v-path*. The *distance* from u to v is the minimum length (number of edges) of a u,v-path. This definition holds both for graphs and for digraphs with the understanding that in a digraph a path must follow the tail-to-head direction along each edge.

Distance is symmetric in graphs but not in digraphs. The *diameter* of a graph or digraph is the maximum, over all vertex pairs (u, v), of the distance from u to v. A digraph is *strong* if its diameter is well-defined; that is, each vertex is reachable from every other. Our model of "graph" has no loops or multiple edges; for computation of distances they are irrelevant.

An *orientation* of a graph is a digraph obtained from it by assigning a direction to each edge; that is, each edge becomes an ordered pair of vertices. A graph is 2-*edge-connected* if at

*National Opinion Research Center, kwok-peter@norc.org

†Wharton School, University of Pennsylvania, qiliu@wharton.upenn.edu

‡Department of Mathematics, University of Illinois, Urbana, IL 61801, west@math.uiuc.edu. Work supported in part by NSA Grant H98230-06-1-0065.

least two edges must be deleted to leave a disconnected subgraph; being 2-edge-connected is an obvious necessary condition for the existence of a strong orientation. Robbins [8] proved that this condition is also sufficient. We thus seek orientations with small diameter.

Let $f(G)$ denote the minimum diameter over all orientations of a graph G; this is the *oriented diameter* of G. Let $f(d)$ denote the maximum of $f(G)$ such that G is a 2-edge-connected graph with diameter d. It is not immediately obvious that $f(d)$ is finite. In 1978, Chvátal and Thomassen [1] proved that $\frac{1}{2}d^2 + d \le f(d) \le 2d^2 + 2d$ for $d \ge 2$. There seem to have been no improvements in the general bounds. The known exact values are $f(1) = 3$ and $f(2) = 6$ [1]. For $d = 3$, the general result reduces to $8 \le f(3) \le 24$. In this paper, we improve these bounds to $9 \le f(3) \le 11$. We also give a slight correction to [1]; their construction for $f(d) \ge \frac{1}{2}d^2 + d$ is not correct for odd d, but a minor modification fixes it.

Bounds on oriented diameter have also been studied in terms of other parameters and in special classes of graphs. Fomin et al. [2] showed that the oriented diameter is bounded above by $9\gamma(G)-5$ and by $k+4$, where $\gamma(G)$ is the domination number of G and k is the minimum size of a dominating set that induces a 2-edge-connected subgraph. For asteroidal-triple-free graphs with diameter d, this yields $f(G) \le 9d - 5$, and they improve this upper bound to $2d+11$. McCanna [6] and Konig, Krumme, and Lazard [5] studied graphs where the oriented diameter equals the diameter (such as hypercubes and many discrete tori). Other studies of oriented diameter in special classes include [3, 4, 7, 9].

2 Lower Bounds

The lower bound on $f(d)$ in [1] arises from a sequence of graphs. Let $H_1 = C_3$, and designate any vertex as the "root". For $r > 1$, form H_r from two disjoint copies of H_{r-1} by adding a new root adjacent to the two old roots and adding another path joining the old roots through $2r - 2$ additional new vertices. Figure 1 contains two copies of H_3 (plus a 9-cycle); the roots of the copies of H_3 are the central vertices in the picture.

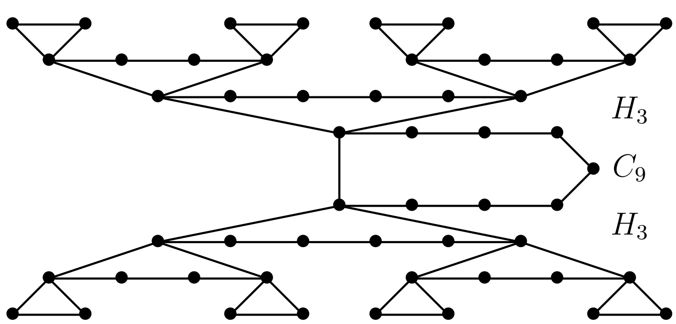

Figure 1: Graph of diameter 7 with oriented diameter 32

Let G_r consist of two edge-disjoint copies of H_r with a common root vertex. Chvátal and

Thomassen observed that G_r has diameter $2r$ and oriented diameter $2(r^2 + r)$, and hence $f(d) \geq \frac{1}{2}d^2 + d$ when d is even. For odd d, they stated that the graph consisting of copies of H_r and H_{r+1} with a common root vertex has diameter $2r + 1$, but it has diameter $2r + 2$.

Instead, we use two disjoint copies of H_r plus a cycle of length $2r + 3$ consisting of the two root vertices plus $2r+1$ new vertices, with the root vertices of the copies of H_r adjacent on the cycle. Figure 1 shows the resulting graph for $r = 3$, with diameter 7.

Since each block is a cycle, a strong orientation must orient each block as a directed cycle. From any peripheral vertex in one copy of H_r to an appropriate peripheral vertex in the other copy, one may be forced to go the long way around a cycle in each level. Hence the distance may be $(2\sum_{i=1}^{r} 2r) + (2r + 2)$. With $d = 2r + 1$, this equals $\lceil \frac{1}{2}d^2 \rceil + d$, as desired.

For $d = 3$, the resulting lower bound for $f(3)$ is 8. We improve this by 1.

Proposition 2.1 *$f(3) \geq 9$.*

Proof. Consider a graph G obtained from K_4 by subdividing the three edges incident to one vertex, replacing each with a path of length d. We prove that G has diameter d and that every strong orientation has diameter at least $3d$. For $d = 3$, this yields $f(3) \geq 9$.

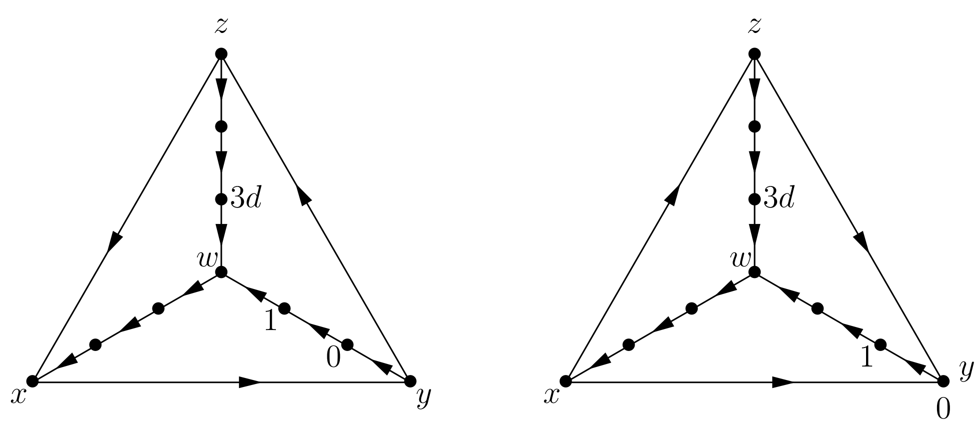

Figure 2: Subdivision of K_4

Let D be a strong orientation of G. There can be no source and no sink, so each path joining vertices of degree 3 in G is a path in D. Since diameter is unchanged when a strong orientation is reversed, we may assume that the central vertex w has outdegree 1, with a path of length d to the lower-left vertex x (see Figure 2). There are two cases; x may have outdegree 1 or 2, and in the first case we may choose its outneighbor to be the lower-right vertex y. In each case the remainder of the orientation is forced by avoidance of sources and sinks (and a choice by symmetry for the edge zy in the second case). In each case the path from the vertex labeled 0 to the vertex labeled $3d$ is the shortest (the only!) path from its origin to its terminus; it is a spanning path. □

Note that $d = 2$ and $d = 3$ are the only cases where the construction in Proposition 2.1 improves the lower bound $\lceil \frac{1}{2}d^2 \rceil + d$. Plesník [7] showed that for $d = 2$ this construction has the fewest vertices among graphs with diameter 2 and oriented diameter 6.

3 Upper Bound: Constructing the Orientation

Henceforth we confine our attention to graphs with diameter 3. We use $d_G(u, v)$ and $d_D(u, v)$ to denote the distance from u to v in a graph G or digraph D.

Lemma 3.1 (Chvátal–Thomassen [1]) *If G is a 2-edge-connected graph such that every edge lies in a cycle of length at most k, then G has an orientation D such that*

$$d_D(u, v) \leq [(k-2)2^{\lfloor (k-1)/2 \rfloor} + 1]d_G(u, v)$$

for all $u, v \in V(D)$. □

Let G be a 2-edge-connected graph of diameter 3. If every edge of G lies in a triangle, then Lemma 3.1 yields $f(G) \leq 9$. In proving that $f(G) \leq 11$, we may therefore assume that G has an edge uv that lies in no triangle. Our method is to construct an orientation D so that every vertex has a short path to u and a short path from v. If we can make each have length at most 5, then $d_D(x, y) \leq 11$ for all $x, y \in V(D)$. In some cases we will have $d_D(x, u) \geq 6$ or $d_D(v, y) \geq 6$, and then we must find a shorter x, y-path directly.

To begin this plan, we define most of the desired orientation D based on distances to u and v in G. By $N(S)$, we mean the set of all vertices in G having at least one neighbor in S.

Definition 3.2 *Specification of D, part 1.* Fix $uv \in E(G)$. Let $S_{j,k} = \{w \in V(G) \colon d_G(w, u) = j \text{ and } d_G(w, v) = k\}$. For $x \in \{u, v\}$ let $T_i(x) = \{w \in V(G) \colon d_G(w, x) = i\}$. With G having diameter 3 and uv lying in no triangle, we have

$$V(G) = \{u, v\} \cup S_{1,2} \cup S_{2,1} \cup S_{2,2} \cup S_{2,3} \cup S_{3,2} \cup S_{3,3};$$

these sets correspond to the ellipses in Figure 3. We further partition these sets as follows:

$$\begin{array}{ll}
A = S_{1,2} \cap N(T_2(u)) & A^* = S_{1,2} - A \\
B = S_{2,1} \cap N(T_2(v)) & B^* = S_{2,1} - B \\
I = S_{2,3} \cap N(T_3(u) \cup S_{2,2}) & A' = S_{2,3} - I \\
J = S_{3,2} \cap N(T_3(v) \cup S_{2,2}) & B' = S_{3,2} - J \\
X = S_{3,3} \cap (N(I) - N(J)) & Y = S_{3,3} \cap (N(J) - N(I)) \\
Z = S_{3,3} \cap N(I) \cap N(J) & C = S_{3,3} - (X \cup Y \cup Z)
\end{array}$$

Given this partition, we define part of the orientation D, as shown in Figure 3. Start with $u \to v$. For vertex sets R and S, we use the notation $R \to S$ to mean that all edges with endpoints in R and S are oriented from R to S. Thus for each list below, all edges with endpoints in two successive sets are oriented from the first set to the second.

$$\{v\} \to B \to S_{2,2} \to A \to \{u\},$$
$$B \to J \to Y \to S_{2,2} \to X \to I \to A,$$
$$J \to (Z \cup S_{2,2}) \to I, \qquad B' \to J \to I \to A',$$
$$A^* \to A, \qquad B \to B^*, \qquad X \to C \to Y.$$

□

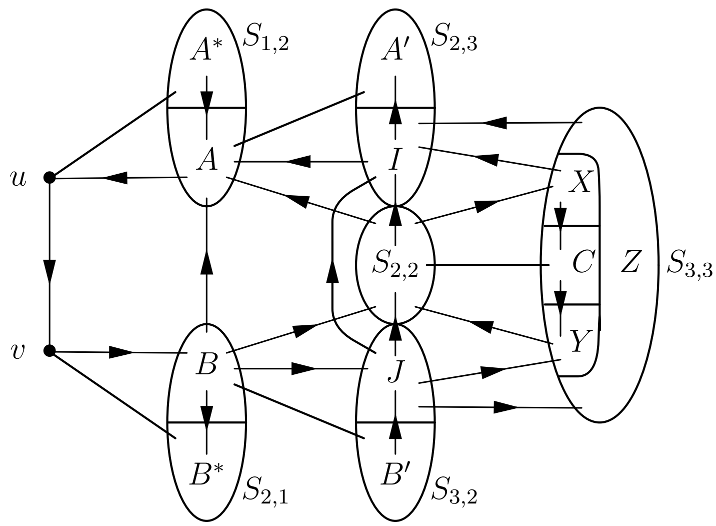

Figure 3: Vertex partition of G

Before we specify the rest of D, we pause to observe how the orientations defined so far fit into the plan we have described. Blank spots in the table refer to cases where we have not yet established a sufficiently small bound. Some of these will be handled easily in the next phase of defining the orientation; others will be more problematic and require additional attention.

Lemma 3.3 *With G partially oriented as specified in Definition 3.2, upper bounds on the distances of vertices in various sets from v or to u are as listed below. These are distances using only the edges that have been oriented so far.*

for w in	A	B	I	J	$S_{2,2}$	X	Y	Z	C	A'	B'	A^*	B^*
$d_D(w,u) \le$	1		2		2	3	3	3					
$d_D(v,w) \le$		1		2	2	3	3	3					

Proof. Consider first the vertex sets in Figure 3. Note that $T_1(u) = S_{1,2} \cup \{v\}$, since uv is in no triangle and $|d_G(w,v) - d_G(w,u)| \le 1$. Similarly, $T_1(v) = S_{2,1} \cup \{u\}$. The vertices with distance 2 from $\{u,v\}$ are those in $S_{2,3} \cup S_{2,2} \cup S_{3,2}$, and the remaining vertices are in $S_{3,3}$.

By definition, $S_{1,2} \subseteq N(u)$ and $S_{2,1} \subseteq N(v)$. Vertices of A^* have no neighbors in $T_2(u)$, but every vertex of $T_2(u) - S_{2,1}$ has a common neighbor with u, and hence $S_{2,3} \cup S_{2,2} \subseteq N(A)$. Similarly, $S_{2,2} \cup S_{3,2} \subseteq N(B)$.

For $w \in S_{3,3}$, we have $d_G(w, u) = d_G(w, v) = 3$, and hence w has a neighbor both in $S_{2,3} \cup S_{2,2}$ and in $S_{2,2} \cup S_{3,2}$. By construction, none of those neighbors are in $A' \cup B'$. Based on whether w has neighbors in I and/or J, we put w into X, Y, Z, or C. Thus to have neighbors in both $T_2(u)$ and $T_2(v)$, each vertex of $X \cup Y \cup C$ must have a neighbor in $S_{2,2}$.□

If vertices x and y lie in distinct ellipses in Figure 3, and no edge is drawn joining the two sets that contain them, then x and y are nonadjacent in G; this follows from the distance requirements and from the definitions of the sets.

The edges drawn in Figure 3 do not generally indicate complete bipartite subgraphs; they merely specify the orientation of whichever edges occur joining the sets at the head and tail. In particular, Figure 3 has ten more "vertical" arrows that we have not explained. Pairs of vertices in the relevant sets need not be adjacent and need not be nonadjacent; the distance conditions in G are satisfied by the edges we have already oriented. We orient any such edge that is present as shown. Edges joining distinct regions within $S_{3,3}$ may be oriented arbitrarily, except as indicated from X to C and from C to Y.

There remain five undirected edges in Figure 3; we use the next lemma to orient some of the edges they represent. The idea of the lemma comes from [1]. The edges will be oriented to bound $d_D(w, u)$ by 2 for $w \in A^*$, by 3 for $w \in A'$, and by 4 for $w \in C$. Similarly, we will have $d_D(v, w)$ bounded by 2 for for $w \in B^*$, by 3 for $w \in B'$, and by 4 for $w \in C$. The difficult cases that require additional care will be getting to u from the lower sets B, B^*, J, B' and from v to the upper sets A, A^*, I, A'. Not all of these will be achievable in distance at most 5, and hence in some cases from vertices of one set to another we will need to find a more direct route.

Given vertex sets S and S' in a graph, we use $[S, S']$ to denote the set of edges having endpoints in S and S'. A graph is *nontrivial* if it has at least one edge.

Lemma 3.4 *In a graph H, let S and S' be disjoint vertex sets such that $S' \subseteq N_H(S)$. If the induced subgraph $H[S']$ is connected and nontrivial, then there is an orientation F of $H[S'] \cup [S, S']$ such that $d_F(S, w) \le 2$ and $d_F(w, S) \le 2$ for every $w \in S'$.*

Proof. Let T be a spanning tree of $H[S']$. Let P and Q be the partite sets of T as a bipartite graph. Orient each edge of T from its endpoint in P to its endpoint in Q. Each vertex x in P has a neighbor in S; orient some such edge toward x. Each vertex y in Q has a neighbor in S; orient some such edge away from y. Now each edge of T lies on a path or cycle of length 3 departing from and returning to S. Since each vertex of S' belongs to such an edge, the conclusion follows. □

Definition 3.5 *Specification of D, part 2.* Call the orientation produced in Lemma 3.4 a *short orientation for $H[S']$ relative to S.* For vertices of A^*, B^*, A', B', and C, the *parent set* is $\{u\}$, $\{v\}$, A, B, or $S_{2,2}$, respectively. Let $S' = A^* \cup B^* \cup A' \cup B' \cup C$. For each nontrivial component H of $G[S']$, use in D a short orientation for H relative to its parent set. Also orient $[C, Z]$ arbitrarily.

It remains to consider isolated vertices in $G[S']$; let w be such a vertex. Since G is 2-edge-connected, w has at least two neighbors in G. Distance requirements give w at least one neighbor in its parent set. For $w \in A^*$, orient uw from u to w; already the other incident edges are oriented toward A. Similarly, for $w \in B^*$, orient wv from w to v. For $w \in A'$, orient $[w, A]$ so that w has exactly one successor, and this successor is in A; we postpone to Definition 3.7 the choice of the successor vertex in A. Similarly, for $w \in B'$, orient $[w, B]$ so that w has exactly one predecessor, which is in B. For $w \in C$, orient at least one edge of $[S_{2,2}, w]$ in each direction, unless it has only one edge, in which case orient it to form a path of length 2 with another edge incident to w. □

Lemma 3.6 *If $w \in A^* \cup B^* \cup A' \cup B'$, and S is the parent set of w, then $d_D(S, w) \le 2$ and $d_D(w, S) \le 2$, except that when w is an isolated vertex of $G[A']$ or $G[B']$, it may fail that $d(A, w) \le 2$ or $d(w, B) \le 2$.*

Proof. For vertices not isolated in $G[A^* \cup B^* \cup A' \cup B']$, Lemma 3.4 applies. Otherwise, the orientation produced by the last part of Definition 3.5 gives the specified vertex a successor or predecessor in the desired set, except for the case excluded. □

As indicated earlier, we seek short paths to u and from v for every vertex w. If $|B'| > |A'|$, then we can reverse all of D and interchange the roles of u and v to reduce to the case where $|A'| \ge |B'|$; hence we may assume that $|A'| \ge |B'|$. There are three main cases depending on whether neither, one, or both of these sets are empty. The case where $A' \ne \varnothing$ and $B' = \varnothing$ requires a modification of D and further specification of D within A. We pause to complete the definition of D before beginning our analysis of distances.

Definition 3.7 *Specification of D, part 3.* We define further subsets of A and B.

Vertices of B seek neighbors in sets close to u to form short paths to u in D. Let $B_4 = B \cap N(A)$. Let $B_3 = B \cap N(A \cup S_{2,2})$. Let $B_2 = B \cap N(A \cup S_{2,2} \cup J)$. Let $B_1 = B - B_2$. For an edge xy with $x \in B_i$ and $y \in B_j$, orient $x \to y$ if $i < j$.

Similarly, let $A_4 = A \cap N(B)$, $A_3 = A \cap N(B \cup S_{2,2})$, $A_2 = A \cap N(B \cup S_{2,2} \cup I)$, and $A_1 = A - A_2$. For an edge xy with $x \in A_i$ and $y \in A_j$, orient $y \to x$ if $i < j$.

We have postponed choosing the unique successor (in A) for a vertex y in A' that is isolated in $G[A']$. Let it be in A_i for the least possible index i; furthermore, let the successor be in $A_1 - N(A_2)$ if y has any neighbor there (in preference to $A_1 \cap N(A_2)$). Make the analogous choice of unique predecessor (in B) for any vertex of B' that is isolated in $G[B']$.

For two special types of vertices in $A_1 - N(A_2)$, we change the orientation of some incident edges. Consider $w \in A_1 - N(A_2)$. If all neighbors of w in A' lie in nontrivial components of $G[A']$, then orient all edges of $[w, A']$ toward w. If every edge of $[w, A']$ points toward A' (at this point this means that some successor of w in A' is an isolated vertex in $G[A']$), then reverse the orientation of wu to point toward w, and reverse each u, w-path through an isolated vertex of $G[A^*]$ so that it now travels from w to u. (These reversals are not needed until Lemma 4.8, and we will not need an analogous reversal step for vertices in B.)

All edges not yet specified are oriented arbitrarily. □

4 Upper Bound: Distance Analysis

In this section, we prove that the diameter of the orientation D defined in the preceding section is at most 11: that is, $d_D(x, y) \le 11$ for all $x, y \in V(D)$. The key vertices are u and v. We first use them to show that vertices of $A^* \cup B^*$ cause no trouble. Note that A and B are nonempty in all cases, since otherwise uv is a cut-edge. To see the arguments clearly, refer to Figure 3.

Lemma 4.1 *If $x \in A^*$, $y \in B^*$, and $w \in V(D)$, then*

$$d_D(w, x) \le d_D(w, u) + 2, \qquad d_D(y, w) \le d_D(v, w) + 2,$$
$$d_D(x, w) \le d_D(u, w) + 2, \qquad d_D(w, y) \le d_D(w, v) + 2.$$

Proof. This follows from the triangle inequality and Lemma 3.6. □

The simplest entries in the next table were already explained in Lemma 3.3 for motivation.

Lemma 4.2 *Distances in D to u or from v are bounded as in the table below. Furthermore, equality in the large values for I and J require the vertex to have no neighbor in $S_{2,2}$ or in the other member of $\{I, J\}$.*

for w in	A_2	B_2	I	J	$S_{2,2}$	X	Y	Z	C	A'	B'	A^*	B^*
$d_D(w, u) \le$	1	5	2	4	2	3	3	3	4	3		2	
$d_D(v, w) \le$	5	1	4	2	2	3	3	3	4		3		2

Proof. The distance classes in Definition 3.2 yield the smaller values in columns up to Z.

When $w \in J$, the partition of $S_{3,2}$ implies that $N(w)$ intersects $I \cup S_{2,2} \cup S_{3,3}$, and hence $d_D(w, u) \le 4$, with equality only when w has no neighbor in $I \cup S_{2,2}$. Similarly, $d_D(v, w) \le 4$ when $w \in I$, with equality only when w has no neighbor in $J \cup S_{2,2}$.

Similarly, a vertex $w \in B_2$ has a neighbor in $A \cup S_{2,2} \cup J$. Prepending w to a path from such a neighbor yields $d_D(w, u) \le 5$, with equality only when w has no neighbor in $A \cup S_{2,2}$. The analogous statement holds for $w \in A_2$.

The values in the last five columns use Lemma 3.6 for nonisolated vertices of $G[C \cup A' \cup B' \cup A^* \cup B^*]$. Also every isolated vertex of $G[A']$ has a successor in A, and every isolated vertex of $G[B']$ has a successor in B. □

Other bounds require certain sets to be nonempty.

Lemma 4.3 *If $A' \neq \varnothing$, then $d_D(w,u) \le 4$ for all $w \in B$ (with equality only when w has no neighbor in $A \cup S_{2,2}$); also $d_D(w',u) \le 6$ for all $w' \in B'$, and $d_D(v,u) \le 5$ (with equality only when both $[B,A]$ and $S_{2,2}$ are empty). Similarly, if $B' \neq \varnothing$, then $d_D(v,w) \le 4$ for all $w \in A$ (with equality only when w has no neighbor in $B \cup S_{2,2}$), and $d_D(v,w') \le 6$ for all $w' \in A'$.*

Proof. Consider $w \in B$. If w has a neighbor in $A \cup S_{2,2}$, then $d_D(w,u) \le 3$ (Definition 3.7 prevents arriving in A at a vertex of A_1). Hence we may assume that any path P of length at most 3 from w to A' in G starts along an edge to $J \cup B$; let x be the neighbor of w on P.

If $x \in J$, then reaching A' in two more steps requires P to next visit I. The first two edges of P now form a path in D from w, and appending two more edges yields $d_D(w,u) \le 4$.

If $x \in B$, then reaching A' in two more steps requires x to have a neighbor in A, so $x \in B_4$. Since $w \notin B_4$, the edge wx is oriented toward x, by Definition 3.7. Now the first two edges of P form a path in D from w to A, and appending one edge yields $d_D(w,u) \le 3$.

For $w' \in B'$, if there exists $w \in B$ such that $d_D(w',w) \le 2$, then $d_D(w,u) \le 4$ yields $d_D(w',u) \le 6$. Such a vertex w fails to exist only when w' is isolated in $G[B']$ and has only one neighbor in B, a predecessor. Then w' has a successor in J and $d_D(w',u) \le 5$.

Prepending v to a path from B to u yields $d_D(v,u) \le 5$. Equality requires $d_D(w,u) = 4$ for all $w \in B$, which requires both $[B,A]$ and $S_{2,2}$ to be empty.

An argument symmetric to this proves the statements for $B' \neq \varnothing$. □

Lemma 4.4 *If A' and B' are nonempty, then upper bounds on $d_D(w,u)$ and $d_D(v,w)$ are as listed below. Furthermore, the large values for A^* and B^* can hold with equality only when $[B,A]$ and $S_{2,2}$ are empty.*

for w in	A	B	I	J	$S_{2,2}$	X	Y	Z	C	A'	B'	A^*	B^*
$d_D(w,u) \le$	1	4	2	4	2	3	3	3	4	3	6	2	7
$d_D(v,w) \le$	4	1	4	2	2	3	3	3	4	6	3	7	2

Proof. Note first that $A_1 = \varnothing$ when $B' \neq \varnothing$, since $d_G(x,y) \ge 4$ when $x \in A_1$ and $y \in B'$. Hence wu is oriented toward u for all $w \in A$.

The values from Lemma 4.2 are valid whether A' and B' are empty or not.

The large values for A^* and B^* follow from Lemmas 4.1 and 4.3, using $d_D(v,u) \le 5$.

The large values for A, B, A', and B' follow from Lemma 4.3. □

Lemma 4.5 *If A' and B' are nonempty, then $f(G) \leq 11$.*

Proof. Consider $x, y \in V(D)$. By Lemmas 4.1 and 4.4, $d_D(x, y) \leq 9$ when x or y is in $A^* \cup B^*$. Hence we may assume $x, y \notin A^* \cup B^*$.

The triangle inequality yields $d_D(x, y) \leq d_D(x, u) + 1 + d_D(v, y)$. By Lemma 4.4, this bound is at most 11 except when $x \in B'$ and $y \in A'$ with $\max\{d_D(x, u), d_D(v, y)\} = 6$ and $\min\{d_D(x, u), d_D(v, y)\} \geq 5$. We argue that then there is a shorter x, y-path in D.

By symmetry, we may assume that $d_D(x, u) = 6$. By Lemma 4.4, x has no successor in $J \cup B$, and hence x is a source in a nontrivial component of $D[B']$. Its sinks have successors in B, so we may choose $z \in B$ with $d_D(x, z) = 2$. Now $d_D(x, u) = 6$ requires $d_D(z, u) = 4$, and hence z has no neighbor in A or $S_{2,2}$, by Lemma 4.3. Furthermore, z has no neighbor in B that has a neighbor in A; by Definition 3.7 this again would yield $d_D(z, u) \leq 3$.

Now, let P be a z, y-path in G with length 3, with vertices z, a, b, y in order. We have argued that $a \notin A \cup S_{2,2}$ and that $b \notin A$. Therefore, $a \in J$ and $b \in I$. Since $[I, A']$ is oriented toward A', we conclude that P is a path in D, and hence $d_D(x, y) \leq 5$.

We have shown that $d_D(x, y) \leq 5$ if $\max\{d_D(x, u), d_D(v, y)\} = 6$. On the other hand, if both are at most 5, then $d_D(x, y) \leq 11$, as desired. □

The second case is even easier.

Lemma 4.6 *If $A' = B' = \varnothing$, then $f(G) \leq 11$.*

Proof. Since $A' = \varnothing$, we have $A_2 = A$, and all of $[A, u]$ is oriented toward u. If $w \in B$, then by definition w has a neighbor in $A \cup S_{2,2} \cup J$. If $N(w)$ intersects $A \cup S_{2,2}$, then $d_D(w, u) \leq 3$. If w has a neighbor x in J, then x has a neighbor in $I \cup S_{2,2} \cup S_{3,3}$, and $d_D(w, u) \leq 5$. We obtain

for w in	A	B	I	J	$S_{2,2}$	X	Y	Z	C	A^*	B^*
$d_D(w, u) \leq$	1	5	2	4	2	3	3	3	4	2	7
$d_D(v, w) \leq$	5	1	4	2	2	3	3	3	4	7	2

By Lemma 4.1, $d_D(x, y) \leq 9$ when x or y is in $A^* \cup B^*$. Hence we may assume that $x, y \notin A^* \cup B^*$. The table above now combines with $d_D(x, y) \leq d_D(x, u) + d_D(u, v) + d_D(v, y)$ to prove that $d_D(x, y)) \leq 11$ for all $x, y \in V(D)$. □

The most difficult case is $A' \neq \varnothing$ and $B' = \varnothing$. We need several lemmas. The first establishes short paths in D for special vertices under special technical conditions.

Lemma 4.7 *If $y \in A'$, and all edges of $[A_2 \cup N(A_2), y]$ are oriented toward y except possibly one edge yw with $w \in A$, then $d_D(v, y) \leq 5$.*

Proof. Note that $A_2 \cup N(A_2)$ contains I. Thus we are considering all edges joining y to $A \cup I$ except those with endpoints in $A_1 - N(A_2)$. Note first that there is at least one such edge; otherwise, y cannot reach any vertex of B in three steps in G.

In all cases, we find a short v, y-path in G that is also a path in D. Any neighbor of y in I is a predecessor of y, and then $d_D(v, y) \leq 5$, since $d_D(v, x) \leq 4$ for all $x \in I$. Hence we may assume that $N(y) \subseteq A \cup A'$.

Choose $z \in B$, and let P be a shortest z, y-path in G. Since P has length at most 3 and z cannot reach $A' \cup (A_1 - N(A_2)$ in two steps, P must reach y via an edge from $A_2 \cup N(A_2)$. If P is oriented as a path in D, then $d_D(v, y) \leq 4$. By Definition 3.7, P is a path in D unless the last edge in P is wy. Suppose that this is so.

If y is an isolated vertex in $G[A']$, then y has another neighbor x in A, and the choice of the unique successor of y via Definition 3.7 ensures that there is a path from B to y through x that is no longer than P; hence again $d_D(v, y) \leq 4$.

If y is a source vertex in a nontrivial component of $D[A']$, then the last edge of P cannot be directed away from y, and $d_D(v, y) \leq 4$. If y is a sink vertex in such a component, then we append an edge to a path reaching any predecessor in that component to obtain $d_D(v, y) \leq 5$. □

Let A_0 be the set of vertices in $A_1 - N(A_2)$ such that the edge uw points toward w (Definition 3.7). These are the vertices of $A_1 - N(A_2)$ having no predecessors in A'. As we have observed, each vertex of A_0 has a successor that is an isolated vertex in $G[A']$.

Lemma 4.8 *If $A' \neq \varnothing = B'$, then $d_D(v, y) \leq 6$ for $y \in A'$ and $d_D(v, w) \leq 7$ for $w \in A$, with equality for the latter only when $w \in A_1 - N(A_2)$ and all neighbors of w in A' are sinks in nontrivial components of $D[A']$.*

Proof. Note that $A = A_2 \cup A_1$. When $A' \neq \varnothing$, diameter 3 for G yields $B_1 = \varnothing$, so $B_2 = B$.

If R is a nontrivial component of $D[A']$, then Lemma 4.7 applies whenever y is a source vertex of R. Thus $d_D(v, y) \leq 5$ when y is a source vertex of R, and $d_D(v, y) \leq 6$ when y is a sink vertex of R. If y is isolated in $G[A']$, then by construction y has exactly one successor, which lies in A. Now Lemma 4.7 yields $d_D(v, y) \leq 5$.

Now consider $w \in A$. If $w \in A_2$, then $d_D(v, w) \leq 5$, by Lemma 4.2. If $w \in A_1 \cap N(A_2)$, then the orientation of all edges of $[A_2, A_1]$ toward A_1 yields $d_D(v, w) \leq 6$. If $w \in A_0$, then $d_D(v, u) \leq 5$ yields $d_D(v, w) \leq 6$. Hence we may assume that $w \in A_1 - N(A_2)$ and w is a successor of some vertex y in A'. By the cases considered above, we have $d_D(v, w) \leq 6$ unless all neighbors of w in A' are in nontrivial components of $G[A']$. For such a vertex w, Definition 3.7 orients all edges of $[A', w]$ toward w, so we have $d_D(v, w) \leq 7$, with equality only if all neighbors of w are sink vertices of nontrivial components in $D[A']$. □

Theorem 4.9 *If G is a graph with diameter 3, then $f(G) \leq 11$.*

Proof. Lemmas 4.5 and 4.6 handle the cases where A' or B' are both empty or both nonempty. Since we applied symmetry to assume that $|A'| \geq |B'|$ before defining D, the remaining case is $A' \neq \varnothing$ and $B' = \varnothing$.

We must consider $d_D(w,u)$ for $w \in A_0$ (this set is empty when $B' \neq \varnothing$). Such a vertex w has a successor x in $A^* \cup A'$. If $x \in A'$, then by Definition 3.7 x is isolated in $G[A']$ and has a successor in $A - A_0$. Hence $d_D(w,u) \leq 3$.

Note also that all edges from A to u used in forming short paths to u did not use vertices of A_0, so the reversal of wu in Definition 3.7 did not damage any needed short paths.

The preceding lemmas now yield the following table:

for w in	A	B	I	J	$S_{2,2}$	X	Y	Z	C	A'	A^*	B^*
$d_D(w,u) \leq$	3	4	2	4	2	3	3	3	4	3	2	6
$d_D(v,w) \leq$	7	1	4	2	2	3	3	3	4	6	8	2

By Lemma 4.1, $d_D(x,y) \leq 10$ when x or y is in $A^* \cup B^*$. Hence we may assume $x, y \notin A^* \cup B^*$. The table above now combines with $d_D(x,w) \leq d_D(x,u) + 1 + d_D(v,w)$ to prove that $d_D(x,w)) \leq 11$ except when $x \in B \cup J \cup C$ with $d_D(x,u) = 4$ and $w \in A$ with $d_D(v,w) = 7$. Consider such x and w.

As discussed in Lemma 4.8, all neighbors of w in A' are sinks in nontrivial components of $D[A']$. Also, w has no neighbor in A_2. If $x \in C \cup J$ with $d_D(x,u) = 4$, then x has no neighbor in I, so $d_G(x,w) \geq 4$. Hence $x \notin C \cup J$.

If $x \in B$ with $d_D(x,u) = 4$, then x has no neighbor in $A \cup S_{2,2}$, In this case, let y be a neighbor of w in A'. As remarked earlier, y has no neighbor in I. Let P be an x,y-path of length 3 in G, with vertices x, b, a, y in order. If $b \in J$ and $a \in I$, then we obtain a short x,w-path in D. Hence $b \in B$ and $a \in A$, with ay directed toward a. Now $d_D(x,u) \leq 3$. Hence if there exist w such that $d_D(v,w) = 7$, then $d_D(x,u) \leq 3$ for all $x \in V(G) - B^*$. □

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