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Some Remarks on Normalized Matching

DOUGLAS B. WEST

University of Illinois, Urbana, Illinois 61801

LAWRENCE H. HARPER

University of California, Riverside, California 92502

AND

DAVID E. DAYKIN

University of Reading, Berks, England

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A new class of LYM orders is obtained, and several results about general LYM orders are proved. (1) Let $A_1 \subset A_2 \subset \cdots \subset A_r$ be a chain of subsets of $[\mathbf{n}] = \{1,..., n\}$. Let $\langle a_i \rangle$ and $\langle b_i \rangle$ be two nondecreasing sequences with $a_i \leqslant b_i$ for $1 \leqslant i \leqslant r$. Then $\{X \subset [\mathbf{n}]: a_i \leqslant |X \cap A_i| \leqslant b_i\}$, ordered by inclusion, is a poset having the LYM property. (2) The smallest regular covering of an LYM order has $M(P)$ chains, where $M(P)$ is the least common multiple of the rank sizes. (3) Every LYM order has a smallest regular covering with at most $|P| - h(P)$ classes of distinct chains, where $h(P)$ is the height of P. To obtain (3), we discuss "minimal sets" of covering relations between two adjacent levels of an LYM-order.

1. INTRODUCTION

This paper grew out of discussions at the Symposium on Ordered Sets in Banff, 1981. We have several remarks to make on the subject of normalized matching in partially ordered sets (henceforth posets). These include proving that a certain class of posets have the LYM property, computing the size of the smallest regular coverings of LYM orders, and discussing the properties of "minimal" LYM orders. Any terms not explicitly defined herein can be found in [3] or [12].

There are three widely-used equivalent definitions of LYM orders. These are the normalized matching property, the LYM property, and the existence of a regular covering of the poset by chains. Kleitman [8] proved the

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equivalence of these properties: we will define them as needed. The most commonly used definition is via the LYM inequality [10, 11, 13]. A ranked poset has the LYM *property* if any antichain $F \subset P$ satisfies the following inequality, called the LYM *inequality*:

$$\sum_{x \in F} \frac{1}{N_x} \leqslant 1,$$

where N_x is the size of the rank containing x. In this paper, the other equivalent characterizations will be of more use.

Our new class of posets includes a class shown to be LYM by Lih [9]. Let $S(X)$ consist of the subsets of a set X, ordered by inclusion, and let $[\mathbf{n}] = \{1,..., n\}$. Given a fixed subset $A \subset [\mathbf{n}]$, Lih proved that the subposet $P \subset S([\mathbf{n}])$ defined by $P = \{X \subset [\mathbf{n}]: X \cap A \neq \varnothing\}$ is an LYM order. He proved this using an early theorem due to Harper [6] (see also Hsieh and Kleitman [7]), which gave sufficient conditions for the poset given by the direct product of two LYM orders to be an LYM order. We will use this theorem, henceforth called the "product theorem," to show that posets in a more general class of posets including Lih's posets are all LYM orders. First, we note that Griggs [5] used the product theorem to obtain a different generalization of Lih's result. Take a partition of $[\mathbf{n}]$ into blocks $B_1,..., B_r$. For each block take an arithmetic sequence $\{a_i + jb_i: j \geqslant 0\}$. Let P be the subsets $X \subset [\mathbf{n}]$ such that for all i the size of $X \cap B_i$ is in the ith arithmetic sequence. Griggs proved that P is always an LYM order. Our generalization of Lih's result is Theorem 1. There are similarities in statement and proof between Griggs' theorem and this theorem. However, we have not found a common generalization.

THEOREM 1. *Let $A_1 \subset A_2 \subset \cdots \subset A_r$ be a chain in $S([\mathbf{n}])$. Let $\langle a_i \rangle$ and $\langle b_i \rangle$ be two nondecreasing sequences with $a_i \leqslant b_i$ for $1 \leqslant i \leqslant r$. Let $P = \{X \subset [\mathbf{n}]: a_i \leqslant |X \cap A_i| \leqslant b_i\}$. Then P is an* LYM *order, and in fact the rank sizes $N_0, N_1,...$ of P satisfy $N_k^2 \geqslant N_{k-1} N_{k+1}$.*

After proving this theorem, we will discuss other aspects of LYM orders. We show that the smallest regular coverings of an LYM order have $M(P)$ chains, where $M(P)$ is the least common multiple of the rank sizes. This leads us to discuss "minimal" LYM orders, which are posets from which no covering relations can be discarded without losing the LYM property. We point out several characteristics of such posets. This enables us to obtain an upper bound on the smallest number of distinct chains in a regular covering of any LYM order. If $h(P)$ is the height of P (one less than the number of ranks), then P has a regular covering with at most $|P| - h(P)$ distinct chains. Finally, we note that minimal LYM orders might be useful in studying a well-known open question about LYM orders. It is conjectured that all LYM orders have "completely saturated partitions" (see [2, 4, 12]). This is true if and only if all *minimal* LYM orders have completely saturated partitions, which was our original motivation for studying these orders.

2. Proof of Theorem 1

We prove Theorem 1 by induction on n and r. We will use the normalized matching property instead of the equivalent LYM property, since it is a local property more susceptible to proof by induction. Let P_k be the set of elements in the kth rank of P, and let $N_k = |P_k|$. A ranked poset P has the *normalized matching property* (introduced by Graham and Harper [1]) if, whenever $F \subset P_k$ and F^* is the subset of P_{k+1} consisting of the elements of P_{k+1} covering elements of F, it holds that

$$\frac{|F|}{N_k} \leqslant \frac{|F^*|}{N_{k+1}}.$$

Our induction is an easy consequence of the product theorem, which we now describe. The *direct product* of two posets Q and R is $\{(a, b): a \in Q, b \in R\}$, ordered by $(a, b) \leqslant (c, d)$ iff $a \leqslant b$ and $c \leqslant d$. The theorem states that the product of two LYM orders satisfying $N_k^2 \geqslant N_{k-1}N_{k+1}$ for all k is also an LYM order satisfying that property. This numerical property of the rank sizes is called *log-concavity*.

Let P be the poset specified in the statement of the theorem. For $r = 0$ there is no restriction, and P is $S(\mathbf{[n]})$, which is a log-concave LYM order. Assume $r > 0$, and let $\bar{A}_r = \mathbf{[n]} - A_r$. There are no restrictions on the elements of $\mathbf{[n]}$ not contained in A_r. That is, if $X \subset \bar{A}_r$ and $Y \subset A_r$, with $Y \in P$, then $(X \cup Y) \in P$. Hence, P is the direct product of $S(\bar{A}_r)$ and the subposet of $S(A_r)$ generated by the given conditions. Call the latter poset P'.

Since $S(\bar{A}_r)$ is a log-concave LYM order, the product theorem will yield the desired result if P' also has those properties. If $\bar{A}_r$ is nonempty, we apply induction on n. If $A_r = \mathbf{[n]}$, let P'' be the subposet of $S(A_r)$ obtained by dropping the restriction $a_r \leqslant |X \cap A_r| \leqslant b_r$. By induction on r, P'' is a log-concave LYM order. The poset P' is obtained from P'' merely by deleting top ranks and bottom ranks to conform to the restriction $a_r \leqslant |X| \leqslant b_r$, since in this case A_r is the entire set. However, deleding top or bottom ranks of a log-concave LYM order does not destroy log-concavity or the normalized matching property. This completes the proof.

3. Minimum Coverings

The third characterization of LYM orders is by the existence of a regular covering by chains. A ranked poset has a *regular covering by chains* if there is a collection of chains in the poset, not necessarily distinct, such that (1) the chains intersect each rank, and (2) each element of a given rank occurs in the same proportion of the chains in the covering.

To generate regular coverings, one can use the following version of the normalized matching property, proved by Graham and Harper [1]. As before, let P_k be the kth rank of a poset P, and let $N_k = |P_k|$. Consider the bipartite graph generated by the covering relations between P_k and P_{k+1}. Direct the edges upward. Assign a supply of N_{k+1} to each node from P_k and a demand of N_k to each node from P_{k+1}. They proved that P is an LYM order if and only if all these transportation problems between pairs of adjacent levels have solutions. By this we mean that the associated max flow problem has as its maximum flow the sum of the supplies (or demands).

If the transportation problems have solutions, they yield integer flows for each of the edges in the Hasse diagram. Call these the *standard edge labels*. Note that these transportation problems need not have unique solutions; we merely pick one set of solutions. For each maximal chain in the partial order, let its weight be the product of the standard edge labels on its edges. Using these weights as the multiplicities of chains gives a regular covering. The proof of this is essentially the constructive proof that the normalized matching property implies the existence of a regular covering by chains.

Given a particular poset, solving these transportation (flow) problems is more efficient that the matching procedure Kleitman gave in [8] for generating regular coverings. However, both procedures generate regular coverings of size $\prod_k N_k$, which is larger than necessary.

It is easy to find ways to decrease the number of chains in this regular covering. For example, if the standard edge labels give the same label to some minimal set of edges which separate the top level from the bottom level, then those labels can be replaced by 1. Similarly, they could all be divided by a common factor. Figure 1 illustrates this with the product of two

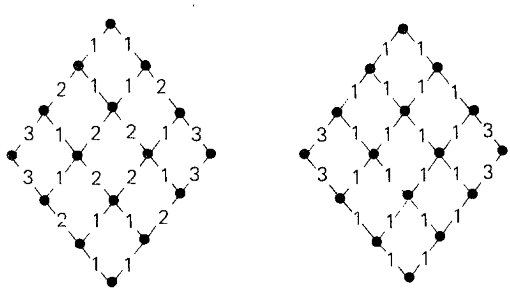

Figure 1

4-element chains. This is the smallest example we found where the number of chains could be reduced by eliminating common factors from the edge labels of separating sets.

To generalize this example, let C_n be an n-element chain. For $C_m \times C_n$ with $n \leqslant m$, the standard edge labels produce a regular covering with $n!\,(n-1)!$ chains, using 2^{2n} distinct chains. Using the product of standard edge labels as chain multiplicities almost never achieves the minimum covering, though it would be interesting to know bounds on the size ratio between such coverings and the smallest regular covering. Dividing out common factors can effect some reduction in the total number of chains, but it can never decrease the number of distinct chains used in a covering.

Now we ask for the minimum number of chains in a regular covering and the minimum number of distinct chains in such a covering. An obvious lower bound on the number of chains is the least common multiple of $\{N_k\}$, which we call $M(P)$. In light of the above examples, it is surprising that this lower bound can always be achieved. In addition to solving the extremal problem, this theorem gives an alternative constructive proof that the normalized matching property implies the existence of a regular covering.

THEOREM 2. *Any* LYM *order* P *has a minimum regular covering with* $M(P)$ *chains.*

Proof. Begin by obtaining edge labels similar to the standard edge labels, but this time use $M(P)/N_k$ as the demand or supply for the elements in P_k. The resulting edge labels between P_k and P_{k+1} will be the standard edge labels multiplied by $M(P)/\mathrm{lcm}(N_k, N_{k+1})$. Also, the *total* of the labels assigned between each pair of adjacent levels will be $M(P)$. Now it is easy to construct the chains. For each point in P_k, a flow of $M(P)/N_k$ units enters from below, and the same flow departs above. Match the units of flow entering to the units leaving, in any manner. The resulting $M(P)$ chains form a regular covering, since each element of P_k lies on $M(P)/N_k$ of them. ∎

Figure 2 illustrates this procedure. The regular covering has 30 chains. In contrast to the method discussed earlier, here the edge labels are not multiplied together. This explains why the covering has fewer chains.

Finding the smallest number of distinct chains in a regular covering is harder. In the next section, we will obtain an upper bound.

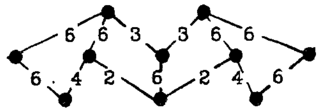

FIGURE 2

4. Minimal LYM Orders

In this section we discuss the characteristics of minimal LYM orders and mention applications to minimal chain coverings and completely saturated partitions. A *minimal* LYM order is one for which the deletion of any covering relation in the Hasse diagram destroys the LYM property. The example at the end of Section 3 is such an LYM order, as is $C_m \times C_n$.

Obviously, any LYM order can be reduced to a minimal LYM order by throwing away some edges. Several properties of such an order follow easily.

REMARK 1. Let G be the bipartite graph of the relations on two adjacent ranks of a minimal LYM order, where the rank sizes are m and n. If $m > n$, then G is a forest having all its vertices of degree 1 in the rank with m elements. If $m = n$, then G is a complete matching.

Proof. Assign to the edges of G a set of standard edge labels obtained by solving the Graham–Harper transportation problem on these two ranks. Suppose that G has some cycle, on which the smallest label is s, appearing on edge e. The cycle has even length, so we can subtract s from the label on every edge at even distance from e and add s to every label at odd distance from e. The resulting set of labels is also a set of standard edge labels, since the flow in or out of any vertex is the same as it was before. However, now e is labeled 0 and can be dropped from G without destroying the LYM property. After eliminating all the cycles, G is a forest. If the smaller rank has a vertex of degree 1, then the set consisting of that vertex alone violates the normalized matching property.

If $m = n$, we eliminate cycles as before and consider a vertex of degree 1. Its neighbor has the same supply or demand requirement, so the flow on this edge completely satisfies both vertices. Thus, any other edge incident to the neighbor receives flow 0 and can be deleted from G. ∎

REMARK 2. Let G be as above. If G consists of more than a single tree, then the two rank sizes m and n have a greatest common factor $d > 1$, and each tree of G contains vertices from the two ranks in the same ratio as m/n.

Proof. If some tree of G contains r vertices from the part of size m and s vertices from the part of size n, then by the normalized matching property we have $s/n \geqslant r/m$ and $r/m \geqslant s/n$. Thus $r/s = m/n$, and m and n cannot be relatively prime. ∎

Unfortunately, the converse does not hold. In particular, the trees or forests that can occur are not uniquely determined by the rank sizes. The smallest instance of nonuniqueness is $(m, n) = (6, 4)$; Fig. 3 also illustrates that the number of trees in G is not uniquely determined by the rank sizes.

Although this result is negative, we still hope for a characterization of the

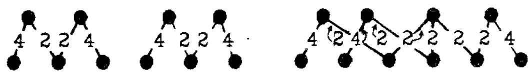

FIGURE 3

forests that can arise in minimal LYM orders. Meanwhile, we give an application of Remark 1. Let $h(P)$ be the *height* of P. (This is one less than the number of ranks in P.)

THEOREM 3. *Any* LYM *order P has a minimum regular covering in which the $M(P)$ chains fall into $|P|-h(P)$ classes of distinct chains.*

Proof. Given P, find a minimal LYM order Q within it by deleting some relations, if necessary. Any (minimum) regular covering of Q is also a (minimum) regular covering of P. Obtain edge labels for Q as in the proof of Theorem 2. As in that proof, we construct the chains by matching the units of flow entering each element to the units of flow leaving it.

We proceed from the bottom up, constructing the chains level by level. At P_0, we partition the $M(P)$ chains into N_0 classes by letting each element of P_0 be the bottom element of $M(P)/N_0$ chains. Having reached the kth rank, we extend the chains to the next rank as follows. Let x be any element of P_k. Place a linear order on the classes of distinct chains entering x from below, and place a linear order on the edges extending upward from x. Since the regular covering has $M(P)$ chains, the multiplicities of the chains entering x from below and the edge labels on the edges from x to P_{k+1} both sum to $M(P)/N_k$. Extend the chains reaching x, using the chosen linear orders, by iteratively extending the "least" unextended chain along the "least" unassigned unit of flow from x to the next level. After $M(P)/N_k$ steps, all the chains have been extended, and the number of times each edge from x to P_{k+1} has been used equals its edge label.

We need to count how many classes of distint chains are produced by this procedure. The linear order on the chains entering x induces a set of partial sums of the chain multiplicities; call this set A. Similarly, the order on the edges from x induces a set B of partial sums for the flow multiplicites. Note that $|B|$ is the number of elements covering x. Since the matching procedure is consistent with both orderings, $A \cup B$ is the set of partial sums for the classes of distinct chains reaching P_{k+1} via x. The number of distinct chains is $|A \cup B|$, with the chain multiplicities being the difference of consecutive partial sums. If $|B|=s$, then $|A \cup B|-|A| \leqslant s-1$, since $M(P)/N_k$ belongs to both A and B.

Summing this bound of $s-1$ over all $x \in P_k$, we obtain an upper bound on the increase in the number of classes of distinct chains when the construction proceeds from P_k to P_{k+1}. By Remark 1, the relations between the levels form a forest, so there are at most $N_k + N_{k+1} - 1$ of them. Since we count $s(x) - 1$ for each of the N_k elements in P_k, the total increase is at most $N_{k+1} - 1$. When the top rank is reached, the chains we have constructed belong to at most $N_0 + \sum_{i \geqslant 1} (N_i - 1) = |P| - h(P)$ distinct classes. ∎

Since we know only the size and height of the poset, this bound is best possible. For any LYM poset in which at most one rank has more than one element, the minimum number of distinct chains in a regular covering is exactly $|P| - h(P)$. In the example at the end of Section 3, $|P| - h(P) = 8$, and it is easy to see that this poset has no regular covering with less than 8 distinct chains. It would be interesting to characterize the posets for which $|P| - h(P)$ is the exact answer. We do not know an efficient algorithm to compute the minimum number of distinct chains in a regular covering of a fixed LYM order, or even a fixed minimal LYM order.

REFERENCES

1. R. L. GRAHAM AND L. H. HARPER, Some results on matching in bipartite graphs, *SIAM J. Appl. Math.* **17** (1969), 1017–1022.
2. C. GREENE AND D. J. KLEITMAN, The structure of Sperner k-families, *J. Combin. Theory Ser. A* **20** (1976), 41–68.
3. C. GREENE AND D. J. KLEITMAN, Proof techniques in the theory of finite sets, *in* "Studies in Combinatorics" (G.-C. Rota, Ed.), MAA Studies in Math, Vol. 17 pp. 22–79, 1978.
4. J. R. GRIGGS, Sufficient conditions for a symmetric chain order, *SIAM J. Appl. Math.* **32** (1977), 807–809.
5. J. R. GRIGGS, Collections of subsets with the Sperner property, *Trans. Amer. Math. Soc.* **269** (1982), 575–591.
6. L. H. HARPER, The morphology of partially ordered sets, *J. Combin. Theory Ser. A* **17** (1974), 44–58.
7. W. N. HSIEH AND D. J. KLEITMAN, Normalized matching in direct products of partial orders, *Stud. Appl. Math.* **52** (1973), 285–289.
8. D. J. KLEITMAN, On an extremal property of antichains in partial orders: The LYM property and some of its implications and applications, *in* "Combinatorics" (Hall and J. H. Van Lint, Eds.), *Math. Centre Tracts* **56** (1974), 77–90.
9. K.-W. LIH, Sperner families over a subset. *J. Combin. Theory Ser. A* **29** (1980), 182–185.
10. D. LUBELL, A short proof of Sperner's lemma, *J. Combin. Theory* (1966), 299.
11. L. D. MESHALKIN, A generalization of Sperner's Theorem on the number of subsets of a finite set, *Teor. Verojatnost. i Primenen.* **8** (1963), 219–220 [Russian]; *Theory Probab. Appl.* **8** (1963), 203–204. [English]
12. D. B. WEST, Extremal problems in partially ordered sets, *in* "Ordered Sets" (I. Rival, Ed.), pp. 473–521, Reidel, Dordrecht, 1982.
13. K. YAMAMOTO, Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343–353.